6. Examples

In this section, we deal with the mathematics of Examples 2, 3, 4, 5, 6 and 11.

Example 2. *The visibility of surfaces of revolution*
A surface of revolution is generated by rotating a planar curve γ about an axis in the plane of the curve. We may assume that γ is in the x^1x^3 plane and has a parametric representation

$$\vec{x}(t) = (r(t), 0, h(t)) \ (t \in I \subset \mathbb{R}) \text{ where } r(t) > 0 \text{ on } I,$$

and that the axis of rotation is the x^3–axis. Furthermore we assume

$$(r'(u^1))^2 + (h'(u^1))^2 \neq 0 \text{ on } I, \tag{6.1}$$

which is a natural assumption connected with the smoothness of γ. We put $u^1 = t$ and write u^2 for the angle of rotation. Then the surface of rotation $RS(\gamma)$ generated by γ in this way has a parametric representation

$$\vec{x}(u^i) = (r(u^1)\cos u^2, r(u^1)\sin u^1, h(u^1)) \qquad ((u^1, u^2) \in D = I \times (0, 2\pi)) \tag{6.2}$$

Putting $\vec{u} = \vec{u}(u^2) = (\cos u^2, \sin u^2, 0)$ and $\vec{e}^{\,3} = (0, 0, 1)$, we may write (6.2) as

$$\vec{x}(u^i) = r(u^1)\vec{u} + h(u^1)\vec{e}^{\,3} \quad ((u^1, u^2) \in D). \tag{6.3}$$

First we determine the intersections of the surface of rotation $RS(\gamma)$ with a straight line g given by a parametric representation

$$\vec{y}(t) = \vec{p} + t\vec{v} \quad (t \in \mathbb{R}) \tag{6.4}$$

where $\vec{p} = (p^1, p^2, p^3)$ and $\vec{v} = (v^1, v^2, v^3)$, that is we find $(u^1, u^2) \in D$ and $t \in \mathbb{R}$ with

$$r(u^1)\vec{u} + h(u^1)\vec{e}^{\,3} - (\vec{p} + t\vec{v}) = \vec{0}. \tag{6.5}$$

This implies

$$h(u^1) - (p^3 + tv^3) = 0. \tag{6.6}$$

First we consider the case $v^3 \neq 0$ when g is not orthogonal to the axis of rotation of $RS(\gamma)$. Then (6.6) implies

$$t = \frac{h(u^1) - p^3}{v^3}. \tag{6.7}$$

We put

$$\vec{a} = \vec{p} - \frac{p^3}{v^3}\,\vec{v} \text{ and } \vec{b} = \frac{1}{v^3}\,\vec{v}$$

and, taking the square in (6.5) and substituting (6.7), we obtain

$$r^2(u^1) + h^2(u^1) = \left(\vec{p} - \frac{p^3}{v^3}\,\vec{v} + \frac{h(u^1)}{v^3}\,\vec{v}\right)^2 = (\vec{a} + h(u^1)\vec{b})^2.$$

Hence we must find the zeros $u^1 \in I$ of

$$f(u^1) = r^2(u^1) + h^2(u^1) - (\vec{a} + h(u^1)\vec{b})^2 = 0. \tag{6.8}$$

Using the zeros u_0^1 of (6.8), we determine the values $t_0 = t(u_0^1)$ in (6.7) and finally the values $u_0^2 \in (0, 2\pi)$ from

$$\cos u_0^2 = \frac{p^1 + t_0 v^1}{r(u_0^1)} \text{ and } \sin u_0^2 = \frac{p^2 + t_0 v^2}{r(u_0^1)}. \tag{6.9}$$

Now a point $P \in RS(\gamma)$ is invisible if and only if, with $\vec{p} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{PC}$, there is a solution $u_0^1 \in I$ of (6.8) with corresponding $t_0 > 0$ from (6.7) and $u_0^2 \in (0, 2\pi)$ from (6.9).

