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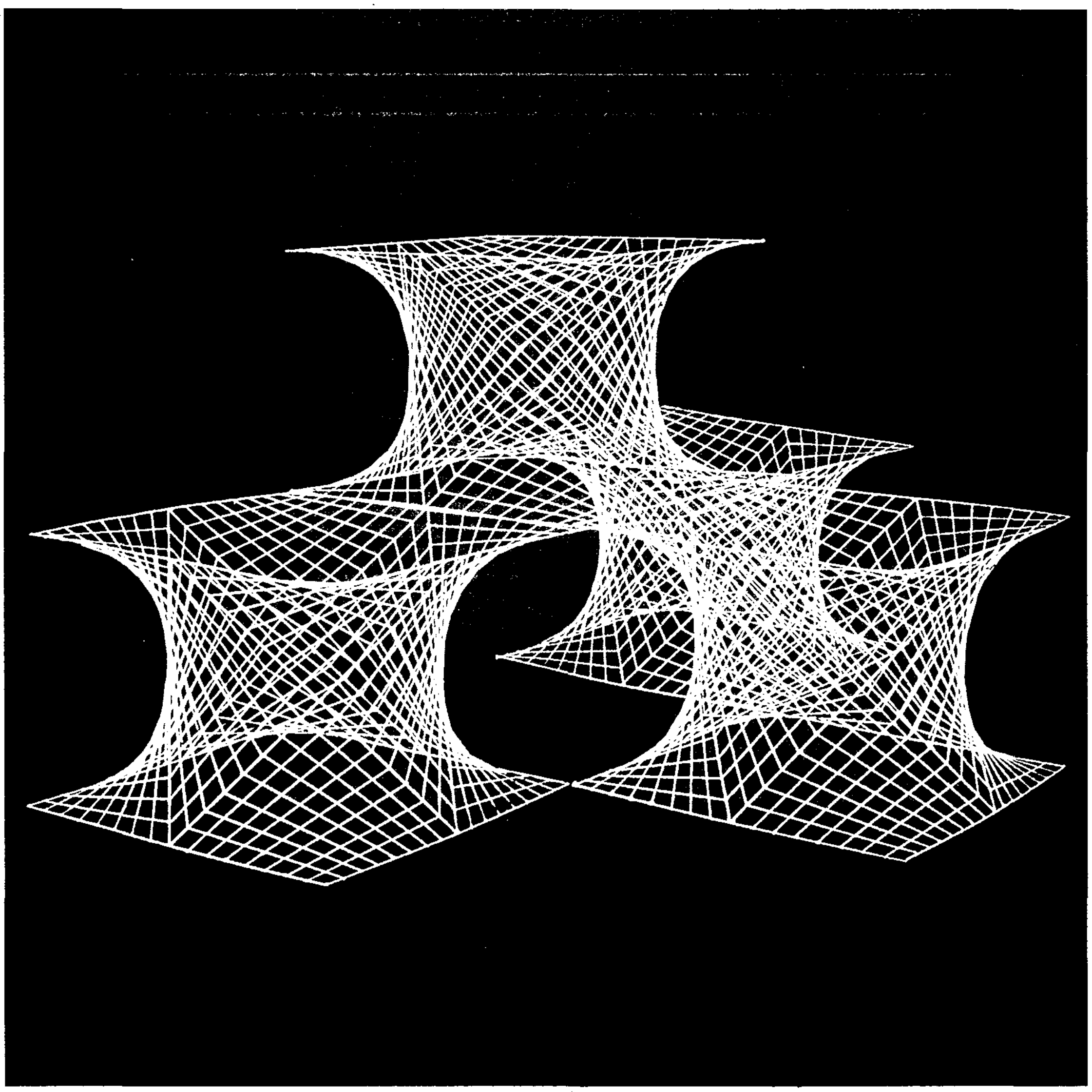

