THE PROJECTION SPHERE

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Abstract. In this paper we show a way of viewing normal surfaces in a one-vertex triangulation of a three-manifold using the unique vertex-linking two sphere and projections of the normal surface.

1. Normal Surface Theory

Let M be a three-manifold with a triangulation $\mathcal{T}$. A **normal surface** in M is a properly embedded surface such that the intersection of the surface with each 3-simplex of M is a collection of properly embedded disjoint disks with either three or four edges that span three or four distinct faces of the 3-simplex. A disk with three edges is called a **t-disk**, and a disk with four edges is called a **q-disk**. A **normal isotopy** of M is an isotopy that leaves the simplicies of $\mathcal{T}$ invariant. Up to normal isotopy, there are four distinct t-disks and three distinct q-disks in each 3-simplex. We refer to t-disks and q-disks as **elementary disks**. A normal surface can consist of any combination of the four t-disk types but can have at most one q-disk type in each tetrahedron. Figure 1 illustrates the t-disks and the q-disks. An **arc type** is the normal isotopy class of an arc in which an elementary disk meets a 2-face of a 3-simplex of $\mathcal{T}$. A **normal arc** is made up of a union of arcs from given arc types.

Corresponding to a triangulation $\mathcal{T}$ is a set of **matching equations**. The elementary disk types relative to $\mathcal{T}$ are arbitrarily assigned labels $(a_1, a_2, a_3, ..., a_{7t})$, where t is the number of tetrahedra in $\mathcal{T}$. We then can assign to a normal surface F a 7t-tuple $\vec{F} = (x_1, x_2, x_3, ..., x_{7t})$, called the **normal coordinates** of F, where x_i denotes the number of elementary disks in F of type a_i . A 7t-tuple of non-negative integers $\vec{x} = (x_1, x_2, x_3, ..., x_{7t})$ corresponds to a normal surface if it satisfies two constraints. The first constraint is that there can be only one q-disk type in each tetrahedron. The second constraint concerns the matching of elementary disks that meet along an interior 2-simplex of M. In a tetrahedron there are two elementary disk types that meet a face in a given arc type. In two tetrahedra that meet along a common 2-face there are three matching equations; one for each arc type. Each equation is of the form: $x_h + x_i = x_j + x_k$.

The system of matching equations, subject to the constraints that there can be only one q-disk type in each tetrahedron and all the variables are nonnegative, has a solution space in which each integral solution corresponds to a normal surface. Projecting that solution space onto the unit sphere gives us the **projective solution space**. Thus the Projective Solution Space is the solution space to the system we get by adding the additional equation $x_1 + x_2 + ... + x_{7t} = 1$

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We will refer to a normal surface and its corresponding vector solution to the matching equations with the same symbol F. A **vertex surface** is a connected, two-sided normal surface whose projection class is a vertex in the projective solution space. A **fundamental surface** is a normal surface with a vector solution that is not the sum of two other solutions.

It has been shown that a connected normal surface only depends on which Q-disks are in that surface [9]. This has developed into Q-normal surface theory, where we can restrict our attention to just the Q-disks. There are 3t types of Q-disks and we will arbitrarily label them $\{b_1, b_2, b_3, ..., b_{3t}\}$. We describe this theory now, allowing us to represent a normal surface by the 3t-tuple $\vec{F_Q} = (y_1, y_2, y_3, ..., y_{3t})$ where y_i denotes the number of q-disks in F of type y_i.

