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ORDER / DISORDER
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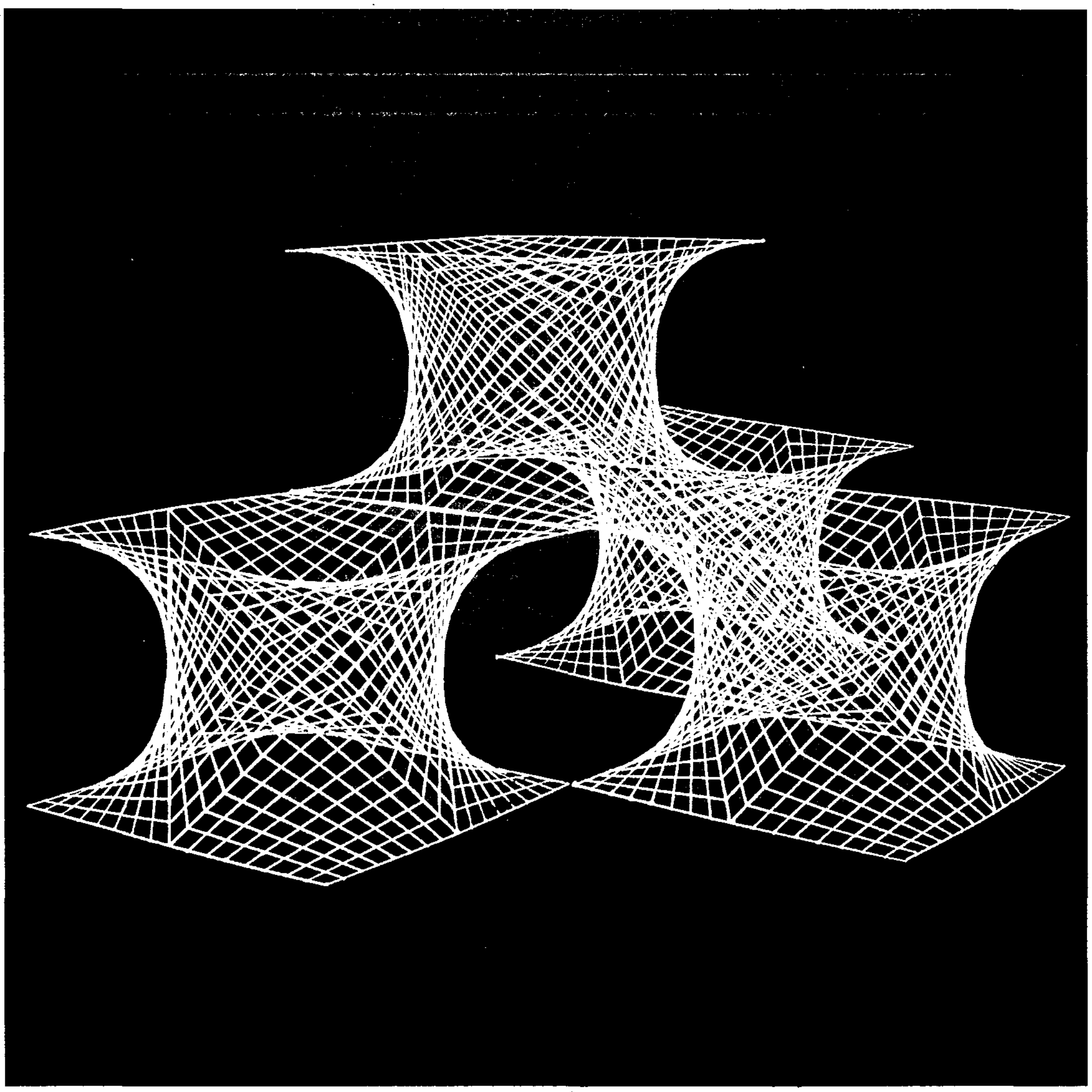