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OPTIMIZING LOTUS FLOWERS

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Abstract: *It has been discovered that in many cases the fruits in the receptacle of the sacred lotus* (Nelumbo nucifera) *are arranged in accordance with the solution to the following geometrical problem: How must* n *non-overlapping equal circles be packed in the unit circle so that the diameter of the circles will be as large as possible? In the paper an account of this problem (its putative solutions, and related configurations in lotus receptacles, traditional Japanese mathematics and family crests) is presented.*

1. INTRODUCTION

Design of ropes and cables has raised the following mathematical problem [1] which became one of the classical problems of discrete geometry: How must n non-overlapping equal circles be packed in the unit circle so that the diameter d of the circles will be as large as possible? [2, 3] This problem originated in engineering can be investigated in an engineering way. To an arrangement of circles there corresponds a *graph*. The vertices of the graph are the centres of the circles, and the edges of the graph are straight-line segments joining the centres of the touching circles. This graph is considered as a bar-and-joint structure supported along the boundary of a circle of radius $1 - d/2$, and a configuration is looked for where the bar length is a maximum, the distances between joints are not smaller than the bar lengths, the structure is rigid and it can have a stable state of self-stress without tensional force in bars. The solution of the structural problem corresponds to a locally optimal solution of the mathematical problem.

Over the years other methods have also been applied to solve this problem. Very recently Graham et al. [4] applied a billiard algorithm with which they produced putative solutions of the above packing problem for $n \leq 65$. Interestingly enough the solutions for small values of n can be seen in old Wasan (Japanese classical mathematics) books [5, 6] and Mon (Japanese family crest) collections [7] made long ago. In ancient Egypt, packing of equal circles in a circle was also used to measure the area of the circle [8]. Recently we discovered that the carpels in the receptacle of lotus flowers are arranged in accordance with dense packing [9]. This property is revealed even better for the arrangement of the lotus fruits in the receptacle.

The aim of this paper is to make a short comparison between the lotus fruit arrangements and the solutions of the circle packing problems, and to have a look at the related configurations in Wasan, and Mon design.

2. LOTUS

Lotus (*Nelumbo nucifera*) flowers (Figure 1(a)) have a conical receptacle with a circular even upper surface. In that circular face there are several small depressions, each of which contains one carpel. At first sight it seems that the carpels are arranged spirally [10]. On close inspection, however, we have found that the arrangement of carpels, and later the arrangement of fruits, in the receptacle is in accordance with the solution to the above circle packing problem. The solution provides the most economical configuration of circles because the area of the given circular domain covered by the circles is a maximum. So, *Nelumbo* optimizes the fruit arrangement in the receptacle. Since in living objects there are always imperfections and mutations, this is not a rigid law but only a prevalent tendency. Comparing actual fruit arrangements (Figure 2) with the mathematical solutions (Figures 6 and 7), the agreement is striking, disagreement is found mainly in those cases where the upper surface of the receptacle is distorted and/or the fruits are not equal. This is in accordance with the fact that the solutions of mathematical packing problems are sensitive for changes in data. Close packings of n circles, for instance, result in different configurations in a circle and in a domain different from a circle.

It is interesting to mention that in artistic representations of lotus sometimes not the optimal fruit arrangements are given. For example, the decoration in the Golden Pavilion, Horyuji temple in Nara, Japan, shows lotus flowers with 19 carpels (Figure 1(b)) arranged concentrically with carpel numbers: 1, 7, 11. In the optimal case, however, the arrangement is: 1, 6, 12 (Figure 2, n = 19).

