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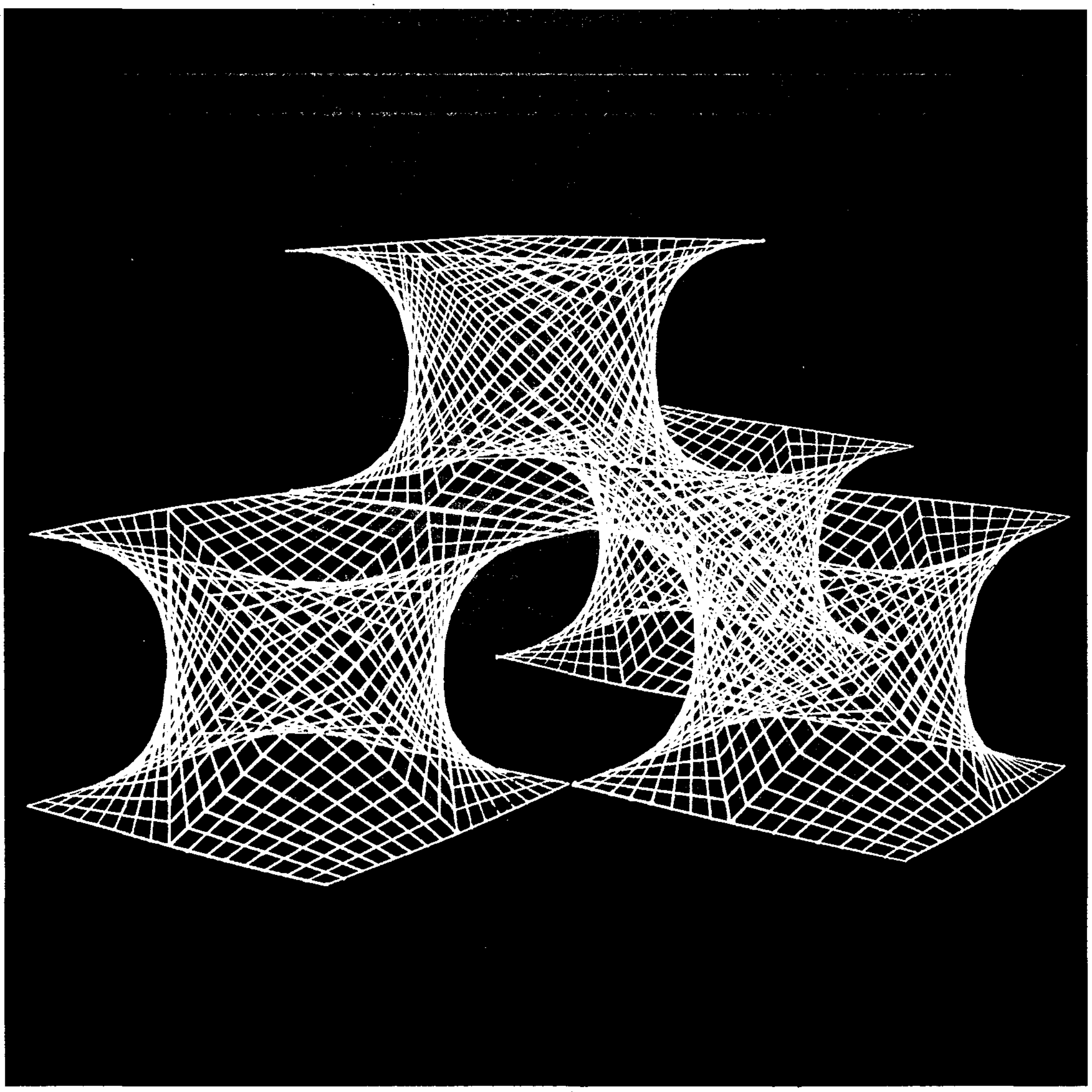