Now we consider the case $v^3 = 0$ when the straight line g is orthogonal to the axis of rotation of $RS(\gamma)$. Then (6.6) implies

$$f(u^1) = h(u^1) - p^3 = 0. \tag{6.10}$$

We determine the zeros $u_0^1 \in I$ of (6.10). For each u_0^2 there are at most two intersections with the corresponding parallel, that is with the u^2 line corresponding to u_0^1. The corresponding values $t_0 = t(u_0^1)$ are the solutions of

$$t^2\vec{v}^2 + 2t\vec{p} \bullet \vec{v} + \vec{p}^2 - \left(r^2(u_0^1) + h^2(u_0^1)\right) = 0. \tag{6.11}$$

Finally we have to determine the values u_0^2 from (6.9). Now a point $P \in RS(\gamma)$ is invisible if and only if, with $\vec{p} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{PC}$, there is a solution $u_0^1 \in I$ of (6.10) with corresponding $t_0 > 0$ from (6.11) and $u_0^2 \in (0, 2\pi)$ from (6.9). We observe, that if P is a point of the surface of revolution then $\vec{p}^2 = r^2(u_0^1) + h^2(u_0^1)$ and (6.11) reduces to a linear equation.

The algorithm described above is implemented in the procedure *RotST. Visibility*.

The procedure *RotST.NotHidden* is very similar with the single exception that now, in general, the point P under consideration is not a point of the surface of revolution, and so we need to find the solutions of the quadratic equation (6.11) in the special case $v_3 = 0$.

Example 3. *(a) The contour of surfaces of revolution*
We consider a surface of revolution RS with a parametric representation (6.2). Then

$$\vec{n}(u^i) = r(u^1)\left(r'(u^1)\vec{e}^3 - h'(u^1)\vec{u}(u^2)\right).$$

So condition (2) for $P \in RS$ with position vector $\overrightarrow{OP} = \vec{x}(u^i)$ to be a contour point is

$$-r(u^1)h'(u^1) + h(u^1)r'(u^1) - r'(u^1)\vec{c} \bullet \vec{e}^3 + h'(u^1)\vec{c} \bullet \vec{u}(u^2) = 0, \tag{6.12}$$

since $r(u^i) \neq 0$. Taking into account the symmetry of rotation, we may assume $\vec{c} = \|\vec{c}\|(\cos\Theta, 0, \sin\Theta)$ with $\|\vec{c}\| > 0$ and $\Theta \in [0, 2\pi)$.
First we study the case $\vec{c} \bullet \vec{u}(u^2) = 0$ when the centre of projection is on the axis of rotation. Then (6.12) yields

$$g_1(u^1) = r(u^1)h'(u^1) - r'(u^1)(h(u^1) - \vec{c} \bullet \vec{e}^3) = 0. \tag{6.13}$$

Contour lines are the parallels that correspond to the solutions $u_0^1 \in I$ of (6.13).
Now we consider the case $\vec{c} \bullet \vec{u}(u^2) \neq 0$ when the centre of projection is not on the axis of rotation. First we determine the zeros $u_0^1 \in I$ of

$$g_2(u^1) = h'(u^1) = 0. \tag{6.14}$$

Since $r'(u_0^1) \neq 0$ by the condition in (6.1), it follows from (6.12) that

$$h(u_0^1) = \vec{c} \bullet \vec{e}^3. \tag{6.15}$$

Therefore each parallel corresponding to a solution $u_0^1 \in I$ of both (6.14) and (6.15) is a contour line. Now we consider the interval I without the solutions of (6.14). By the choice of $\vec{c}$, the condition in (6.12) is equivalent with

$$h'(u^1)\|\vec{c}\|\cos\Theta\cos u^2 = r(u^1)h'(u^1) + r'(u^1)\left(\|\vec{c}\|\sin\Theta - h(u^1)\right).$$

Since $\vec{c} \bullet \vec{u^2} \neq 0$ implies $\cos\Theta \neq 0$, the last condition is equivalent with

$$\cos(u^2) = a(u^1) = \frac{r'(u^1)\left(\|\vec{c}\|\sin\Theta - h(u^1)\right) + r(u^1)h'(u^1)}{h'(u^1)\|\vec{c}\|\cos\Theta} \tag{6.16}$$

From (6.16) we can determine $u^2(u^1)$ for those values $u^1 \in I$ for which $|a(u^1)| \leq 1$.