Figure 1: Lotus flower and its representation as decoration in Horyuji temple (courtesy of Professor K. Miyazaki)

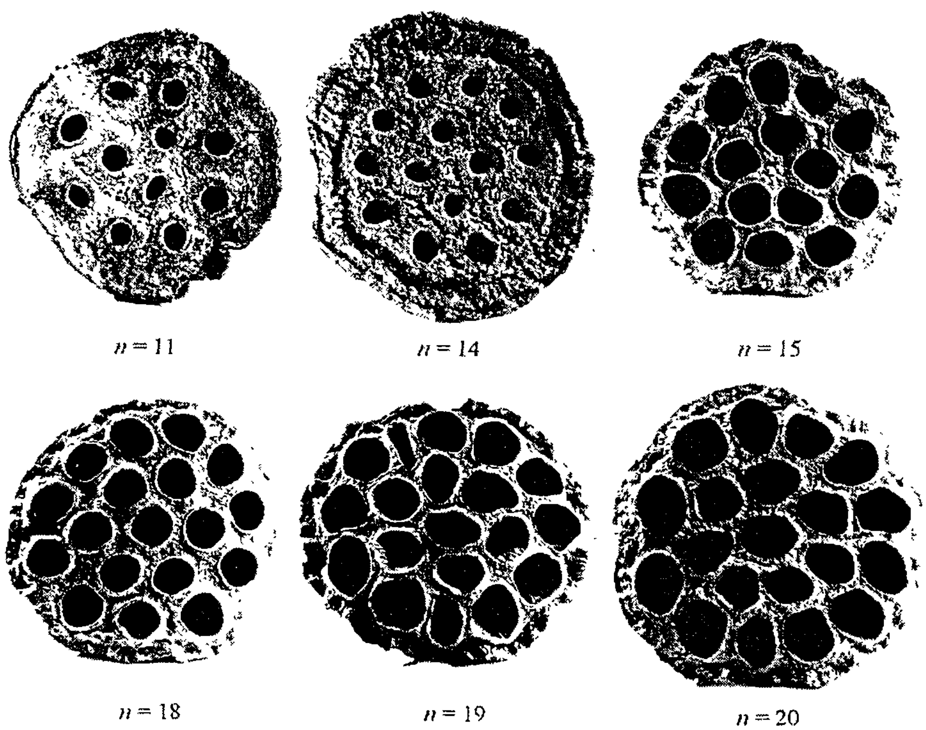

Figure 2: Arrangement of different numbers of fruits in the lotus receptacle (depressions in dried specimens)

3. MON (JAPANESE FAMILY CREST)

Circles have been very frequently used as motifs in Japanese family crests [7], often in close packings. Packing of circles in a circle is quite common (Figure 3). Many of them are closely related to the solution to the problem of the densest circle packing in a circle for $n = 2$ to 9. Although lotus is an important flower in Japan, an important symbol in Buddhism, it seems that its carpel does not appear among the family crests, circles do not represent fruits of lotus, at least it is not declared. Circles in Figure 3 have different meanings: stars (1, 2, 4-6, 14-17, 22-30), dragon's eye or snake's eye (7-12), weight used on balance-type scales (13), plum blossom (3, 15), tomoe (18-21).