We fix an orientation to each of the interior edges of M^3. Let $\vec{a}$ be an oriented edge. We describe how a Q-disk, q_1, that meets $\vec{a}$ is assigned a value at that corner of +1 or -1. Using a right-hand rule we rotate around $\vec{a}$ in the positive direction. As we leave the tetrahedron that contains q_1, if the normal arc of q_1 in that face separates the vertex at top end of $\vec{a}$ from the other two vertices in that face, we assign that corner of q_1 the value of +1; otherwise the corner is assigned a value of -1. Then for each edge $\vec{e_k}$, there is an equation: $\sum_{i=1}^{3t} \epsilon_{k,i} y_i = 0$, where $0 \leq y_i$ is the number of q-disks in F of type b_i and $\epsilon_{k,i} = -1, 0$, or $+1$ is the value assigned to b_i at the edge $\vec{e_k}$. $\epsilon_{k,i} = 0$ if the q-disk of type b_i does not meet the edge $\vec{e_k}$.

The above system of matching equations together with the conditions that there is only one q-disk type in each tetrahedron and all the variables are nonnegative give rise to the **Q-solution space**. By adding the equation $y_1 + y_2 + ... + y_{3t} = 1$ gives us the **Q-Projective Solution Space** which is the projection of the Q-Solution Space onto the unit sphere. A normal surface F corresponds to the vector $\vec{F_Q} = (y_1, y_2, ..., y_{3t})$ in the Q-projective solution space. The only normal surfaces not accounted for in the Q-projective solution space are disks and 2-spheres made up of only t-disks.

2. 0-efficient Triangulations and One-vertex Triangulations

A **0-efficient triangulation** of a closed 3-manifold is a triangulation in which the only normal 2-spheres are vertex linking 2-spheres. Jaco and Rubenstein have recently proved the following result.

Theorem 1. [4] *Every closed, orientable, irreducible 3-manifold with the exception of RP^3 has a 0-efficient triangulation. Furthermore, a compact, orientable, irreducible, ∂-irreducible 3-manifold with non-empty boundary, no component of which is a 2-sphere, admits a triangulation having all of its vertices in the boundary and precisely one vertex in each boundary component.*

In order to construct a 0-efficient triangulation, they modify a given triangulation by collapsing normal 2-spheres and maintaining a triangulation. In chapter 4 we show that we can construct a 0-efficient triangulation of a 3-manifold M if M has a strongly irreducible Heegaard splitting of genus 2.

A closed 3-manifold with a 0-efficient triangulation will have just one vertex, unless the manifold is either S^3 or $L(3, 1)$ [4]. It must be irreducible [1].

A one-vertex triangulation of a surface has a nice property concerning normal curves that bound a disk.

Lemma 1. [6] *If $\mathcal{G}$ is any one-vertex triangulation of a surface G then the only normal simple closed curve bounding a disk is the vertex-linking curve.*

3. The Projection Sphere

In this paper we assume that M is a three-manifold triangulated with a one-vertex triangulation. We denote the triangulation by $\mathcal{M}$. If M has boundary, it must be connected and contain the vertex. Recall that a 0-efficient triangulated manifold with at most one boundary component contains just one vertex (with the exception of S^3 and $L(3,1)$).

We begin by defining the **Projection Sphere** of M.

Definition 1. *The projection sphere of M (or projection disk of M, in the case $\partial M \neq \emptyset$) is a fixed copy of a normal vertex linking sphere (or normal vertex linking disk) in $\mathcal{M}$.*

Since $\mathcal{M}$ has one vertex the projection sphere is unique up to normal isotopy. It consists of one copy of each of the t-disks of $\mathcal{M}$. We give the projection sphere this inherited triangulation. Looking at this triangulation of the projection sphere we can see how the faces of the tetrahedra in $\mathcal{M}$ meet. If two of the t-disks in the projection sphere share a common edge, the tetrahedra that contain the two t-disks meet in the face that contains the common edge. Also, the 1-simplices of $\mathcal{M}$ are present in the projection sphere as vertices of the t-disks. Since both ends of the edges meet at the single vertex, there is a pair of vertices of the projection sphere that correspond to each of the edges of $\mathcal{M}$.

The projection sphere gives us an alternate way to view normal surfaces in the manifold M. We will show how we can represent the normal surfaces of M by normal one-manifolds in the projection sphere. First we describe a method for projecting a normal surface onto the projection sphere. Consider a single tetrahedron v of $\mathcal{M}$. For the tetrahedron v, we arbitrarily label the six edges a, b, c, d, e, f, and the four t-disks x_1, x_2, x_3, x_4 as shown in Figure 2.

