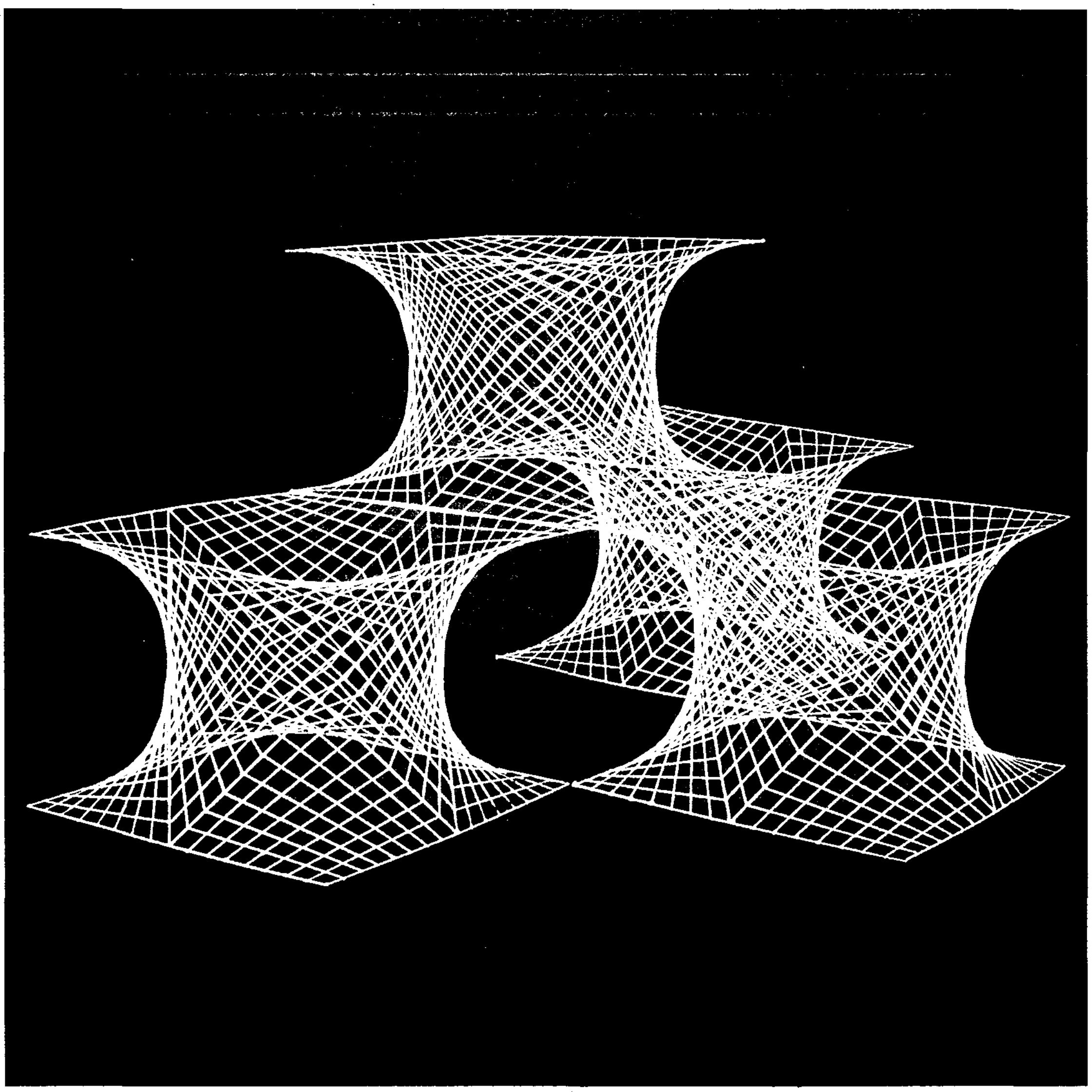
Symmetry: Culture and Science

ORDER / DISORDER
Proceedings, 4th Congress

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Volume 9, Numbers 2 - 4, 1998



