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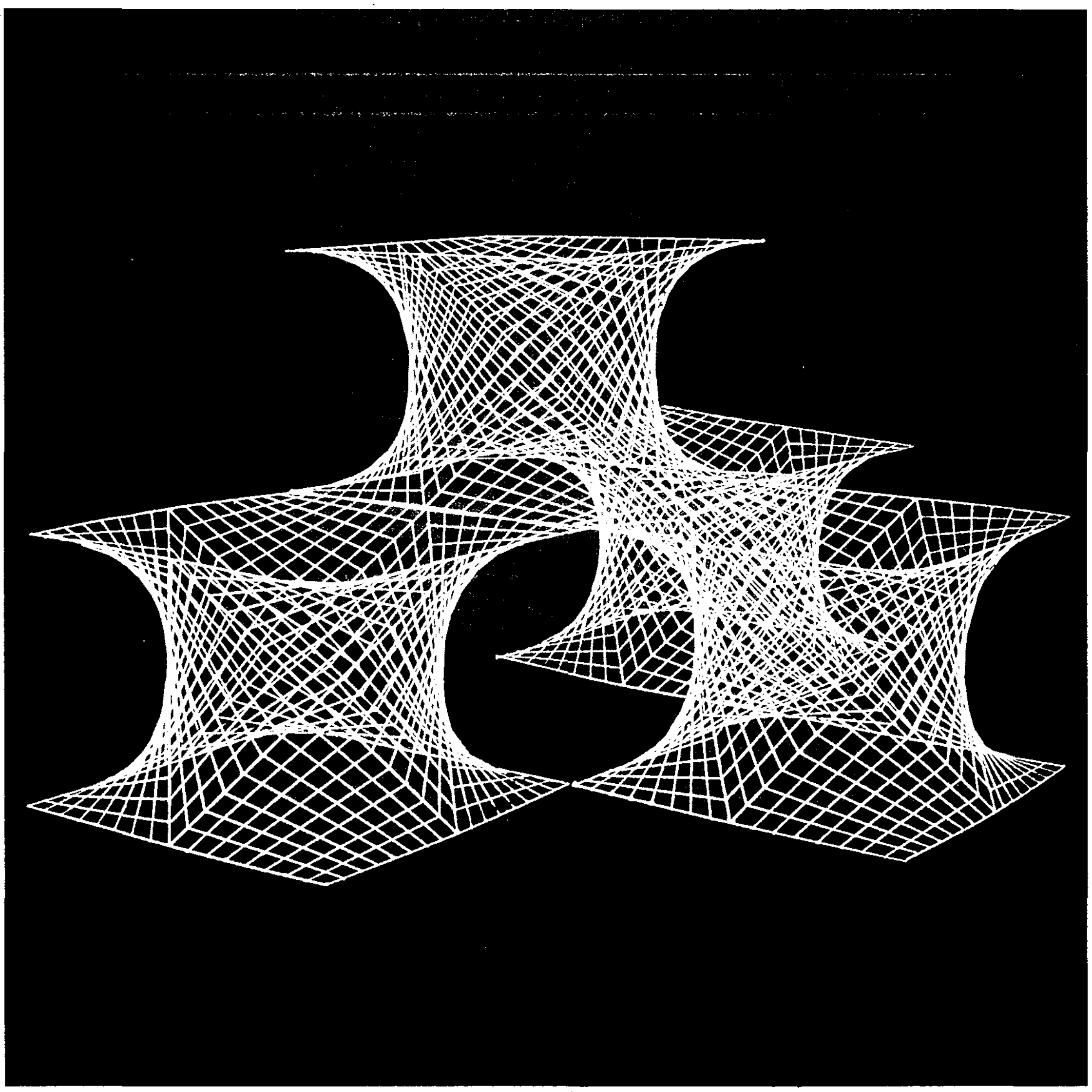

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GEOMETRICAL INTERPRETATION: RECURRENCE FORMULA OF NATURAL NUMBER'S POWER SERIES SUMMATION

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Abstract: *The geometrical interpretation of natural number's* n*-th power series summation is presented by the superposition of the* j*-dimensional lattice of hyperspace regular simplexes,* α_j *in which* α_{j-1} *stacks with a linear increment weight of the lattice points. The superposition of* α_j *is carried out every* (j+*1) times rotation around the* (j+*1)-fold axis of* α_j. *The superpositions result in homogeneous density in* α_j. *The recurrence formula of sum of the power series of natural numbers is presented using the mean value of the superposed weight of the lattice points in the* α_j.

1. INTRODUCTION

Some geometrical interpretations of natural numbers' n-th power series summation were presented for n = 1, 2, 3 (Gardner 1986). On the other hand, a geometrical properties are discussed for second moment of inertia of α_j Voronoi cell around the origin (Conway & Sloane 1993). However, a geometrical interpretation of generalized number theory using α_j has not been proposed so far as the author is aware. In this paper a new geometrical scheme for the sum of the n-th power of natural numbers n = 1, 2, 3 is presented by using the geometry of the closest packing of circles (1- and 2-D cases) and that of spheres (3-D case) which is called face-centered close packed structure (Conway & Sloane 1993). As Coxeter pointed out, any j+1 points with equal distance between any two points which do not lie in a (j-1)-space are the vertices of a j-D simplex, as point α_0, line-segment α_1, regular triangle α_2, regular tetrahedron α_3, regular pentatope

α_4, ... , regular simplex α_j (Coxeter 1973). It is shown that higher-order power series of natural numbers can be obtained by considering further extension of the dimension, namely the lattice points in the hypersimplex α_j.