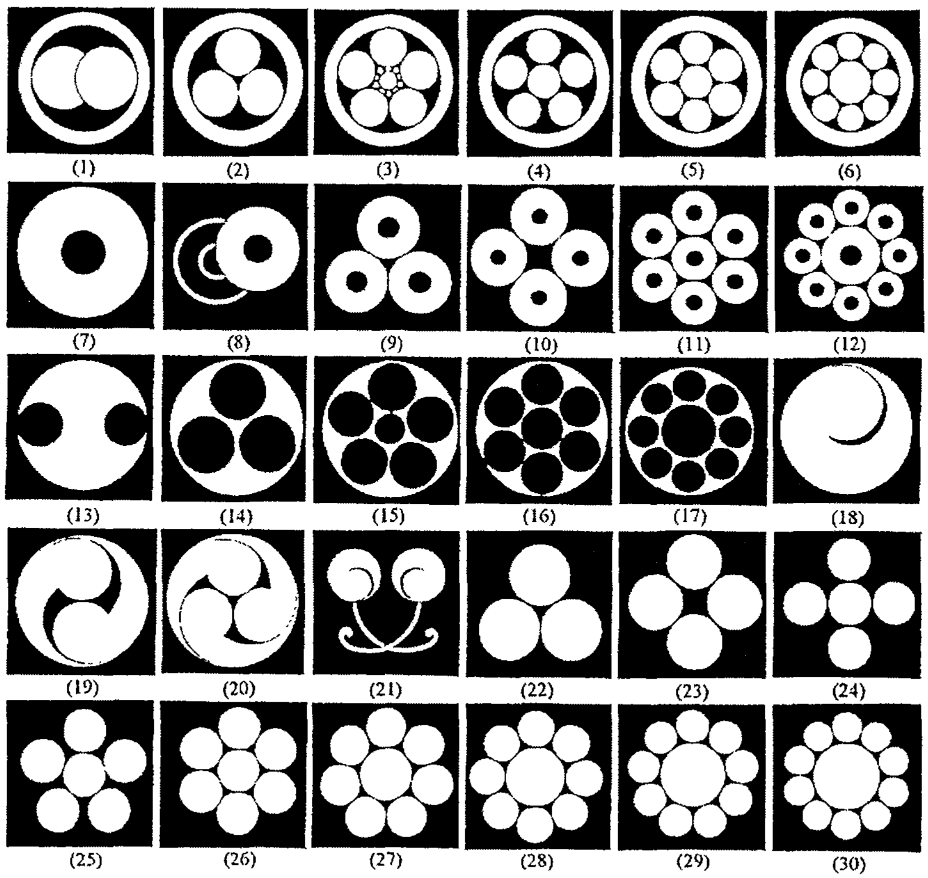

Figure 3: Circle packings in a circle in Japanese family crests

4. WASAN (JAPANESE CLASSICAL MATHEMATICS)

The Japanese mathematics wasan in the 17^{th}-19^{th} centuries discussed different problems where one had to find an unknown quantity from given quantities under certain conditions. In geometrical problems such quantities were, for instance, length of line segments or diameter of circles. At that time if someone had a nice mathematical discovery, then he put it down on a wooden board, called *sangaku*, dedicated it to gods and hung it up under the roof of a shrine or a temple [11]. Most sangaku problems are geometrical. The sangaku problems were often collected and published in a book. Many of these geometrical problems are related to circles, quite frequently to close packings of circles, particularly to densest packings of equal circles in a circle. In general the problem is not what the densest packing is, but to determine a distance, a circle radius or area of a domain in a close packing under constraints. Figure 4 shows figures of problems in different old wasan books. (a) [6] and (b) [5] are the densest packings of 5 and 7 equal circles. (c) [5] gives the densest packing of 9 equal circles if the diameter of the central circle is reduced. (d) [12] provides the densest packing of 10 equal circles with two axes of symmetry. It is interesting to mention that Kravitz [1] has considered this configuration as conjectured solution of the densest packing problem without symmetry constraints. The correct solution was given later by Pirl [13] (Figure 6, n = 10). (e) [14] shows a 3-dimensional configuration, but in the picture there is a packing of 12 circles of two different sizes in a circle. If the diameter of the central three circles is increased and that of the nine circles along the boundary is decreased, then with a slight modification, the densest packing of 12 equal circles (Figure 6, n = 12) is obtained. (f) [14] is one of the densest packings of 18 equal circles (Figure 7(a)).

Why did the Japanese have an interest in the densest packing problems? D. Nagy [15] asks this question in one of his papers analysing the connection between old Japanese mathematics and modern discrete geometry. He thinks that one of the probable reasons can be that the Japanese posed practical problems concerning economical cuttings and arrangements of things. One of the earliest problems published in wasan books [16] is counting barrels piled up in a triangular form. If the pile contains k rows of barrels, then the solution is the triangular number $t(k) = k(k+1)/2$. Knowing this result one can answer the question: How many bamboo sticks are there in a bundle containing one stick in the middle and k concentric layers of sticks hexagonally packed about it? [16] The answer is the hexagonal number $h(k) = 3k(k+1)+1$. In the case k = 1, we have $h(1) = 7$ (Figure 4(b)); in the case k = 2, we have $h(2)$ = 19 (Figure 4(f), one circle in the middle is added). The interesting point is that the Japanese have known in practice that equal bamboo sticks of hexagonal number can be arranged in a circular cylinder form (Figure 5) [16, 17].