Symmetry: Culture and Science
Vol. 9, Nos. 2-4, 457-463, 1998

TANGENTIAL SYMMETRIES OF PLANAR CURVES AND SPACE CURVES

Bernd Wegner

Address: Fachbereich Mathematik, TU Berlin, Strasse des 17. Juni 135, D - 10623 Berlin, E-mail. wegner@math.tu-berlin.de

1. INTRODUCTION

Mathematically symmetries occur for many objects and they have a quite general notion. They are automorphisms of a mathematical structure leaving invariant characteristic properties or quantities defined for this structure. For example, in Euclidean geometry symmetric figures mostly admit a non-trivial isometric self-map of the ambient Euclidean space, which also maps the figure onto itself. Hence this map, restricted to the figure, obviously preserves all the metric properties of that figure. But also combinatorial symmetries are considered where the self-map of the figure only preserves the combinatorial structure, while metric relations may change after having applied the map. Lots of different types of such symmetries are known, and all of them distinguish the shape of such a figure from the shape in the general case in a way which more or less immediately can be noticed by looking at that figure.

In the case of planar curves reflectional or rotational symmetries are considered as remarkable properties. Also self-similarities like in the case of spirals are of interest, and they are used to characterize certain spirals by admitting a big family of self-similarities. These properties have consequences for utilizing certain curves in the applications of geometry. Spatial generalizations of these notions are obvious, and in space the possibilities of possessing symmetries are even richer for curves. Compare a helix with a circle, for example. The aim of this short note is to explain the impact of so-called tangential symmetries on the shape of curves in the Euclidean plane and in Euclidean 3-space. As symmetry notions they seem to be quite weak, but nevertheless they have visible consequences for the shape of these curves.

2. TANGENTIAL SYMMETRIES IN THE PLANE

For closed smooth curves in the plane three types of tangential symmetries are of interest. These symmetries always consist of a group of smooth self-maps on the domain S^1 for the parametrization $c : S^1 \to E^2$ of the curve. The maps are assumed to assign points with parallel tangents to each other, and may be subject to additional conditions.

The general case without restrictive conditions has been studied by J. Shaer (unpublished) and F. J. Craveiro de Carvalho and S.A. Robertson [CR1]. Some classification results are obtained for open curves by the latter while J. Shaer provided a complete description of the shape of closed curves having a non-trivial group of tangent preserving self-maps. Mathematically spoken, the asignment of their unit tangents factorizes through a multiple covering map of the unit sphere for these curves, after having identified tangent vectors with opposite directions. For example in the locally convex case, the group Z_2 characterizes strictly convex ovals. Centrally symmetric curves may serve as examples for the nonconvex case. More general groups lead to curves with a finite number of loops, directed to the same "interior" side of the curve in the locally convex case and possessing symmetries concerning the shape of "interior" and "exterior" loops and bumps in the general case. This shape can be visualized very easily, and it obviously determines a fairly special structure for these curves.

If the tangential symmetry is assumed to preserve the normal lines in addition, then we get what has been introduced by H. Farran and S.A. Robertson [FR] as exterior self-parallelism in the general context of immersions. For closed curves this reduces to the notion of rosettes of constant width. In particular, these curves are locally strictly convex, only the cyclic group of order two can appear as a non-trivial group of self-parallelisms, and the rosettes can be generated in a very special way: A rigid line segment could be moved along the curve, connecting point and image point for the only non-trivial tangential symmetry, such that this segment always will be a normal to the curve at its endpoints. It is easy to imagine the shape of such curves, though they are more general than the classical rosettes, where a rotational symmetry can be observed. In this general case the rotational symmetry only refers to the loop structure. Details on rosettes of constant width can be found in the paper of W. Cieslak and W. Mozgawa [CM] and in [W1].

Finally, assuming as a stronger requirement, that any normal line of the curve can intersect the curve as a normal line only, we arrive at the notion of transnormality.