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# MORPHOLOGICAL TRANSFORMATION OF HYPERBOLIC PATTERNS

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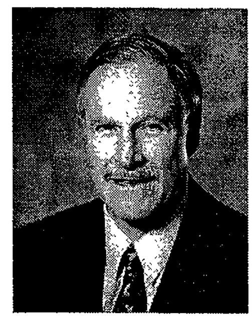
*Fields of interest:* Hyperbolic geometry, M. C. Escher, computer graphics algorithms, computer generated art.
Publications: Families of Escher patterns, presented at *M. C. Escher 1898-1998, An International Congress* (1998) June, Rome, Italy, *The Symmetry of hyperbolic Escher patterns, *Proceedings of the Third Interdisciplinary Symmetry Congress* (of the International Society for Interdisciplinary Studies of Symmetry: ISIS-Symmetry) (1995), August, Alexandria, VA; *Artistic patterns with hyperbolic symmetry, *Proceedings of Symmetry of Structure, An Interdisciplinary Symposium* (1989) August, Budapest, Hungary; *Creating hyperbolic Escher patterns, *M. C. Escher, Art and Science*, Coxeter, H.S.M. et al., eds., North-Holland, New York (1986) pp. 241-248; *Hyperbolic symmetry, *Computers & Mathematics with Applications* 12B (1986) nos. 1/2, pp 139-153.



**Abstract:** *The Dutch artist M. C. Escher is known for his repeating patterns of interlocking motifs. Most of Escher's patterns are Euclidean patterns, but he also designed some for the surface of the sphere and others for the hyperbolic plane, thus making use of all three classical geometries: Euclidean, spherical, and hyperbolic. In some cases it is evident that he applied a morphological transformation to one of his patterns to obtain a new pattern, thus changing the symmetry of the original pattern, sometimes even forcing it onto a different geometry. In fact Escher transformed his Euclidean* Pattern Number 45 *of angels and devils both onto the sphere,* Heaven and Hell *on a carved maple sphere, and onto the hyperbolic plane,* Circle Limit IV. *A computer program has been written that converts one hyperbolic pattern to another by applying a morphological transformation to its motif. We will describe the method used by this program.*

# 1 INTRODUCTION

M. C. Escher created many repeating patterns of the Euclidean plane. In a few cases he distorted or "morphed" these patterns to obtain new patterns in other geometries: spherical or hyperbolic. Escher's Pattern Number 45 of angels and devils is the only one that he converted to both the sphere and the hyperbolic plane. These three related patterns are shown in (Schattschneider 1990) on pages 150, 244 and 296; see Figure 1 below for *Circle Limit IV*. Professor Coxeter discusses the three patterns on pages 197-209 of Coxeter 1981.

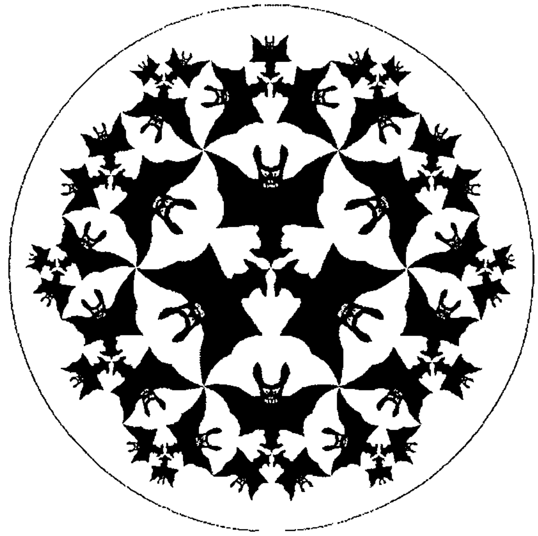


**Figure 1:** This is Escher's hyperbolic pattern *Circle Limit IV* of angels (the white background) and devils (foreground).

There are probably many ways to distort or "morph" one pattern into another. The method we will describe applies to repeating patterns based on the regular tessellations, $\{p; q\}$, composed of regular $p$-sided polygons meeting $q$ at a vertex. Thus, given one repeating pattern, we could theoretically create a doubly infinite family of related patterns by morphing the original pattern into others based on different values of $p$ and $q$. Many of Escher's Euclidean patterns and all of his spherical and hyperbolic patterns are based on $\{p; q\}$. For example, his Euclidean Pattern Number 45 and the related spherical and hyperbolic patterns mentioned above are based on the tessellations $\{4; 4\}$, $\{4; 3\}$ and $\{6; 4\}$ respectively. Figure 2 shows the tessellation $\{6; 4\}$ superimposed on *Circle Limit IV*. In these patterns, $p$ is twice the number of angels/devils that meet at their feet and $q$ is the number of wing tips that meet at a point. The meeting point of feet is the intersection of lines of bilateral (reflection) symmetry – hence the need to double the number of angels/devils to obtain $p$.