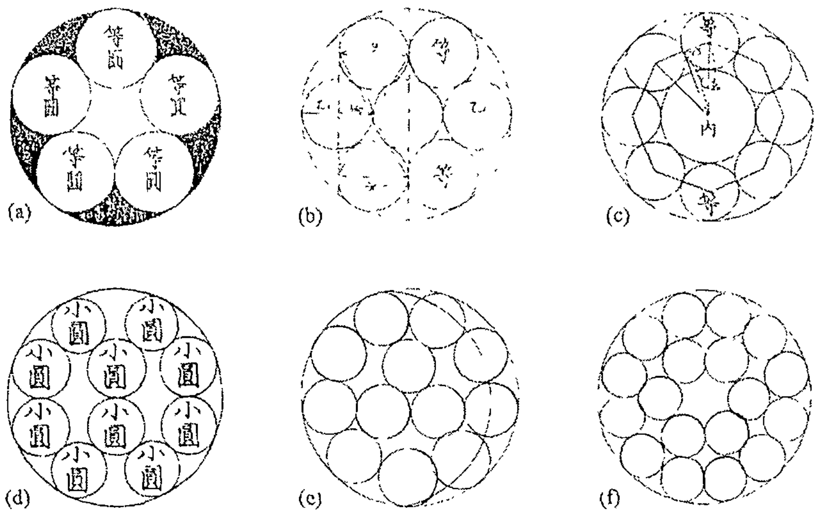

Figure 4: Packings of circles in a circle illustrations in old wasan books

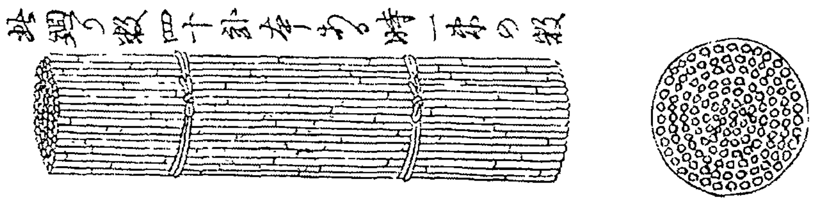

Figure 5: Curved hexagonal packing of equal bamboo sticks: illustrations in old wasan books

5. DISCRETE GEOMETRY

Discrete geometry [2, 3, 22] discusses among others tilings, packings and coverings in 2, 3 or higher dimensions. Extremal problems like densest packings and thinnest coverings are of the utmost importance. Problems of densest packing of equal circles in domains of different shapes in the plane are intensively studied nowadays thanks to the rapidly developing computer-aided methods.

Proven solutions of the problem of densest packing of n non-overlapping equal circles in the unit circle are known up to n = 11 circles [13, 18, 19] and conjectured solutions are known for $12 \leq n \leq 24$ [1, 13, 20, 21]. Graham et al. [4] extended this range recently and provided putative solutions up to n = 65. The packing algorithm used by them is based on the *billiards* model of computational physics, worked out by B.D. Lubachevsky. They compiled the numerical data in a table and presented the figures of the configurations. Subfigures of our Figures 6-8 are also taken from there [4].