This has been introduced by S.A. Robertson [R1] for the more general situation of immersions into Euclidean spaces, but in the simple case of a closed curve in the plane it leads immediately to convex curves of constant width bounding planar convex domains of constant width. This characterization is a classical result. There is a vast amount of literature on ovals of constant width. For, example a comprehensive survey on classical results already could be found in the book of T. Bonnesen and W. Fenchel [BF]. Later surveys are given in handbooks on convexity. These curves may be considered as the trivial case of rosettes of constant width where no loops occur. Hence the kinematic interpretation is simple. They are frequently used in applied geometry. Most famous is their application to the construction of the cylinder and the piston for the Wankel engine.

3. TANGENTIAL SYMMETRIES FOR SPACE CURVES

For closed smooth curves in Euclidean 3-space the most general type of tangential symmetry described above will be too general to lead to conclusions on the shape of the curve. Hence we start immediately with the notions of parallelism and self-parallelism [FR]:

The *exterior parallelism* of two smooth closed curves c_1, $c_2 : S^1 \to E^3$ is defined by the following condition: For every parameter $t \in S^1$ the affine spaces normal to c_1 at $c_1(t)$ and c_2 at $c_2(t)$ coincide. This condition has been shown to be equivalent to the condition that both curves are connected by a parallel section of their normal bundles (see [W2]), i.e., there is a smooth normal vector field e_1 along c_1 such that

$$c_2(t) = c_1(t) + \lambda e_1(t) \text{ and } prn(\nabla_{\dot{c}_1(t)} e_1) = 0 \tag{1}$$

for all $t \in S_1$, where $\dot{c}_1(t)$ denotes the tangent vector field of c_1 as usual and *prn* denotes the orthogonal projection to the corresponding normal (vector) space of c_1. Previous investigations of this notion for curves could be found in the paper [CR2] by F.J. Craveiro de Carvalho and S.A. Robertson and in [W3].

Hence the existence of a parallel mate for $c : S^1 \to E^3$ has been reduced to the search for a global parallel normal vector field along c. Generally, these vector fields only exist locally along c. Parallel transfer of the normal plane along one period of c with respect to the normal connection leads to a rotation of the normal plane, which is characterized (up to integer multiples of 2π) by an oriented angle $\alpha(c)$, which we call the *total normal twist* of c. For Frenet curves this quantity is given by their total torsion up to integer multiples of 2π. Looking at general orthonormal frame fields $\{T, e_1, e_2\}$ along c, where T denotes the unit tangent field of c, and setting

$$\omega_{12} = < \nabla_T e_1, e_2 > = - < \nabla_T e_2, e_1 > = - \omega 21, \tag{2}$$

we get for the total normal twist of c (up to integer multiples of 2π)

$$\alpha(c) = \int_{S^1} \omega_{21}(t)dt \tag{3}$$

where there is no need to parametrize c with arclength.

A *self-parallelism* of c is given by a diffeomorphism $\delta : S_1 \rightarrow S_1$ such that c and $c \cdot \delta$ are parallel in the exterior sense. This is the notion of tangential symmetry to be discussed now. Clearly only closed curves with vanishing total twist possibly will admit such a tangential symmetry. The variety of these curves has been studied in much detail in a joint paper with T.F. Mersal [MW]. But already in a previous paper [W3] curves with non-trivial tangential symmetries have been related to a center curve with total normal twist being a rational multiple of 2π in the following way: Take the non-vanishing normal vector to the center curve, connecting the center curve and the original curve at some point, and apply normal parallel transfer to this vector along several periods of the center curve, until the trace of the endpoint of this vector will lead to a closed curve. This will restore the original curve. Moreover, starting the same procedure with any curve, where the total normal twist is a rational multiple of 2π, and with a suitable normal vector such that the construction will avoid singularities, we get a curve exhibiting Z_n as its group of tangential symmetries, where n is the number of periods until the trace of the end point of the vector will provide a closed curve.