2. CASE STUDY OF SUPERPOSITION OF SIMPLEXES IN 1-D TO 4-D

First, the geometrical interpretation of the sum of low-dimensional (n = 1, 2, 3) power of natural numbers is presented concretely. Consider 1-D lattice L_1 with linear increment weight 1, 2, 3, ... , n on the lattice points. Let $R(\emptyset)$ be the rotation operator of α_1 with rotation angle $\emptyset_1$ around the [1, 1] zone axis in the 2-D square lattice. α_1 operated on by $R(\emptyset_1)$ becomes $\alpha_1(\emptyset_1)$, if α_1 is rewritten as $\alpha_1(0)$, then

$$\alpha_1(\emptyset_1) = R(\emptyset_1) \cdot \alpha_1(0). \tag{1}$$

The superposition of $\alpha_1(\emptyset_1) + \alpha_1(0)$ is represented as

$$S_1 \cdot \alpha_1(0) = \{R(\emptyset_1) + I\} \cdot \alpha_1(0) \tag{2}$$

where S_1 is the superposition operator and I is the identity operator of $\alpha_1(0)$. In the 1-D case $\emptyset_1$ is restricted to π, then

$$\cos \emptyset_1 = -1 \tag{3}$$

$$S_1 = R(\pi) + R(0) = R(\pi) + I \tag{4}$$

The weight W_1 of all the lattice points. $S \cdot \alpha_1(0)$ is the same weight $W_1 = n$ as shown in Figure 1.

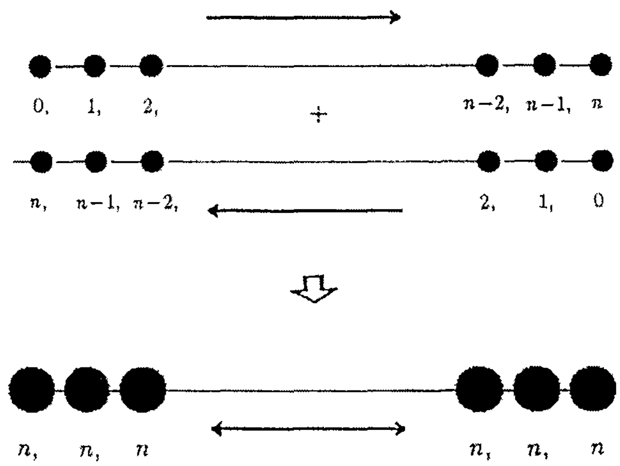

Figure 1: Line segment α_1 with linear increment weight and superposed α_1 with homogeneous density n.

If $\alpha_1(0)$ has nonlinear increment weight, $S_1\alpha_1(0)$ should have an inversion center or 2-fold rotational symmetry, but it has homogeneous density $W_1 = n$ for $\alpha_1(0)$ with linear increment weight. Let the number of lattice points of α_1 be l_1 and the multiplicity of superposition be M_1, so $l_1 = n + 1$ and $M_1 = 2$ for two superpositions.

The mean mass of S · $\alpha_1(0)$,

$$l_1 = n + 1 \qquad W_1 = n \qquad M_1 = 2 \qquad \overline{m_1} = \frac{l_1 W_1}{M_1} = \frac{1}{2}n(n+1). \tag{5}$$

$\overline{m_1}$ m is equal to the mass of $L_1(0)m_1$, since the weight of the k-th point is k and the number of k-th points l_0 is 1,

$$l_0 = 1 \qquad m_1 = \sum_{k=0}^{n} k \bullet l_o. \tag{6}$$

Therefore,

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1). \tag{7}$$

Hereafter the notations of α_j, S_j, W_j, M_j, l_j, m and $\overline{m_j}$ are introduced for j-D parameters.

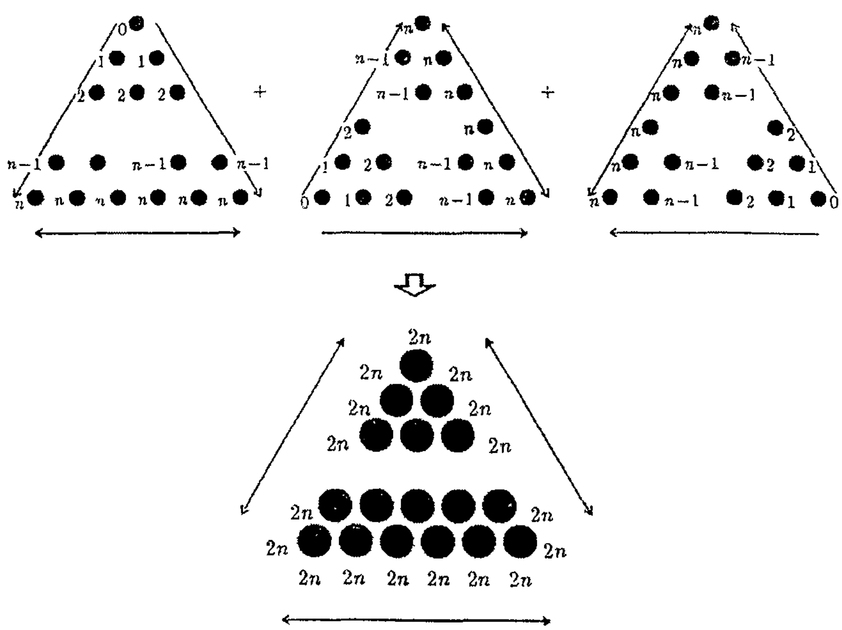

Figure 2: Regular triangle α_2 operated on by 0, $2\pi/3$ and $4\pi/3$ rotations and superposed α_2 with homogeneous density $2n$. Stacking layers of line segments α_1 with linear increment weight are shown by black circles.