(b) *Lines of intersection of planes and surfaces of revolution*
If E is a plane with normal vector $\vec{N}$ and P a point in E then the intersection of E with the surface of revolution RS is given by

$$\left((r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)) - \overrightarrow{OP}\right) \bullet \vec{N} = 0,$$

by (3). In view of the symmetry of rotation, we may assume that $n^2 = 0$ for the second component of the vector $\vec{N}$. Putting $a_0 = \overrightarrow{OP} \bullet \vec{N}$, we conclude

$$n^1 r(u^1)\cos u^2 + n^3 h(u^1) - a_0 = 0. \tag{6.17}$$

First we consider the case when $g_2(u^1) = n^1 r(u^1) = 0$. Then $\vec{N}$ is parallel to the axis of rotation, since $r(u^1) \neq 0$. The lines of intersection are the parallels corresponding to the values u^1_0 that are the zeros of

$$g_1(u^1) = n^3 h(u^1) - a_0 = 0.$$

Otherwise, if $g_2(u^1) = 0$ then we can solve (6.17) for

$$\cos u^1 = -\frac{g_1(u^1)}{g_2(u^1)} \tag{6.18}$$

and obtain $u^2(u^1)$ for those $u^1 \in I$ for which

$$\left|\frac{g_1(u^1)}{g_2(u^1)}\right| \leq 1.$$

(c) *Lines of intersection of surfaces of revolution*
Let $RS(\gamma)$ and $RS(\gamma^*)$ be surfaces of revolution generated by the smooth curves γ and γ^* which are given by the parametric representations $(r(t), 0, h(t))$ $(t \in I)$ and $(r^*(t^*), 0, h^*(t^*))$ $(t^* \in I^*)$ with $r(t) > 0$ on I, $r^*(t^*) > 0$ on I^*,

$$(r'(t))^2 + (h'(t))^2 > 0 \text{ on } I \text{ and } (r^{*'}(t^*))^2 + h(^{*'}(t^*))^2 > 0 \text{ on } I^*. \tag{6.19}$$

For the lines of intersection of $RS(\gamma)$ and $RS(\gamma^*)$ we must have

$$r(u^1)\cos u^2 = r^*(u^{*1})\cos u^{*2},\ r(u^1)\sin u^2 = r^*(u^{*1})\sin u^{*2}$$

and

$$h(u^1) = h^*(u^{*1}) \text{ for all } u^1, u^{*1} \in I \cap I^* \text{ and } u^2, u^{*2} \in (0, 2\pi).$$

Squaring the first two equations and adding them yields $(r(u^1))^2 = (r^*(u^1))^2$, hence $r(u^1) = r^*(u^{*1})$, since $r(u^1), r^*(u^{*1}) > 0$ on $I \cap I^*$, and then also $u^2 = u^{*2}$ from the first two equations, since the map $v \mapsto (\cos v, \sin v)$ is one to one on $(0, 2\pi)$. Furthermore it follows from the conditions in (6.19) that at every point $u^1 \in I$, at least one of the functions r' or h' is unequal to zero. We assume that $r'(u_0^1) \neq 0$ for some $u_0 \in I$. By the continuity of r' there is a neighbourhood $N_0 = N(u_0^1) \subset I$ such that r' is unequal to zero on N_0, hence the inverse function ϕ of r exists on N_0. Thus $u^1 = \phi(r^*(u^{*1}))$ on N_0, and so the line of intersection is locally given by the zeros of the function

$$f(u^{*1}) = h(\phi(r^*(u^{*1}))) - h^*(u^*) \text{ on the set } \phi(N_0) \cap I^*.$$

The other cases are treated in exactly the same way.

Example 4. *Some algebraic curves*
An important class of two–dimensional or *planar* curves is that of algebraic curves of order n, given by equations, that is the class

$$\mathcal{C}_n = \left(X = (x^1, y^1) \in \mathbb{R}^2 : \sum_{0 \le k+m \le n} a_{km}(x^1)^k (x^2)^m = 0 \ (a_{km} \in \mathbb{R}) \right).$$

The most familiar algebraic curves are the *conic sections*, that is the curves in the family $\mathcal{C}_2$.
As a first example, we consider *Cassini curves* ; they are curves in $\mathcal{C}_4$ and can geometrically be defined as the set of all points for which the product of the distances from two given points is constant. If the product has the value a^2 and the distance between the given two points is equal to $2c$, then the corresponding Cassini curve is given by the equation

$$f(x^1, x^2; a, c) = \left((x^1)^2 + (x^2)^2\right)^2 - 2c^2\left(x^2 - y^2\right)^2 + c^4 - a^4 = 0.$$