We will begin with a brief review of hyperbolic geometry. Next we discuss repeating patterns and regular tessellations, and the morphological transformation process, showing an example. Finally we suggest possible further directions of research.

## 2 HYPERBOLIC GEOMETRY

Unlike the Euclidean plane and the sphere, the entire hyperbolic plane cannot be isometrically embedded in 3-dimensional Euclidean space. Therefore, any model of hyperbolic geometry in Euclidean 3-space must distort distance. The *Poincaré circle model* of hyperbolic geometry has two properties that are useful for artistic purposes: (1) it is conformal (i.e., the hyperbolic measure of an angle is equal to its Euclidean measure) – thus a transformed object has roughly the same shape as the original, and (2) it lies within a bounded region of the Euclidean plane – allowing an entire hyperbolic pattern to be displayed. The "points" of this model are the interior points of a *bounding circle* in the Euclidean plane. The (hyperbolic) "lines" are interior circular arcs perpendicular to the bounding circle, including diameters. The sides of the hexagons of the {6; 4} tessellation shown in Figure 2 lie along hyperbolic lines as do the backbone lines of the fish in Figures 3 and 4.

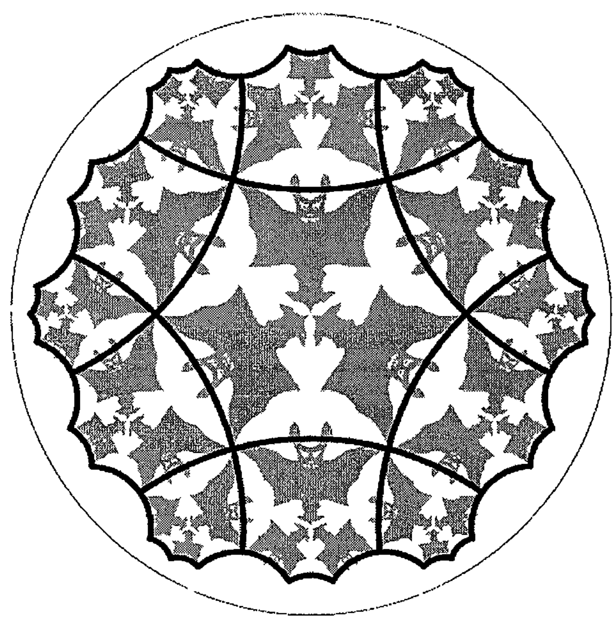


**Figure 2:** This is *Circle Limit IV* showing the underlying {6, 4} tessellation

## 3 REPEATING PATTERNS AND REGULAR TESSELLATIONS

A *repeating pattern* of the Euclidean plane, the hyperbolic plane, or the sphere is a pattern made up of congruent copies of a basic subpattern or motif. For instance, a black half-devil plus an adjacent white half-angel make up a *motif* for Figure 1.

An important kind of repeating pattern is the *regular tessellation*, $\{p; q\}$, of the plane by regular $p$-sided polygons, or *p-gons*, meeting $q$ at a vertex. The values of $p$ and $q$ determine which of the three "classical" geometries, Euclidean, spherical, or hyperbolic, the tessellation lies in. The tessellation $\{p; q\}$ is spherical, Euclidean, or hyperbolic according as $(p-2)(q-2)$ is less than, equal to, or greater than 4. This is shown in Table 1 below. Note that most of the tessellations are hyperbolic. In the spherical case, the tessellations $\{3; 3\}$, $\{3; 4\}$, $\{3; 5\}$, $\{4; 3\}$ and $\{5; 3\}$ correspond to versions of the Platonic solids (the regular tetrahedron, octahedron, icosahedron, cube, and dodecahedron respectively) "blown up" onto the surface of their circumscribing spheres. One can interpret the tessellations $\{p; 2\}$ as two hemispherical caps joined along p edges on the equator; similarly $\{2; q\}$ is a tessellation by $q$ lunes. Escher's only use of these latter tessellations appears to be the carved beechwood sphere with 8 grotesques (Schattschneider 1990, p. 244) based on $\{2; 4\}$. The tessellations $\{3; 6\}$, $\{4; 4\}$ and $\{6; 3\}$ are the familiar Euclidean tessellations by equilateral triangles, squares, and regular hexagons, all of which Escher used extensively.