Second, consider 2-D lattice $\alpha_2(0)$ which is the stacking of $\alpha_1(0)$, the length of which increases linearly as shown in Figure 2 to form a regular triangle lattice $\alpha_2(0)$. The superposition is applied to $\alpha_2(0)$ in the same way as in the 1-D case. $R\emptyset_2$ is considered using the [1 1 1] plane of the cubic lattice in the first quadrant which takes the form of a regular triangle. The superposition could be performed every $2\pi/3$ rotation successively around the [1 1 1] axis through the center of gravity of a regular triangle, that is,

$$\cos \emptyset_2 = -1/2 \tag{8}$$

$$\mathbf{S}_2 \cdot \alpha_2(0) = \{\mathbf{R}^2(\phi_2) + \mathbf{R}(\phi_2) + \mathbf{I}\} \cdot \alpha_2(0). \tag{9}$$

Here $R^2(\emptyset_2)$ is equal to $R(2\emptyset_2)$, thus

$$\mathbf{S}_2 = \mathbf{R}(\frac{4\pi}{3}) + \mathbf{R}(\frac{2\pi}{3}) + \mathbf{I}. \tag{10}$$

For S_2, three superpositions of $\alpha_2(0)$, a regular triangle with homogeneous density is obtained as shown in Figure 2; here,

$$M_2 = 3, \quad l_2 = \sum_{k=0}^{n}(k+1) = \frac{1}{2}(n+1)(n+2), \quad W_2 = 2n \tag{11}$$

are given as shown in Fig.2. The mean mass of $S_2 \cdot \alpha_2(0)$ is

$$\overline{m_2} = \frac{l_2 W_2}{M_2} = \frac{1}{3}n(n+1)(n+2). \tag{12}$$

The number of lattice points on the k-th layer line-segment in the triangle $\alpha_2(0)$ is l_2 and each lattice point of l_2 has unit weight, thus

$$l_0 = 1 \quad l_1 = \sum_{l_0=0}^{k} l_0 = k+1, \quad m_2 = \sum_{k=0}^{n} k l_1 = \sum_{k=0}^{n} k(k+1) \tag{13}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1). \tag{14}$$

Third, consider 3-D lattice $\alpha_3(0)$ by analogy with the 2-D lattice. A regular triangle of $\alpha_2(0)$ in 2-D space corresponds to a tetrahedron in which a face-centered close-packed lattice is formed. This geometry of $\alpha_3(0)$ in a regular tetrahedron is interpreted as n stacking layers of a regular triangle of $\alpha_2(0)$. That is,

$$M_3 = 4, \quad l_3 = \sum_{k=0}^{n}\frac{1}{2}(k+1)(k+2) = \frac{1}{6}(n+1)(n+2)(n+3), \quad W_3 = 3n. \tag{15}$$

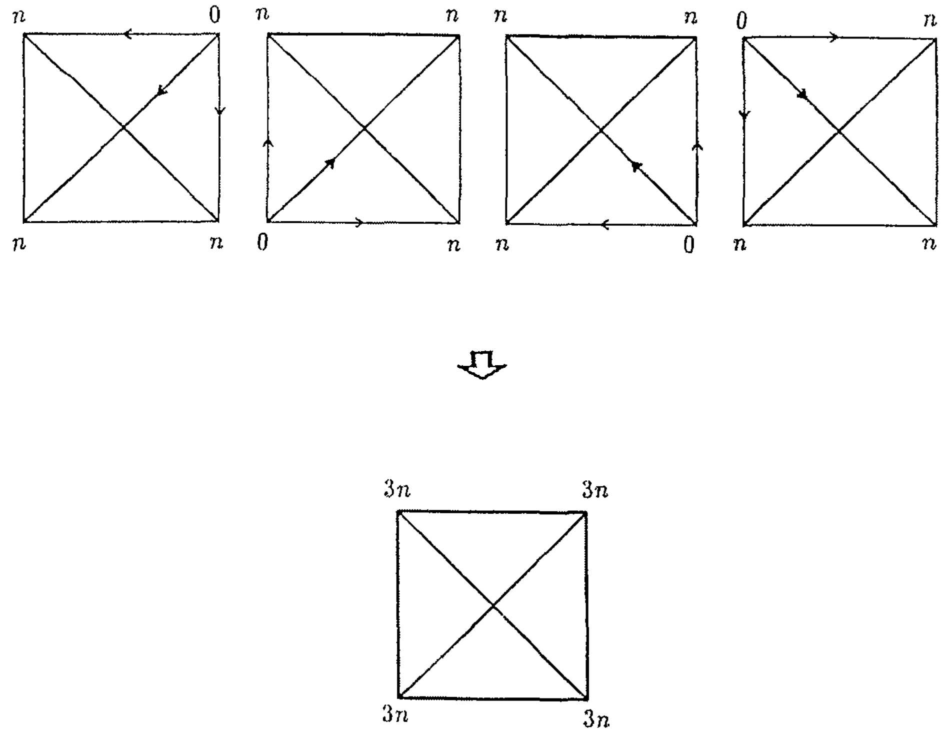

Figure 3: Regular tetrahedra (α_3) operated on by 0, $\emptyset_3$, $2\emptyset_3$, $3\emptyset_3$ rotations (cos $\emptyset_3$ = -1/3) and superposed α_3 with homogeneous density $3n$. A sense of stacking layers of regular triangle (α_2), with increment weight is denoted by arrows

Thus, the mean mass of $S_3 \cdot \alpha_3$ is

$$\overline{m_3} = \frac{l_3 W_3}{M_3} = \frac{1}{8} n(n+1)(n+2)(n+3). \tag{16}$$

Let the number of lattice points in the k-th stacking layer be l_2, and the weight of all the lattice points in the k-th layer be k, then

$$l_2 = \sum_{l_1=0}^{k} l_1 = \frac{1}{2}(k+1)(k+2), \quad m_3 = \sum_{k=0}^{n} k l_2 = \sum_{k=0}^{n} k \sum_{l_1=0}^{k} \sum_{l_0=0}^{l_1} l_0 = \sum_{k=0}^{n} \frac{1}{2} k(k+1)(k+2). \tag{17}$$

From equations (14), (16) and (17),

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2. \tag{18}$$

As for $R(\varnothing_3)$, consider the rotation of a tetrahedron in 4-D space, because it is more comprehensive to select the rotation about any plane including the [1 1 1 1] axis in 4-D space than in 3-D space. Four kinds of tetrahedra obtained for every rotation by the tetrahedral angle are superposed as shown in Figure 3.

$$\cos \varnothing_3 = -1/3 \tag{19}$$

$$\mathbf{S}_3 \cdot \alpha_3(0) = \{\mathbf{R}^3(\phi_3) + \mathbf{R}^2(\phi_3) + \mathbf{R}(\phi_3) + \mathbf{I}\} \cdot \alpha_3(0) \tag{20}$$

where $\varnothing_3$ is equal to the tetrahedral angle.