Introducing polar coordinates $x^1 = \rho\cos\phi$ and $x^2 = \rho\sin\phi$, we obtain

$$\rho^4 - 2c^2\rho^2\cos 2\phi + c^4 - a^4 = 0.$$

A *lemniscate* is the special case $a = c$ of a Cassini curve.
Now we consider two fifth order algebraic curves, namely *double egg lines* and *rosettes*. A double egg line has an application in the problem of doubling a cube. It has the following geometric definition. Let $S_r(0)$ be the circle line of radius $r > 0$ and centred at the origin, and A and B be distinct points on $S_r(0)$. Furthermore let F be the intersection of the straight line $\overline{OA}$ with the straight line through B which is orthogonal to $\overline{OA}$ and P be the intersection of the the straight line $\overline{OB}$ with the line through F which is orthogonal to $\overline{OB}$. If B moves along the circle line $C_r(0)$ then a double egg line is the set of all points P that are constructed in the way just described. Introducing Cartesian coordinates with the x^1 axis along the vector $\overrightarrow{OA}$, we obtain

$$f(x^1, x^2; r) = \left((x^1)^2 + (x^2)^2\right)^3 - r^2(x^1)^4 = 0$$

as an equation for the double egg line, or, in polar coordinates

$$\rho = r\cos^2\phi.$$

This yields a parametric representation

$$\vec{x}(t) = r(\cos^3(t), \cos^2(t)\sin(t)) \quad (t \in [0, 2\pi]).$$

A rosette has the following geometric definition. Let $\overline{AB}$ be a straight line segment of length a the end points of which move along the axes of a Cartesian coordinate system with its centre in the origin O. If P is the intersection of $\overline{AB}$ with the straight line through O which is orthogonal to $\overline{AB}$ then a rosette is the set of all points P which are constructed in the way just described. A rosette is given by the equation

$$f(x^1, x^2; a) = \Big((x^1)^2 + (x^2)^2\Big)^3 - a^2(x^1x^2)^2 = 0;$$

a parametric representation is

$$\vec{x}(t) = \frac{a}{2}\sin 2t(\cos t, \sin t) \quad (t \in [0, 2\pi]).$$

Example 5. *The envelope of a family of ellipses*
Let $\alpha > 0$ be fixed. We consider the family $\Gamma^\alpha = \{\gamma_c : c \in (0,1)\}$ of curves γ_c given by the equations

$$\frac{|x^1|^\alpha}{c^\alpha} + \frac{|x^2|^\alpha}{(1-c)^\alpha} - 1 = 0. \tag{6.20}$$

Then (7) becomes

$$\frac{|x^1|^\alpha}{c^{\alpha+1}} - \frac{|x^2|^\alpha}{(1-c)^{\alpha+1}} = 0, \tag{6.21}$$

and (6.20) and (6.21) yield

$$|x^2|^\alpha = (1-c)^\alpha - \frac{(1-c)^\alpha}{c^\alpha}|x^1|^\alpha \tag{6.22}$$

and

$$|x^1|^\alpha = \frac{c^{\alpha+1}}{(1-c)^{\alpha+1}}|x^2|^\alpha. \tag{6.23}$$

Substituting (6.22) in (6.23) and (6.23) in (6.22), we obtain

$$|x^1|^\alpha = c^{\alpha+1} \text{ and } |x^2|^\alpha = (1-c)^{\alpha+1},$$

or, putting $\beta = \alpha/(\alpha+1)$

$$|x^1|^\beta + |x^2|^\beta - 1 = 0.$$

In the special case $\alpha = 2$, Γ^2 is a family of ellipses and its envelope is the astroid given by the equation

$$|x^1|^{2/3} + |x^2|^{2/3} - 1 = 0.$$

Example 6. *Orthogonal trajectories of generalized circle lines*
Let $I \subset (0, \infty)$ $\alpha > 0$ and Γ^α be the family of all curves given by the equations

$$f(x^1, x^2) = |x^1|^\alpha + |x^2|^\alpha = c^\alpha.$$

In the special case of $\alpha = 2$, the curves $\gamma_c \in \Gamma^2$ are circle lines of radius c, centred at the origin. For $\alpha \geq 1$, the curves in Γ^α are the boundaries of the balls of radius c, centred at the origin, with respect to the norm $\| \cdot \|_\alpha$ defined by

$$\|(x^1, x^2)\|_\alpha = \left(|x^1|^\alpha + |x^2|^\alpha\right)^{1/\alpha}.$$

The differential equation (8) for the orthogonal trajectories becomes

$$\alpha \text{sgn}(x^1)|x^1|^{\alpha-1}\frac{dx^2}{dx^1} = \alpha \text{sgn}(x^2)|x^2|^{\alpha-1} \text{ for } x^1, x^2 \neq 0,$$