This gives a fairly clear picture of space curves admitting this kind of symmetries. But there are other advantages related to the visual perception of these curves. There is an obvious one, coming from the kinematic interpretation of the parallel transfer in the normal bundle of a curve. Consider a normal frame as a rigid two-dimensional figure, moving without acceleration freely along the curve, with the constraint to remain in the normal plane forever. Then the motion will be described by the parallel transfer in the normal bundle of the curve. Hence, considering the intersection of a tangentially symmetric curve, having Z_n as its symmetry group, with its normal plane as the vertex set of a regular n-gon, this motion of the frame along the central curve will preserve this figure after one period, though a permutation of the vertices may have happened.

Interpreting the original curve as a figure located on a tube around the central curve, it carries information for the visualization of the center curve as follows: The tube may be taken as a more solid image of the center curve. The twist of the center curve may be visualized by drawing families of curves on the surface of this tube. The most appropriate curves for this will be those obtained by parallel transfer in the normal bundle of the center curve, and they will be closed curves only, if they exhibit a tangential symmetry. Clearly, then the surface of the torus bounding the tube can be foliated by curves with this kind of tangential symmetry. Furthermore, if another profile than a disk should be taken to thicken the center curve to a solid body, this only will be possible if the tangential symmetry of the original curve is respected by this profile. This can be observed in many images where closed space curves are displayed.

The more restricted form of tangential symmetry which is given by the notion of transnormality has been studied by M.C. Irwin [I], and several geometric results have been obtained for them in [W4]. Within the current context the only interesting result is, that in this case the symmetry group can be Z_2 only, and that overmore the central curve cannot avoid singularities. There are no further conclusions than those of the preceding paragraph resp. preceding section for this case.

There also are a lot of considerations concerning tangential symmetries of curves in higher-dimensional spaces. In particular, the case of Minkowski 4-space is of special interest for Relativity. But this is beyond the goal of this presentation.

4. THE CASE OF SPATIAL POLYGONS

For simple explicit constructions and examples the analytic techniques behind the theory presented above will be an essential obstruction. The theory may be reduced to C^1-curves which are piecewise C^2, and then there is a gateway for considering curves composed of pieces of circles with C^1-matchings for these pieces. Here everything can be reduced to the consideration of a finite set of data given by the finte number of circles in space. But the whole theory even can be broken down to the level of spatial polygons, and this opens a wide field for nice constructions which everybody can pursue on his own. Here just this concept should be presented in analogy to the preceding section, leaving explicit constructions to the reader.

Two simply closed polygons in space are called *parallel*, if there is a bijection between their vertices, mapping consecutive vertices to consecutive ones again, such that the angle-bisecting planes at corresponding vertices coincide. (It should be noted that this is more restrictive, than assuming that corresponding line segments are parallel.) A system of normal vectors to the edges of a polygon is called a *parallel field of normals*, if every two vectors belonging to edges with a common vertex are symmetric with respect to the reflection at the angle-bisecting plane of the polygon at this vertex. Then parallel polygons are related by a parallel normal vector field (of constant length) again. A *self-parallelism* or tangential symmetry is just a combinatorial automorphism of the combinatorial structure behind the polygon, such that the original and the relabelled polygon are parallel.

Furthermore, if a closed polygon possesses a non-trivial self-parallelism, then its total normal twist (which is defined in obvious analogy to the smooth case) vanishes mod 2π. We have the same kind of classification for tangentially symmetric polygons like in the smooth case: i) Assume that the polygon P in E^3 admits a parallel section in the normal bundle of its k-fold covering, then the obvious construction of a parallel at suitable distance has Z_k as its group of self-parallelisms. ii) Viceversa, every polygon possessing more than one self-parallelism can be obtained by the preceding construction from a suitable central polygon.

Details for these considerations can be found in [W5]. Very explicit calculations in the case of spatial quadrilaterals as center polygons have been obtained in [W6]. Everything can be reduced to the framework of elementary Euclidean geometry in 3-space. Examples of self-parallel polygons, which are not restricted to a plane, start with 8 vertices at least. Those with 8 vertices bound a PL-Moebius strip, and the central quadrilateral has to satisfy very special constraints for its angles and side lengths. Constructions with pentagons as central polygons are easier to visualize. They provide very simple families of imbedded PL-Moebius strips having a bounding polygon with 10 vertices, such that the strip may be composed from planar pieces of constant breadth. This is the case of symmetry group Z_2.