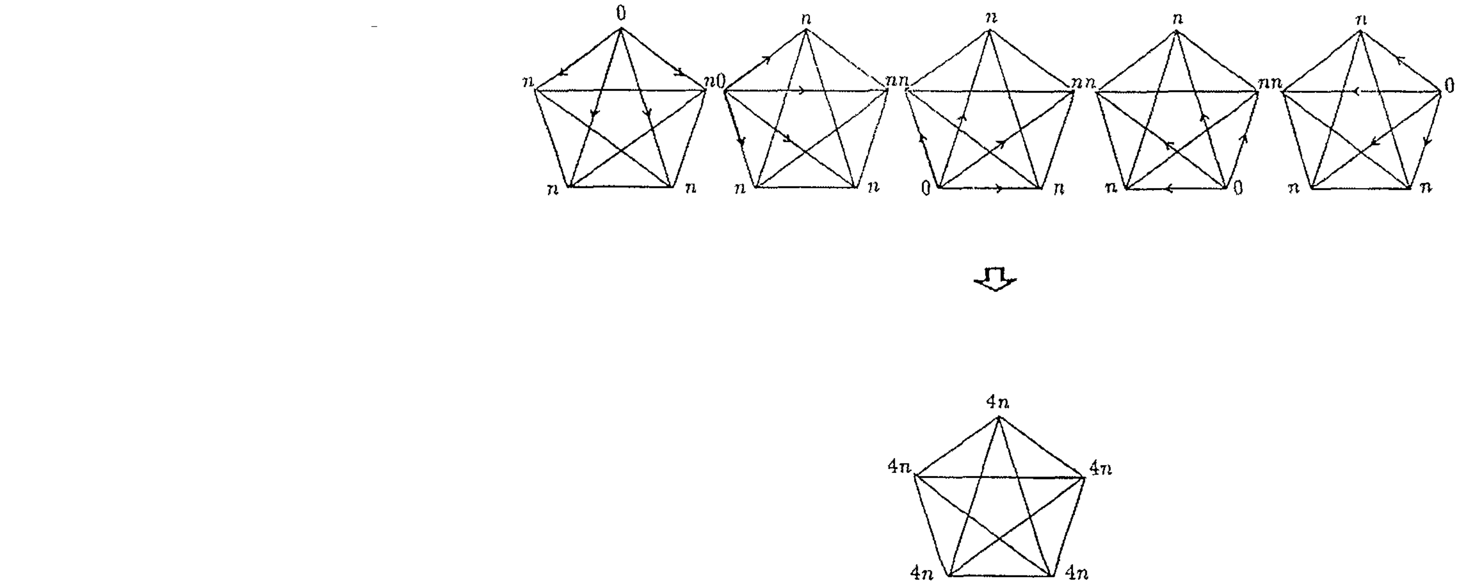

Figure 4: Regular pentatopes α_4 operated on by 0, $\varnothing_4$, $2\varnothing_4$, $3\varnothing_4$ and $4\varnothing_4$ rotations ($\cos \varnothing_4 = -1/4$) and superposed α_4 with homogeneous density $4n$. Sense of increment weight is denoted by arrows.

The practical examples of the 1-D, 2-D and 3-D superpositions of the lattice facilitate deduction of the 4-D and higher-dimensional geometry by analogy. It is necessary that the figure used for the superposition have a high rotational symmetry for the purpose of getting a homogeneous density. A regular polytope (regular hyper-simplex), α_j, is available to satisfy such a condition.

In the 4-D case five regular pentagonal simplexes α_4 are considered as superposition simplexes as shown in Figure 4. Stacking illustrations of j-D simplexes in j+1 dimensional space j = 2, 3, 4 are shown in Figure 5.