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INSTALLATION AND PERFORMANCE FOR REFLECTION OF SOUND AND LIGHT

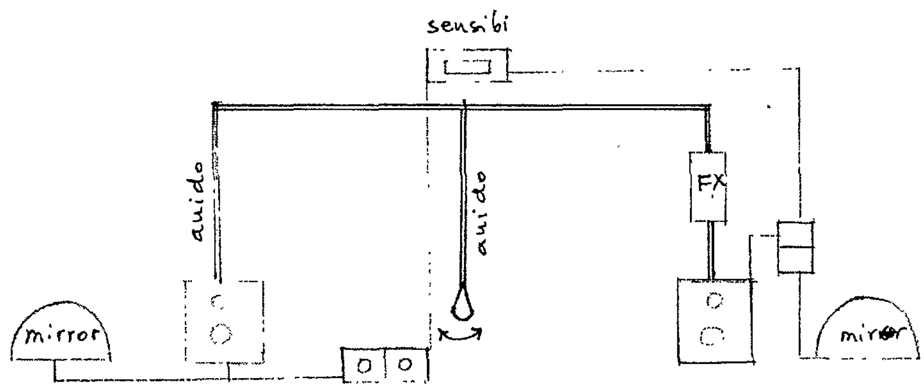
"CONCERTINO FOR ELECTRIC CORDS AND SALT"

Klara Kuchta and Shlomo Dubnov

Addresses: Klara Kuchta Création, 7 rue de la scie, CH-1207 Genéve, Suisse, klara.kuchta@freesurf.ch
Shlomo Dubnov, Hebrew University, Jeursalem, Givat-Ram, dubnov@cs.huji.ac.il

ORDER
- Left mirror half sphere moves.
- Pendium swings: produces a regular sound.

DISORDER
- Right mirror half sphere moves towards the audience
- Sound delay and echo appear.

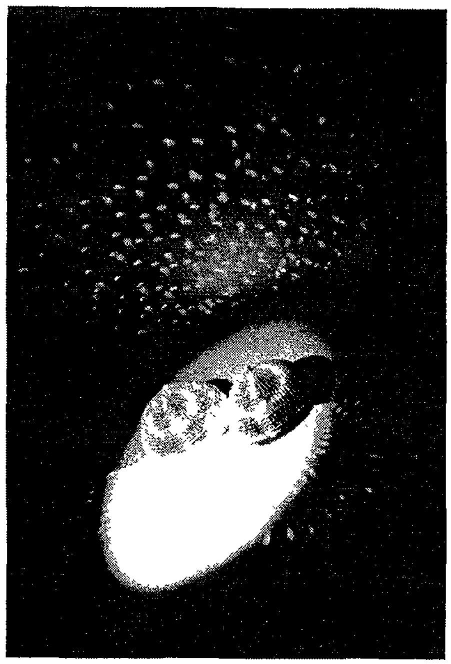


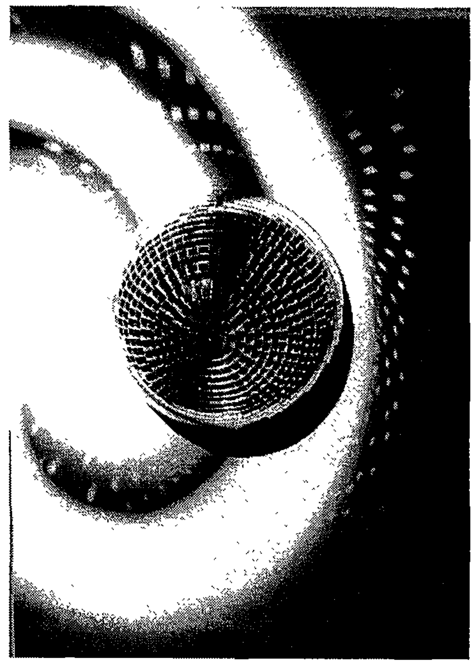
The sounds are generated by a *feedback* effect, which occurs when a microphone is put in front of a loudspeaker.

The microphone is hanged from the ceiling and it swings as a pendulum.

Every time it passes in front of the loudspeaker, feedback sound is added, with many echoes and delays that produce *harmony*, *rhythm*, and light effect.