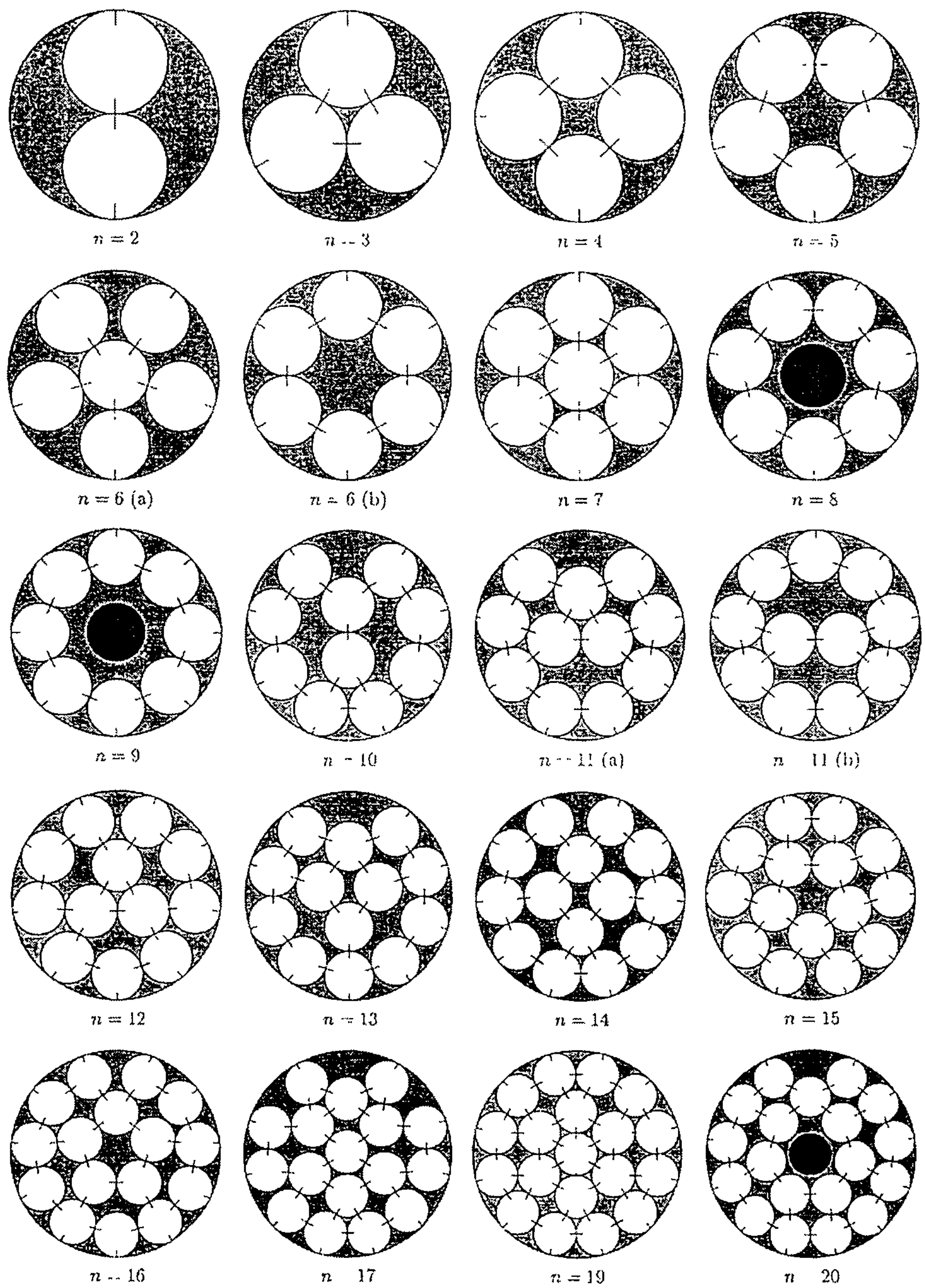

Figure 6: Proven and conjectured best packings of 2 to 17 and 19 to 20 equal circles in a circle (courtesy of Dr. B D. Lubachevsky)