For symmetry group Z_4 and a center quadrilateral we receive recipes for composing four bars with the same square as their profile, such that the result will be a skew frame and such that the edges resulting from the bars form one connected closed polygon (with 16 vertices). The common picture frames have a planar rectangle as their basis, and the edges coming from the bars decompose into four rectangles of different sizes. It is easy to produce wooden models for these skew frames also by cutting the bars into pieces of appropriate lengths, such that planes for the sawing have the right angle with respect to the center line of the bar. Comparing this model with other models of compositions of four bars of variable rectangular profile to a skew framework, the symmetric construction will really appear as the most symmetric (and appealing) solution. Anyway, there will be no other solutions having a square as their costant profile.

The same will be true for other groups of tangential symmetries, only the regular polygon for the profile will change. Hence there will be a more simple solution for Z_3 than for Z_4, but for building a model, it will be easier to get bars with a square profile than with an equilateral triangle. For rectangular profiles which are not squares, the Z_2 models will be good solutions, but then the boundary polygon will decompose into two linked octogons then. Finally it should be observed, that these construction also may be of interest for non-closed polygons: They will provide a solution of the problem to connect two planes in space with bars of the same (regular or semi-regular) profile, such that the starting point and the end point of the connection may be prescribed, the bars start resp. arrive in normal direction at the planes, and the resulting framework fits properly to a polyhedron, i.e., the matchings are done face to face without any prominent pieces.

Conclusion: The preceding considerations show, that a rather general concept of symmetry for curves in the plane and in 3-space leads to interesting versions of visible symmetries for them. These symmetries motivate in many cases why these curves are preferred for applications and for geometric constructions.

REFERENCES

[BF] Bonnesen, T. and Fenchel, W. (1974) *Theorie der konvexen Korper*, Ber Reprint, Springer-Verlag.

[CM] Cieslak, W. and Mozgawa, W. (1987) On rosettes and almost rosettes, *Geom. Dedicata* 24, 221-228.

[CR1] Craveiro de Carvalho, F. J. and Robertson, S A. *The parallel group of a plane curve* (to appear).

[CR2] Craveiro de Carvalho, F. J. and Robertson, S A. (1989) Self-parallel curves, *Math. Scand.* 65, 67-74.

[FR] Farran, H. R. and Robertson S.A. (1987) Parallel immersions in Euclidean space, *J. London Math. Soc.* (2) 35, 527-538.

[I] Irwin, M. C. (1967) Transnormal circles. *J. London Math. Soc.* 42, 545-552.

[MW] Mersal, T F and Wegner, B. (1997) Variation of the total normal twist of closed curves in Euclidean spaces, *Proceedings Conf. on Geometry and Topology*, Braga (to appear)

[R1] Robertson, S A. (1964) Generalized constant width for manifolds. *Michigan Math. J.* 11, 97-105.

[W1] Wegner, B. (1989) Some global properties and constructions for closed curves in the plane, *Geom. Dedicata* 29, 317-326.

[W2] Wegner, B. (1989/1991) Some remarks on parallel immersions, *Coll. Math Soc. J. Bolyai* 56, 707-717.

[W3] Wegner, B. (1991) Self-parallel and transnormal curves, *Geom. Dedicata* 38, 175-191.

[W4] Wegner, B. (1972) Globale Sätze uber Raumkurven konstanter Breite I, II, *Math. Nachr.* 53, 337-344, ibid. 67 (1975), 213-223.

[W5] Wegner, B. (1991) Exterior parallelism for polyhedra, *Math. Pannonica* 2, 95-106.

[W6] Wegner, B. (1991) *Self-parallel polygons and polyhedra*, Geometry, Proc. Conf. Thessaloniki / Greece, 444-451.