The direction of movement of the light reflection is produced by the two mirror hemispheres. The sound direction also corresponds, i.e., order is on the left (no effect) and disorder comes from the right (sound with echo and delay).





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ON THE DEFINITION OF SYMMETRY

Roza Leikin [(1)] – Abraham Berman [(2)] – Orit Zaslavsky [(3)]

Address: [(1)] Faculty of Education, University of Haifa, Haifa, 31905, Israel (to whom correspondence should be addressed.)
[(2)] Technion - Israel Institute of Technology
[(3)] Technion - Israel Institute of Technology

DEFINITION OF SYMMETRY

Many mathematicians and mathematics educators emphasize the role of definition of a mathematical concept for developing conceptual understanding. The analysis of mathematical and educational literature indicates that the meaning of symmetry is not precisely defined. In addition, there are several different approaches to the definition of symmetry, depending on the content, the perspective taken, and consequently on the types of symmetry. Thus, symmetry is often viewed as a collection of disconnected concepts. In this paper we propose a formal definition of symmetry reflecting a unifying approach to the concept of symmetry in mathematics. This definition lends itself to considering a hierarchy of the different types of symmetry.

BACKGROUND

Symmetry is a very important scientific concept. It appears in different branches of mathematics and connects them. Symmetry often helps in making mathematical proofs elegant and can be considered as one of the important heuristics in mathematical problem solving. Nevertheless, the notion of symmetry is not precisely defined (Lowrey 1989). There are several different approaches to the definition of symmetry, depending on the perspective taken. For example, in the Russian dictionary of foreign words (Pchelkina 1988) four different meanings for the Greek word *Συμμετρια* are given:

- corresponding proportions between parts of a whole or of a body;
- a property of a geometric figure for which the figure can coincide with itself, in a way that not all its points remain in the same place;
- a global property of nature connecting the laws of conservation of energy, of movement, of atom and molecular structure, and of crystals' structure;
- a mutual relationship between the parts of the body with respect to an axis, a point or a plane.

Mathematicians treat symmetry in various ways: as a property of an object, as a special relation between objects, or as a special kind of transformation. In teaching mathematics, teachers and textbooks usually distinguish between symmetry in geometry and symmetry in other branches of mathematics. Even in geometry they deal separately with symmetrical geometric figures and with different transformations (reflection, rotation, and translation), neglecting to point out the underlying common feature of all these figures, transformations, and relationships (Eccles 1972; Ellis-Davies 1986; Fehr, Fey & Hill 1973; Lowrey 1989; Marcus 1989; Seneschal 1989; Skopets 1990; Yaglom 1962). In algebra and *calculus symmetry* is defined differently for different kinds of objects (e.g., functions, systems of equations, matrices, groups, mathematical problems) (Daintith & Nelson 1989; Dreyfus & Eisenberg 1990; Polya 1973; Waterhouse 1983). For example, according to Polya (1973) the expression $xy+yz+zx$ is symmetrical because an exchange of variables does not change its value. Thus, the variables have symmetrical roles. Polya used the notion *role symmetry* to express this type of symmetry. Another type of role symmetry, i.e., logical symmetry, can be found in the mathematical and educational literature. For example, Dhombres (1993) claims that symmetry exists between any two different proofs for one mathematical statement. These proofs have symmetrical roles with respect to the statement they prove. This logical symmetry of proofs has many implications. For example, Silver, Mamona-Downs, Leung & Kenney (1996) refer to symmetrical change as one of the strategies in posing mathematical problems.

As a result of this variety of types of symmetry and the differences between them symmetry is often viewed as a collection of disconnected concepts. An analysis of mathematical dictionaries supports this claim (Borowski & Borwein 1991; Clapman 1990; Daintith & Nelson 1989; Schwartzman 1994). For example, Daintith & Nelson (1989) define separately a symmetric function, a symmetric matrix, and a symmetric relation and only then define symmetry as follows:

"*Symmetry:* In general, a figure or expression is said to be symmetric if parts of it may be interchanged without changing the whole. For example $x^2+2xy+y^2$ is symmetric in x and y. *A symmetric operation (symmetry)* is an operation on a figure or expression that produces an identical figure or expression...." (Daintith & Nelson 1989, p. 313)