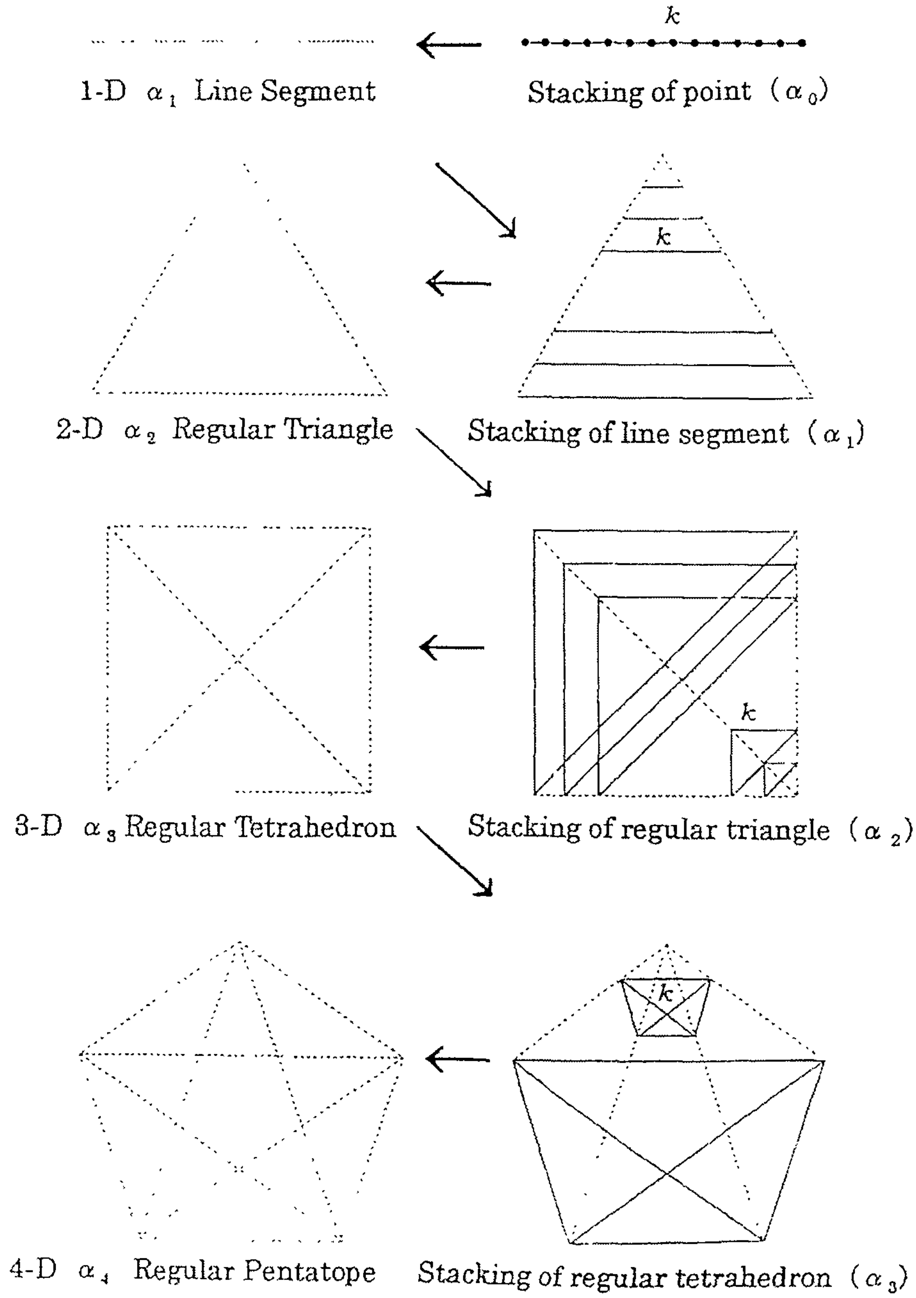

Figure 5: Stacking illustration of α_{j-1} in α_j. Stacking of line segments (α_1) in regular triangle (α_2), those of α_2 in regular tetrahedron (α_3) and those of α_3 in regular pentatope α_4.

3. DERIVATION OF $\overline{m_j}, m_j$

The sum of 1st, 2nd and 3rd power of natural numbers is derived by the superposition of line segment, triangle and tetrahedron, respectively. Here, consider j-D simplex α_j; since α_j is composed of n stacking layers of α_{j-1} and all lattice points in its k-th layer l_j have the same weight k, $l_{j,n}$ is represented as follows.

$$l_{j,n} = \sum_{k=0}^{n} l_{j-1,k} \tag{21}$$

where $l_{j-1,k}$ is the lattice point of the k-th layer with linearly increasing weight in the α_{j-1}.

The j-D mean mass, $\overline{m_j}$, is deduced as follows from the superpositions of the 1-D to 3-D regular simplex; here l_j is a lattice point in α_j.

$$M_j = j+1, \quad l_j = \frac{1}{j!}(n+1)(n+2)\cdots(n+j-1)(n+j), \quad W_j = jn \tag{22}$$

$$\overline{m_j} = \frac{l_j W_j}{M_j} = \frac{j}{(j+1)!} n(n+1)(n+2)\cdots(n+j) \tag{23}$$

The parameters of superposed hyper regular simplexes are summarized in Table 1.

j	W_j	$l_{j,n}$	M_j	$\overline{m_j}$
1	n	$n+1$	2	$\frac{1}{2}n(n+1)$
2	$2n$	$\frac{1}{2}(n+1)(n+2)$	3	$\frac{1}{3}n(n+1)(n+2)$
3	$3n$	$\frac{1}{6}(n+1)(n+2)(n+3)$	4	$\frac{1}{8}n(n+1)(n+2)(n+3)$
4	$4n$	$\frac{1}{24}(n+1)(n+2)(n+3)(n+4)$	5	$\frac{1}{30}n(n+1)(n+2)(n+3)(n+4)$
j	jn	$\frac{1}{j!}(n+1)(n+2)\cdots(n+j)$	$j+1$	$\frac{j}{(j+1)!}n(n+1)(n+2)\cdots(n+j)$

Table 1: Parameters of superposed hyper regular simplex

On the other hand, m_j is formulated by stacking of the layers of (j-1)-th regular simplexes with weight k in each layer of the j-D space. For $j \geq 2$,

$$m_j = \sum_{k}^{n} k l_{j-1,k} = \sum_{k}^{n} k \sum_{l_{j-1}}^{k} \sum_{l_{j-2}}^{l_{j-1}} \cdots \sum_{l_3}^{l_4} \sum_{l_2}^{l_3} l_2 = \sum_{k}^{n} k \frac{1}{(j-1)!} \sum_{l}^{k} (l+1)(l+2)\cdots(l+j-2). \tag{24}$$

In order to obtain the summation term of the right side of equation (24), consider the product function $f_j(x)$,

$$f_j(x) = (x+1)(x+2)\cdots(x+j-1). \tag{25}$$

From equation (25),

$$f_{j+1}(n) - f_{j+1}(n-1) = (n+1)(n+2)\cdots(n+j) - n(n+1)\cdots(n+j-1) = j f_j(n). \tag{26}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{n} f_j(k) &= \frac{1}{j} \sum_{k=0}^{n} \{f_{j+1}(k) - f_{j+1}(k-1)\} \\ &= \frac{1}{j}\{f_{j+1}(n) - f_{j+1}(-1)\} = \frac{1}{j} \cdot \frac{(n+j)!}{n!}. \end{aligned} \tag{27}$$