Suppose N is a normal surface in M that has some elementary disks in the 3-simplex v. We will project those elementary disks onto all four t-disks that are part of the projection sphere. For this we view the t-disk x_1 from the vertex at the corner of v that is separated by x_1 from the other three corners. For each ray from that vertex which meets an elementary disk of N, we mark the intersection of the ray with x_1. If the elementary disk of N is the same disk type as x_1, all of x_1 is filled. For any other t-disk, the corner of x_1 whose vertex corresponds to the edge that the x_1 shares with the other t-disk is filled by these marks. See Figure 3. For a q-disk, everything but the corner of x_1 whose vertex corresponds to the edge that the q-disk does not share with x_1 is filled. We project the normal surface onto each of the t-disks of the projection sphere in the same manner.

We are interested in slightly simplifying the projection of a normal surface on the projection sphere. We will show that we can represent the normal surface by just recording the boundary of the projection of the normal surface in the interior of each of the t-disks that form the projection sphere. We call this boundary the **projected track of a normal surface** or just the projected track. In the interior of each of the t-disks that make up the projection sphere, we refer to the boundary of a projected normal surface as the **boundary projection arcs** of the elementary disks.

Lemma 2 will help to get a better understanding of a projected normal track.

Lemma 2. *Given the boundary arcs of a normal t-disk or q-disk in the four t-disks of a projection sphere from a given tetrahedron, by looking at one arc in just one of the four t-disks, we can determine the elementary disk type that is being projected.*

Proof. We refer to the tetrahedron v in Figure 2, where we label the six 1-simplicies of the tetrahedron: a, b, c, d, e, f. Each vertex of the t-disks in the projection sphere correspond to one of the six 1-simplicies of v, where each 1-simplices corresponds to two vertices in the projection sphere. Without loss of generality, we assume that we are given a boundary arc λ in the t-disk x_1. In x_1, the arc λ separates one vertex from the other two. Without loss of generality, we suppose that λ separates vertex a from the other two.

If the boundary arc λ is from a q-disk then the q-disk does not meet the 1-simplex a. The q-disk meets the other two 1-simplicies b and c corresponding to the other two vertices of x_1. Knowing that the q-disk does not meet a means the q-disk also misses the edge in v that is opposite to a. We could also find the other two edges that the q-disk meets by looking at the other triangle with the vertex a; the other two vertices of that t-disk corresponds to the two edges that the q-disk meets. Thus if λ is part of a projected q-disk we can determine the elementary disk type of that q-disk.

If the boundary arc λ is from a t-disk, then the t-disk meets the vertex that is being isolated. The other t-disk with that vertex is the t-disk type that is being projected. Thus we are able to determine which elementary disk type is being projected. □

Proposition 1. *If Γ is a projected track of a normal surface in $\mathcal{M}$ then Γ uniquely determines the normal surface, up to normal isotopy.*

Proof. We group the t-disks that comprise the projection sphere by the tetrahedra they are from. Let x_1, x_2, x_3, x_4 be the four t-disks of the projection sphere from the tetrahedron v shown in Figure 2. Figure 4 shows the boundary projection arcs of every elementary disk type in v, where y_1, y_2, and y_3 are the q-disks.