and it follows that

$$\int \text{sgn}(x^1)|x^{1-\alpha}|\,dx^1 = \int \text{sgn}(x^2)|x^{2-\alpha}|\,dx^2$$

with solutions

$$\log|x^1| = \log|x^2| + \delta \text{ for } \alpha = 2$$

and

$$|x^1|^{2-\alpha} = |x^2|^{2-\alpha} + \delta \text{ for } \alpha \neq 2$$

where δ is a constant of integration. Thus the orthogonal trajectories of the family Γ^2 of circle lines are the rays given by

$$|x^2| = k|x^1| \ (k \in (0, \infty))$$

and, for $\alpha \neq 2$, the curves $\gamma_k^{\alpha,\perp}$ given by the equations

$$f(x^1, x^2; k, \alpha) = |x^1|^{2-\alpha} - |x^2|^{2-\alpha} + k = 0 \ (k \in \mathbb{R}).$$

Example 11. *Lines of constant slope on surfaces of revolution*
We determine all curves on surfaces of revolution that have a constant angle $\beta \in [0, \pi)$ with the axis of rotation, that is with the vector $\vec{e}^3$. First, we recall a few well–known notations from the theory of curves and surfaces. Let S be a surface with a parametric representation $\vec{x}(u^i)$ of class C^r $(r \geq 1)$ on some domain $D \subset \mathbb{R}^2$. Then the functions $g_{ik} : D \to \mathbb{R}$ with

$$g_{ik} = \vec{x}_i \bullet \vec{x}_k \quad (i, k = 1, 2) \text{ where } \vec{x}_k = \frac{\partial \vec{x}}{\partial u^k}$$

are called the *first fundamental coefficients of* S. If γ is a curve on S with a parametric representation $\vec{x}(s) = \vec{x}(u^i(s))$ where s is the arc length along γ, then $\|\dot{\vec{x}}(s)\| = 1$ where the dot denotes differentiation with respect to s. We remark that

$$\|\dot{\vec{x}}(s)\| = \vec{x}_i \bullet \vec{x}_k \dot{u}^i \dot{u}^k = g_{ik} \dot{u}^i \dot{u}^k$$

where the sum is taken with respect to $i, k = 1, 2$.
Let the surface of revolution RS be given by the parametric representation (6.2). Then its first fundamental coefficients are given by

$$g_{11} = g_{11}(u^1) = (r'(u^1))^2 + (h'(u^1))^2, \; g_{12} = 0$$

and

$$g_{22} = g_{22}(u^1) = r^2(u^1).$$

Let $\vec{x}(u^i(s))$ be the parametric of a curve γ on RS where s denotes the arc length along γ. If γ is to be a line of constant slope with the angle β to the x^3 axis, then the equation

$$\dot{\vec{x}} \bullet \vec{e}^3 = h'(u^1)\dot{u}^1 = \cos\beta \tag{6.24}$$

must hold. First we consider the case $\beta \neq \pi/2$. Then solutions of (6.24) exist only in subintervals $J \subset I$ for which $h'(u^1) \neq 0$. Since $\|\dot{\vec{x}}\| = 1$ and

$$\frac{1}{(\dot{u}^1)^2} = \frac{(h'(u^1))^2}{\cos^2\beta},$$

it follows that

$$\left(\frac{du^2}{du^1}\right)^2 = \frac{(h'(u^1))^2/\cos^2\beta - g_{11}(u^1)}{g_{22}(u^1)},$$

hence

$$\frac{du^2}{du^1} = \frac{1}{|\cos\beta|}\sqrt{\frac{(h'(u^1))^2 - g_{11}(u^1)\cos^2\beta}{g_{22}(u^1)}}$$

and

$$u^2(u^1) = \frac{1}{|\cos\beta|}\int\sqrt{\frac{(h'(u^1))^2 - g_{11}(u^1)\cos^2\beta}{g_{22}(u^1)}}\,du^1$$
$$= \int \frac{\sqrt{(h'(u^1))^2\tan^2\beta - (r'(u^1))^2}}{r(u^2)}\,du^1$$

in those subintervals J of I in which

$$|r'(u^1)| \leq |\tan\beta \cdot h'(u^1)|.$$