Many mathematics educators emphasize the important role of the definition of a mathematical concept for building mathematical understanding (De Villers 1994; Fischbein 1987; Moore 1994; Vinner 1991; Wilson 1989). Only a small number of authors discuss symmetry in a more general sense (Leikin, Berman and Zaslavsky 1995; Marcus 1989; Rosen 1989; Rosen 1995; Sonin 1987; Stewart & Golubitsky 1992, Weyl 1952). In these cases they refer to symmetry as proportion, harmony, order, or repetition. A general definition of symmetry in science is given by Rosen (1995) as follows: *Symmetry is immunity to a possible change* (p. 2, ibid.).

In this paper we suggest a unifying approach to the definition of symmetry in mathematics that is similar to what Rosen does for symmetry in science. We look at the immunity of a *property* of a mathematical *object* with respect to a possible change. This possible change corresponds to a *transformation* that can be applied to the object. Thus, symmetry has to do with a triplet — an *object*, a *property* and a *transformation* — as proposed in the following definition.

THE PROPOSED DEFINITION OF SYMMETRY

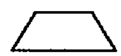
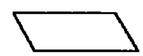
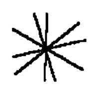
Definition. Symmetry is a triplet (S, Y, M) consisting of an *object* (S), a specific *property* (Y) of the object, and a *transformation* (M) satisfying the following two conditions:

i) The object belongs to the domain of the transformation;

ii) Application of the transformation to the object does not change the property of the object.

The *object S* is said to be *symmetrical* and the *transformation M* is called a *symmetry transformation.*

We refer to *two main types of symmetry:* (i) *Geometric Symmetry* where the object S is a geometric figure, and (ii) *Role Symmetry* where the transformation M is a permutation. We distinguish between different types of geometric symmetry according to the different types of geometric transformations: isometries (reflection, rotation, translation) and nonisometries (homothety). We consider different types of role symmetry according to the type of an object (algebraic, logical, and geometric). Figures 1 and 2 present examples of the different types of geometric and role symmetry.

TYPE OF SYMMETRY	OBJECT (S)	PROPERTY (Y)	TRANSFORMATION* (M)
GEOMETRIC SYMMETRY			
Isometric Symmetry	•An equilateral trapezoid 	• The location of the figure	• Reflection over a line
	• A quadratic function • An even function	• The location of the graph	
	•A parallelogram 	• The location of the figure	• Reflection over a point
	• An odd function	• The location of the graph	
	•A periodic function	• The location of the graph	• Translation
Non-Isometric Symmetry	•A family of straight lines passing through one point 	• The location of the figure	• Homotety

* Note that the given definition of symmetry applies also to three-dimensional geometrical objects where any plane symmetry transformation can be extended to a solid symmetry transformation

Figure 1: Examples of different types of Geometric Symmetry

Observe that our definition of symmetry includes some trivial cases. We will say that a symmetry (*S*, *Y*, *M*) is a *trivial symmetry* if property *Y* is immune to transformation *M* for *any* object *S* having this property. Examples of trivial symmetries are: (a) Any triplet (*S*, *Y*, *I*) where *I* is the identity transformation; (b) Any triplet (*G*, *D*, *IS*) where *G* is a geometric figure, *D* denotes distance, and *IS* is an isometric transformation.

TYPE OF SYMMETRY	OBJECT (S)	PROPERTY (Y)	TRANSFORMATION* (M)
ROLE SYMMETRY			
Algebraic Role Symmetry	• An algebraic expression, e.g. $a+b$,	• The numerical value of the expression	• Permutation of variables
	• A function, e.g. $xy=8$;	• The graph of the function	
	• A systems of equation, e.g.. $\begin{cases} 3x+y+2z=30 \\ 2x+3y+z=30 \\ x+2y+3z=30 \end{cases}$ (Polya, 1981)	• The solution of the system	
Geometric Role symmetry	•An isosceles triangle △	• The location of the triangle • The equality of the sides	• Permutation of the equal sides
Logical Role symmetry	• A symmetrical relation, e.g.: $a \parallel b$, $G \cong F$	• The correctness of the statement	• Permutation of the related objects
	A Symmetrical solution, e.g *Problem*: Of all the triangles inscribed in a given circle which one has the maximal area? *Solution*: If side b and side c of the triangle are not equal, then the area of the triangle can be increased. Hence $b=c$. In the same way, $a=b$. By analytical considerations there exists triangle of maximal area, and by the preceding steps it must be equilateral	• The correctness of the solution	• Permutation of steps in the solution