In each of the three t-disks x_1, x_3 and x_4 we count the number of boundary projection arcs that are parallel to the boundary arc on which x_2 would project; they are indicated by the bold lines in Figure 4. Let the number of these boundary projection arcs be m, n, and o in x_1, x_3 and x_4, respectively. If $m = n = o$ then there are m t-disks of type x_2 and no q-disk types in the tetrahedron v. At least two of the numbers must be equal since there cannot be more than one q-disk type in v. Without loss of generality suppose that $m = n$ but $m \neq o$. Then $o > m$. This implies that there are elementary disks that are q-disks in v with the boundary projection arcs in x_4 that are parallel to the boundary arc on which x_2 would project. By Lemma 2 this classifies which q-disk type is in v. There are $o - m$ q-disks of the type y_3 in v. There are m t-disks of type x_2. Counting the other boundary projection arcs in x_4 will give the number of t-disks of type x_1 and x_3. Counting the boundary projection arcs in x_1 that are parallel to the boundary arc on which x_4 would project gives the number of t-disks of type x_4. □

We end this section with an application of the projection sphere to Q-normal surface theory.

Lemma 3. *The associated sign of a corner of a q-disk can be determined from the projection of the q-disk onto the projection sphere.*

Proof. A q-disk has an associated sign at each of its four corners based on a right-hand orientation. We can find the associated sign of a q-disk in the projected track when we rotate around each vertex of the projection sphere. If we view the 2-sphere outside of the 3-ball containing the 0-simplex of M, we rotate in a clock-wise direction. If the projection of the q-disk has boundary on the side of the triangle we enter then the corner is positive. If on the other hand the boundary arc of the projection of the q-disk is on the side we exit of the triangle, then the corner is negative. See Figure 5. □

The projected track of a normal surface must satisfy the q-matching equations. Since every edge is viewed on the projection sphere as a vertex, we can read off the q-matching equations by traveling around each vertex.

4. Example of the Projection Sphere

We look at an example that we shall deal with throughout the rest of this thesis: the one-tetrahedron triangulation $\mathcal{T}$ of the solid torus T. In the next section we will build upon $\mathcal{T}$ to obtain closed manifolds with a genus one Heegaard splitting. In the next chapter we will use two copies of $\mathcal{T}$ to build genus two handlebody. In this section we view $\mathcal{T}$'s projection disks and the projected tracks from certain normal surfaces.

A one-vertex triangulation of the solid torus is thoroughly discussed in Jaco and Sedgwick [6]. It is the one-tetrahedron triangulation as shown in Figure 6 along with the corresponding triangulation of it boundary. Notice that the boundary is the one-vertex triangulation of the torus.

The projection disk of $\mathcal{T}$ is the normal surface consisting of four t-disks one of each type in the tetrahedron. Figure 7 shows the four t-disks in the tetrahedron and the projection disk obtained after gluing the edges. The vertices of the projection disk are labeled to correspond to the edges in the triangulation $\mathcal{T}$. In any projection sphere or disk, there will be two vertices corresponding to each edge in the triangulated three-manifold. Since some of the tetrahedron's edges are identified to each other, we further distinguish which edge of the tetrahedron the t-disks meet in the pre-image before the identifications.

Figure 8 shows all the planar normal surfaces in T. Figure 9 shows the projected tracks of all the connected planar normal surfaces in the projection disk.

We will look at these normal surfaces again in more detail in Chapter 5 because they play a key role in the surfaces in our triangulation of the genus two handlebody.

5. Lens Spaces

The two lens spaces that have a one-tetrahedron triangulation are $L(4,1)$ and $L(5,2)$. Figures 10 and 11 show these triangulations and their projection spheres. For a one-tetrahedron triangulation of a closed 3-manifold, notice that these are the only two possible triangulation for the projection sphere if there is to be no identification of an edge of a given t-disk to another edge from the same t-disk.

From the one-tetrahedron, one-vertex triangulation of the solid torus T we can layer on multiple tetrahedra and cap off the resulting torus boundary surface with another solid torus to produce any lens space. Each layering is called a Pachner move because the effect of replacing the attaching edge with the new edge in the boundary is to switch the position of one edge in the boundary. The boundary

∂T always remains a torus triangulated by two triangles with three edges and one vertex.