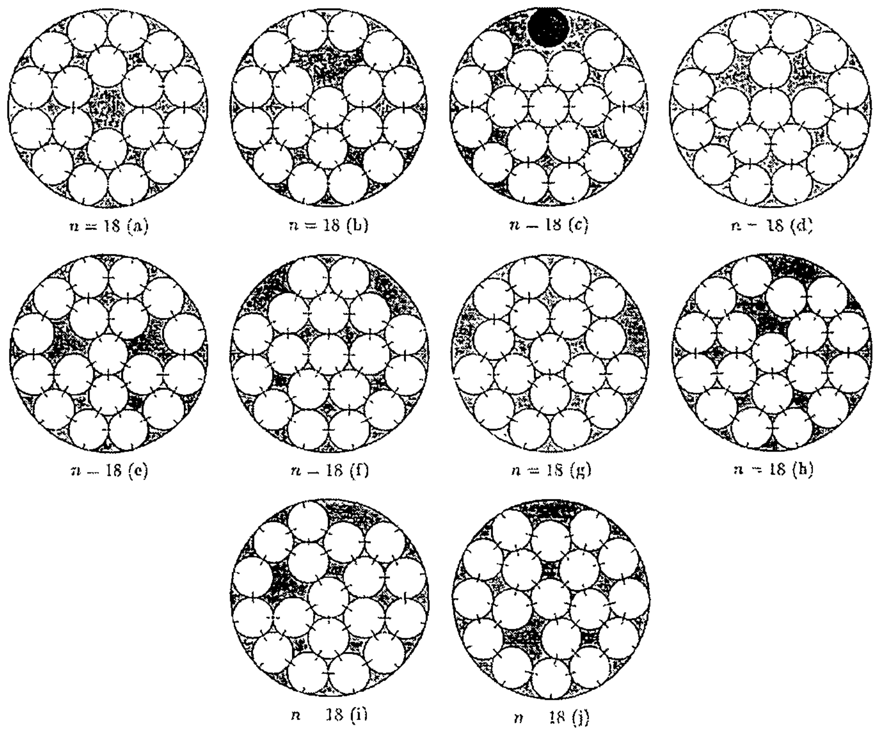

Figure 7: Conjectured best packings of 18 equal circles in a circle (courtesy of Dr. B.D. Lubachevsky)

Lubachevsky and Graham [23] have studied hexagonal dense packings of $h(k) = 3k(k+1)+1$ equal circles in a circle for up to $k = 5$. They have found that a dense circle packing in a circle can be obtained from that in a regular hexagon by deformation such that the number of circles in each circumferential layer is preserved, and the circumferential paths in the graph are closed (Figure 8). This explains why the Japanese could arrange equal bamboo sticks in a circular cylinder form.

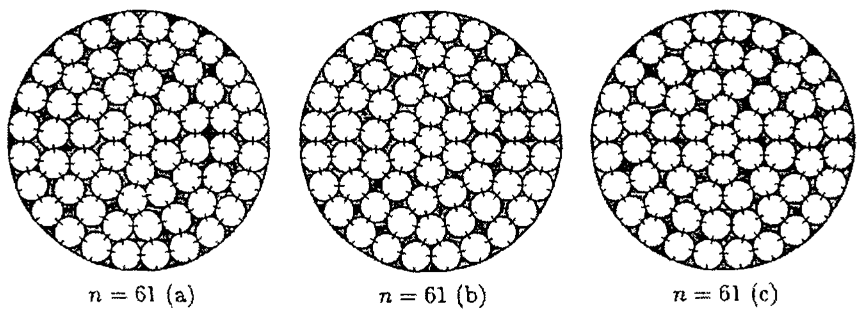

Figure 8: Curved hexagonal packings of 61 circles in a circle (courtesy of Dr B.D. Lubachevsky)

6. CONCLUSIONS

If the circles are represented by their centres, then the problem of the densest packing of equal circles in a circle is equivalent to the following problem: How must n points be distributed in a circle so as to maximize the least distance between any two of them? Therefore, lotus maximizes the minimum distance between fruits. This remarkable extremal property of *Nelumbo* receptacle is similar to that of certain spherical pollen grains, e.g. those of *Fumaria capreolata*, discovered earlier by Tammes, where the orifices on the surface of the pollen grain are arranged so [22]. This is a novel example that living nature can provide solutions to abstract mathematical problems, and so can give inspiration to solve practical (e.g., structural) problems.

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