Consequently,

$$\sum_{k}^{n} (k+1)(k+2)\cdots(k+j-1) = \frac{1}{j} \cdot \frac{(n+j)!}{n!} \tag{28}$$

can be obtained. Here, another product function $4xf_j(x)$ is introduced in order to represent the right side of equation (24) by the parameter n. Equation (29) is derived easily in the same procedure as obtained equation (28).

$$\sum_{k}^{n} k(k+1)(k+2)\cdots(k+j-1) = \frac{1}{(j+1)} \cdot \frac{(n+j)!}{n!} \tag{29}$$

Substituting the right side of equation (29) into equation (24) , equation (30) is obtained.

$$m_j = \frac{1}{(j-1)!}\sum_k^n k(k+1)(k+2)\cdots(k+j-1) = \frac{j}{(j+1)!}(n+1)(n+2)\cdots(n+j). \tag{30}$$

Thus $\overline{m}_j = m_j$ is confirmed by the geometrical interpretation of the superposition of the regular simplexes. The relation is derived from $\overline{m}_j = m_j$ as shown in equation (30).

Equations (23) and (24) are verified by mathematical induction using the stacking geometry. Let the number of lattice points in the k-th layer of (j-1)-D be $l_{j\text{-}1,k}$ as given in equation (24), in which all lattice points have the same weight k. $l_{j,k}$ is derived as follows using equation (28).

$$\begin{aligned} l_{j,k} &= \sum_{l_{j-1}}^{k}\sum_{l_{j-2}}^{l_{j-1}}\sum_{l_{j-3}}^{l_{j-2}}\cdots\sum_{l_2}^{l_3} l_2 = \frac{1}{(j-1)!}\sum_{l}^{k}(l+1)(l+2)\cdots(l+j-2) \\ &= \frac{1}{j!}(k+1)(k+2)\cdots(k+j), \quad (j \geq 2) \end{aligned} \tag{31}$$

Then,

$$\begin{aligned} m_{j+1} &= \sum_k^n k l_{j,k} = \sum_k^n k \sum_{l'}^{k} \frac{1}{(j-1)!}(l'+1)(l'+2)\cdots(l'+j-2) \\ &= \sum_k^n \frac{1}{j!} k(k+1)(k+2)\cdots(k+j-1) \\ &= \frac{(j+1)}{(j+2)!} n(n+1)(n+2)\cdots(n+j+1), \end{aligned} \tag{32}$$

therefore, equation (23) holds for (j+1)-D mass. Here, considering the expanded form of equation (30), the sum of j-th power of natural numbers is represented as a linear combination of sums of (j-1)-th~2nd power of natural numbers k as follows.

$$\sum_k^n k^j = \frac{1}{(j+1)} n(n+1)(n+2)\cdots(n+j) - \{1+2+\cdots+(j-1)\}\sum_k^n k^{j-1}$$
$$-\{1\cdot 2+1\cdot 3+\cdots+2\cdot 3+2\cdot 4+\cdots+(j-2)(j-1)\}\sum_k^n k^{j-2}$$
$$-\{1\cdot 2\cdot 3+1\cdot 2\cdot 4+\cdots+2\cdot 3\cdot 4+2\cdot 3\cdot 5+\cdots+(j-3)(j-2)(j-1)\}\sum_k^n k^{j-3}$$
$$-\cdots\cdots-(j-1)!\sum_k^n k, \qquad (j\geq 2). \tag{33}$$

The coefficients of sums of powers of natural numbers in the right side of equation (33) are given by the sum of combination products, $j\text{-}1C_1$, $j\text{-}1C_2$, ... , $j\text{-}1C_{j\text{-}1}$ of natural numbers 1, 2, 3, ... , j-1.

4. PROJECTION MATRIX FOR ROTATION ANGLE OF SUPERPOSITION

The rotation angle in the superposition operator is determined using body diagonal projection of a hypercube from j-D to (j-1)-D space. Consider a projection of an arbitrary vector of hyper cubic lattice $a(a_1, a_2, a_3, \ldots, a_j)$ along j-D projection vector $p(p_1, p_2, p_3, \ldots, p_j)$. Here projected vector $a\perp$ is given by

$$\mathbf{a}_\perp = \mathbf{a} - \frac{\mathbf{p}}{|\mathbf{p}|}|\mathbf{a}|\cos\theta \tag{34),}$$

where $\frac{p}{|p|}$ is a unit vectors along p and $\emptyset$ is an angle between p and a, then $a\perp$ is written as

$$\mathbf{a}_\perp = \mathbf{a} - \frac{\mathbf{p}(\mathbf{p}\cdot\mathbf{a})}{|\mathbf{p}|^2}. \tag{35}$$

The geometry of a projection along p $\perp$ is shown in Figure 6.