The meridinal disk intersects the boundary ∂T in the three arcs shown in figure 12a. When we layer on a tetrahedron, we have some choices on how to complete the surface in that new tetrahedron. To complete the surface in most cases, we are only able to push through with triangles or quads, and we have not changed the given surface in the three manifold. We must match a quad if the edge that becomes interior would not satisfy the Q-equation corresponding to that edge without it.

Adding a layer changes how the boundary of the meridinal disk intersects the boundary triangles. See Figure 12. After several Pachner moves, we attach another solid torus. The original meridinal curve maps onto the boundary of the new torus with a given slope [6]. This determines which lens space is created.

If we view the layered triangulation of the torus from the perspective of the projection disk, we see a very nice symmetry.

There are only four different normal planar surfaces present in the one tetrahedron, one-vertex solid torus. The projected tracks of all four are in Figure 9. After a Pachner move, the normal surfaces is either pushed through or, if possible, a banding quad might be attached.

Theorem 2. *A presentation of the projection disk for the layered triangulation of T can be presented so as to have a $180°$ rotation symmetry.*

Proof. We start with the one-tetrahedron triangulation of the solid torus T as pictured in Figure 7. Take the center point between the edge formed from the vertices 1 and 1, so that the 6 vertices have a $180°$ rotation symmetry. A layered tetrahedron encloses two of the vertices, but with the same label since the layered tetrahedron is replacing an edge in the boundary with a new edge. So our projection disk gets two new vertices corresponding to the new edge in the layered tetrahedron. We can place these two new vertices in the representation to maintain the symmetry. See Figure 13. The new edges can be placed to maintain the symmetry. For every layered tetrahedron, we can place the two new vertices and additional edges to maintain the $180°$ symmetry. □

Figures 14 and 15 show two different possibilities for the projection disk of a one-vertex triangulation of the solid torus after several layers have been added. Figure 12 shows a solid torus where the attaching edges in the layering are $1, 4, 2, 3,$ and 7 in that order. Since edge 4 was layered over immediately after appearing, the first two layerings canceled each other out in terms of the Pachner moves. Figure 13 shows a solid torus where the attaching edges are $2, 3, 1, 4, 5, 7,$ and 6 in that order.

To create a lens space we just attach another one-vertex, one-tetrahedron triangulation of the solid torus to the boundary of our layered triangulation. This gives us a closed 3-manifold with a one-vertex triangulation. The second solid torus's four t-disks complete the vertex linking two-sphere, so we now have a projection sphere. See figure 15 where the projection disk created in figure 14 gets attached to a solid torus. Notice that no new edges are added when we attach the second solid torus.

The projection sphere for a triangulation created by layering gives us a convenient way of recording the layers.

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References

[1] W. Haken, Some results on surfaces in 3-manifolds, *Studies in Modern Topology*, Math. Assoc. Amer., Prentice-Hall, 1968, 39-98.
[2] J. Hempel, *3-Manifolds*, Princeton University Press, Princeton, N.J., 1976. Ann. of Math. Studies, No. 86.
[3] J. Hempel, *3-manifolds as Viewed from the Curve Complex*, Topology 40 (2001), no. 3, 631-657.
[4] W. Jaco and J. H. Rubinstein, *0-Efficient Triangulations of 3-Manifolds*, Preprint, Submitted to Journal of the American Mathematical Society, 2000.
[5] W. Jaco and J. H. Rubinstein, *PL Equivariant Surgery and Invariant Decompositions of 3-Manifolds*, Advances in Mathematics, **73** (1989), 149-191.
[6] W. Jaco and E. Sedgwick, *Decision Problems in the Space of Dehn Fillings*,Topology, 2000.
[7] W. Jaco and J. Tollefson, *Algorithms for the complete decomposition of a closed 3-manifold*, Illinois J. of Math. 39 (1995), 358-409.
[8] A. Thompson, *Thin Position and the Recognition Problem for* S^3 Math. Research Letters 1 (1994), 613-630.
[9] J. Tollefson, *Normal Surface Q-theory* Pacific J. of Math. 183 (1998), 359-374.

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