Figure 2: Examples of different types of Role Symmetry

RELATIONSHIP BETWEEN ALGEBRAIC AND GEOMETRIC SYMMETRIES

Relationships between different types of geometrical symmetry transformations are widely discussed in the mathematical literature (for example, see Yaglom 1962). In this section we consider relationships between algebraic and geometric symmetry of mathematical objects.

A mathematical object may have different representations. For example, a quadratic function can be represented algebraically as $y = ax^2+bx+c$, or geometrically, as a parabola. We consider an object S to be symmetrical if it is symmetrical at least in one of its representations. If an object is symmetrical in its algebraic (geometric) representation we will say that it is algebraically (geometrically) symmetrical. The relations between these two types of symmetry are given in the following two statements (Leikin 1997).

Statement 1. If a mathematical object has both a geometric and an algebraic representation and if it is algebraically symmetrical, then it is geometrically symmetrical.

Statement 2. Any mathematical object having both a geometric and an algebraic representation is algebraically symmetrical if and only if there exists an isometric transformation (M) satisfying the following conditions: (1) M does not change the location of the object in its geometric representation; (2) M maps each coordinate axis to one of the coordinate axes without change of the axis's direction.

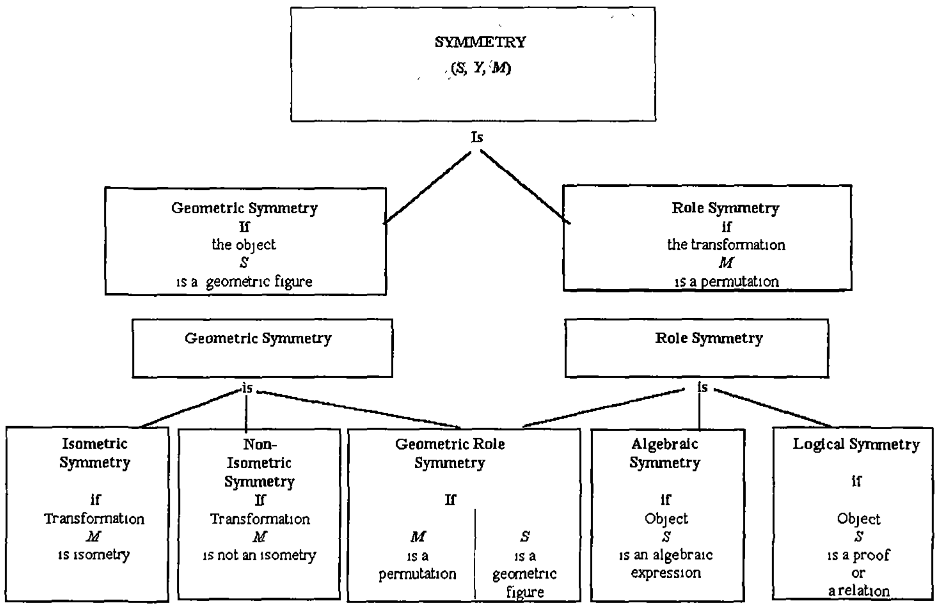


Figure 3: Hierarchy of the Concept of Symmetry in School Mathematics

HIERARCHY OF TYPES OF SYMMETRY

As shown above, the proposed definition of symmetry, which reflects a unifying approach to the concept of symmetry, lends itself to considering its different types as particular cases of the general notion of symmetry. This definition highlights the hierarchical connections between the different types of symmetry. Figure 3 depicts a hierarchy of the concept of symmetry, which depends on the types of symmetrical mathematical objects and on the types of symmetry transformations.

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