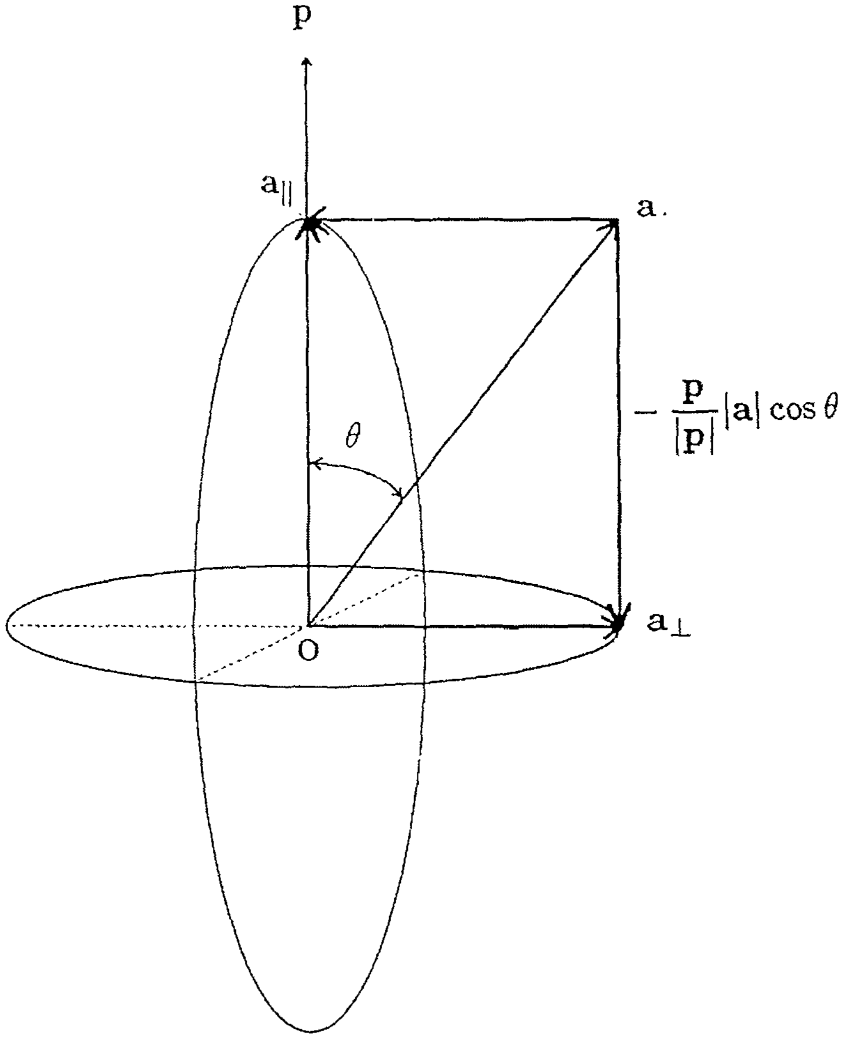

Figure 6: Geometry of j-dimensional orthogonal projection of a to planenormal to projection vector $p(p_1, p_2, p_3, \ldots, p_j)$, $a\perp$ and plane parallel p, $a_{\|}$

Geometry of j-dimensional orthogonal projection of a to plane normal to projection vector $p(p_1, p_2, p_3, \ldots, p_j)$, $a\perp$ and plane parallel to p, $a_{\|}$, j-D orthogonal base vectors a_i ($i = 1\sim j$) and projected vectors $a\perp_i$ are introduced where $p\perp$ is given by equation (36).

$$\mathbf{p} = \sum_{i}^{j} p_i \mathbf{a}_i, \qquad |\mathbf{p}|^2 = \sum_{i=1}^{j} p_i^2 \tag{36}$$

Substituting (36) for (35), a_i ($i = 1\sim j$) and projected vector $a\perp_i$ are introduced and the projection is formulated,

$$\mathbf{A}_\perp = \mathbf{P}_\perp \cdot \mathbf{A} \tag{37}$$

where column vectors of $A\perp$, $A_\parallel$ and projection matrix $P\perp$ are given as follows.

$$\mathbf{A}_\perp = \begin{pmatrix} \mathbf{a}_{1\perp} \\ \mathbf{a}_{2\perp} \\ \mathbf{a}_{3\perp} \\ \cdot \\ \cdot \\ \\ \mathbf{a}_{j-1\perp} \\ \mathbf{a}_{j\perp} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \cdot \\ \cdot \\ \\ \mathbf{a}_{j-1} \\ \mathbf{a}_j \end{pmatrix} \tag{38}$$

$$\mathbf{P}_\perp = \frac{1}{|\mathbf{p}|^2} \begin{pmatrix} |\mathbf{p}|^2 - p_1^2, & -p_1p_2 & -p_1p_3, & \cdot\ \cdot & -p_1p_{j-1}, & -p_1p_j \\ -p_2p_1, & |\mathbf{p}|^2 - p_2^2, & -p_2p_3, & \cdot\ \cdot & -p_2p_{j-1}, & -p_2p_j \\ -p_3p_1, & -p_3p_2, & |\mathbf{p}|^2 - p_3^2, & \cdot\ \cdot & -p_3p_{j-1}, & -p_3p_j \\ \cdot & & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot\ \cdot & \cdot & \\ -p_{j-1}p_1, & -p_{j-1}p_2, & \cdot & \cdot\ \cdot & |\mathbf{p}|^2 - p_{j-1}^2, & -p_{j-1}p_j \\ -p_jp_1, & -p_jp_2 & \cdot & \cdot\ \cdot & -p_jp_{j-1}, & |\mathbf{p}|^2 - p_j^2 \end{pmatrix} \tag{39}$$

The projection of $P_\parallel$ is obtained in the same way as that of $P\perp$. That is, a is decomposed into $a_\parallel$ and $a\perp$ orthogonal to each other. Then $P_\parallel$ is derived easily from the following relation.

$$A_\parallel + A\perp = A \tag{40}$$

$$A_\parallel + P_\parallel \cdot A \tag{41}$$

Substituting (37) and (41) into equation (40) then,

$$P_{\parallel} + P_{\perp} = I \qquad (42),$$

where I is the unit matrix. The conditions of the projection matrix are satisfied as shown in equation (43),

$$P^2{}_{\perp} = P_{\perp}, \qquad P^2{}_{\parallel} = P_{\parallel}, \qquad P_{\perp} \cdot P_{\parallel} = O \qquad (43)$$

where O is the zero matrix.

5. DETERMINATION OF ROTATION ANGLE FOR α_j SUPERPOSITION

First, consider a basis of projection axes of j-D hypercubic lattice to (j-1)-D space. If the body-diagonal axis [1, 1, 1, 1, … , 1, 1] is chosen as a projection axis, the angle between any two basis vectors from center to vertex of j-D hyper regular simplex (α_j) is an equi-solid angle. This can be proven easily α_j using the scalar product of two projected vectors presented by equation (37). Setting projection axis $(p_1, p_2, p_3, \dots, p_{j-1}, p_j)$ in the orthonormal system as follows,

$$p_1 = p_2 = p_3 = \dots = p_{j-1} = p_j = \frac{1}{\sqrt{j}} \qquad (44)$$

substitute (44) into projection matrix (39), then body-diagonal projection matrix (45) is obtained as follows.

$$\mathbf{P}_{\perp} = \begin{pmatrix} 1-\frac{1}{j}, & -\frac{1}{j}, & -\frac{1}{j}, & \cdot & \cdot & -\frac{1}{j}, & -\frac{1}{j} \\ -\frac{1}{j}, & 1-\frac{1}{j}, & -\frac{1}{j}, & \cdot & \cdot & -\frac{1}{j}, & -\frac{1}{j} \\ -\frac{1}{j}, & -\frac{1}{j}, & 1-\frac{1}{j}, & \cdot & \cdot & -\frac{1}{j}, & -\frac{1}{j} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{1}{j}, & -\frac{1}{j}, & \cdot & \cdot & \cdot & 1-\frac{1}{j}, & -\frac{1}{j} \\ -\frac{1}{j}, & -\frac{1}{j} & \cdot & \cdot & \cdot & -\frac{1}{j}, & 1-\frac{1}{j} \end{pmatrix} \qquad (45)$$

The rotation angle $\varnothing_{j-1}$ for superposition of hypersimplexes corresponds to the equi-solid angle between projected basis vectors from j-D to (j-1)-D space. $\varnothing_{j-1}$ is determined by a scalar product of any two projected basis vectors, $A\perp_i$, $A\perp_{i'}$, obtained by equation (45).

$$\cos\phi_{j-1} = \frac{(\mathbf{a}_{\perp i}\cdot\mathbf{a}_{\perp i'})}{|\mathbf{a}_{\perp i}|\cdot|\mathbf{a}_{\perp i'}|} = -\frac{1}{j-1} \qquad (j \geq 2) \tag{46}$$

The verification of equation (46) is given in the case studies of j = 2, 3 and 4, as shown in section 2. That is, $\cos\phi_1 = -1, (\phi_1 = \pi)$, $\cos\phi_2 = -1/2, (\phi_2 = 2\pi/3)$ and $\cos\phi_3 = -1/3, (\phi_3 = 109.47$ deg: tetrahedral angle) are presented as a unit step of superpositional rotation angle for 1-D, 2-D and 3-D, respectively. In general, a body-diagonal projection of a hypercube from j-D to (j-1)-D space generates a (j-1)-D hyper regular simplex which has j-fold rotational symmetry as viewed from [1, 1, 1, 1, ... , 1, 1]. The j-fold rotational symmetry of the hyper regular simplex is confirmed by considering the superposition operator S_j. S_j can be represented as follows.

$$\begin{aligned} \mathbf{S}_j &= \mathbf{R}^j(\phi_j) + \mathbf{R}^{j-1}(\phi_j) + \cdots + \mathbf{R}(\phi_j) + \mathbf{I} \\ &= \mathbf{R}(j\phi_j) + \mathbf{R}\{(j-1)\phi_j\} + \cdots + \mathbf{R}(2\phi_j) + \mathbf{R}(\phi_j) + \mathbf{I} \end{aligned} \tag{47}$$

Equation (47) is equivalent to equation (48) by the Hamilton-Cayley theorem.

$$R^{(j+1)}{}_{(\varnothing j)} = I \tag{48}$$

Equation (48) means that the (j-1)-D hyper regular simplex has the j-D hypercubic body-diagonal axis in the form of a j-fold rotational axis. If j approaches infinity, cos $\varnothing_j$ converges to 0, then the projected vectors in j-D space approach an orthogonal system.

6. CONCLUDING REMARK

A geometrical interpretation of the sum of the j-th power of natural numbers is presented and visualized by considering the superposition of j-D hypersimplexes which are generated by every rotation by angle $\varnothing_j$ about the body-diagonal axis of a (j+1)-D hypercubic lattice. The expanded form of $\sum_{k=1}^{n} k^j$ have already been formulated using Bernoulli polynomials. The relation between the expanded form given in reference (Moriguchi 1957) and that in this work will be discussed elsewhere. This work is partly supported by the Special Coordination Funds for Promoting Science and Technology Agency of the Government of Japan.

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LIST OF SYMBOLS

Symbol		Meaning
$\alpha_j(\phi_j)$	:	j-dimensional hypersimplex at ϕ_j position
$l_{j,k}, l'_{j,k}$	:	Number of lattice points in k-th layer of $\alpha_j(\phi_j)$
$\mathbf{R}(\phi_j)$	:	Rotation operator of α_j
$\mathbf{I}$	:	Identity operator of α_j
$\mathbf{S}_j$	:	Superposition operator of $\alpha_j(\phi_j)$
ϕ_j	:	Rotation angle for superposition of α_j
W_j	:	Weight of lattice points of superposed α_j
M_j	:	Multiplicity of superposition of α_j
m_j	:	Mass of α_j
$\overline{m_j}$	:	Mean mass of superposed α_j
k	:	Ordinal layer number of stacking α_j
n	:	Natural number
$\mathbf{a}$	:	Arbitrary vector of hypercubic lattice
$\mathbf{p}$	:	j-dimensional projection vector
$\mathbf{P}_\perp$	:	j-dimensional projection matrix of $\mathbf{p}$
$\mathbf{P}_\parallel$	:	j-dimensional projection matrix of a perpendicular to $\mathbf{P}_\perp$
$\mathbf{a}_\perp$	:	Projected vector of a along $\mathbf{p}$
$\mathbf{A}_\perp$	:	Column vectors of $\mathbf{a}$
$\mathbf{A}_\parallel$	:	Column vectors of $\mathbf{a}_\parallel$ perpendicular to $\mathbf{a}_\perp$
θ	:	Angle between $\mathbf{p}$ and $\mathbf{a}$