| $q$ \ $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ⋯ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ⋮ | | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | |
| 11 | | o | * | * | * | * | * | * | * | * | * | ⋯ |
| 10 | | o | * | * | * | * | * | * | * | * | * | ⋯ |
| 9 | | o | * | * | * | * | * | * | * | * | * | ⋯ |
| 8 | | o | * | * | * | * | * | * | * | * | * | ⋯ |
| 7 | | o | * | * | * | * | * | * | * | * | * | ⋯ |
| 6 | | o | □ | * | * | * | * | * | * | * | * | ⋯ |
| 5 | | o | ○ | * | * | * | * | * | * | * | * | ⋯ |
| 4 | | o | ○ | □ | * | * | * | * | * | * | * | ⋯ |
| 3 | | o | ○ | ○ | ○ | □ | * | * | * | * | * | ⋯ |
| 2 | | o | o | o | o | o | o | o | o | o | o | ⋯ |
| 1 | | | | | | | | | | | | |

□ - Euclidean tessellations
○ - "Platonic" spherical tessellations
o - spherical tessellations for which $p = 2$ or $q = 2$
$*$ - hyperbolic tessellations

**Table 1:** This table shows the relation between the values of $p$ and $q$, and the geometry of the tessellation $\{p; q\}$.

# 4 MORPHOLOGICAL TRANSFORMATION OF A PATTERN

The basic version of the computer program that performs the morphological transformation requires that the motif be contained in one of the $p$ isosceles triangles formed by the radii of a $p$-gon. Figure 3 below shows the isosceles triangles within a 6-gon of {6; 4} that is the basis of Escher's *Circle Limit I* (shown in gray). A natural motif in *Circle Limit I* is composed of a black half-fish and an adjoining white half-fish, however such a motif has part of a white fish fin protruding outside its isosceles triangle. This motif can be modified to the required form by clipping off the protruding part and "gluing" it back between the tail and the back edge of the fin of the black fish.

The program has been extended slightly so that this modification is often not necessary. The extended program also seems to work reasonably well with a motif that overlaps two adjacent isosceles triangles (with roughly half the motif in each triangle) – as is the case with *Circle Limit IV* (Figure 1).

The basic morphing process makes use of the Klein model of hyperbolic geometry. As with the Poincaré model, the points are interior points of a bounding circle, but the hyperbolic lines are represented by chords. We let $I$ denote the isomorphism that maps the Poincaré model to the Klein model. Then $I$ maps a centered $p$-gon with its isosceles triangles to a regular $p$-sided polygon which also contains corresponding isosceles triangles. Different tessellations $\{p; q\}$ produce different isosceles triangles in the Klein model, but an isosceles triangle from $\{p; q\}$ can be mapped onto an isosceles triangle from {p'; $q'$} by a simple (Euclidean) differential scaling, since those isosceles "Klein" triangles are represented by isosceles Euclidean triangles. Thus the morphological transformation from a $\{p; q\}$ pattern to a $\{p'; q'\}$ pattern can be accomplished by (1) applying $I$ to a motif in an isosceles triangle of $\{p; q\}$, (2) applying the differential scaling to that transformed triangle, and finally (3) applying the inverse of $I$ to the re-scaled triangle containing the motif. The entire pattern can then be formed by replicating the morphed motif. Replication algorithms are discussed in Dunham 1986a and Dunham 1986b. Figure 4 shows the result of morphing the *Circle Limit I* pattern to a {4; 6} pattern – with a transformed isosceles triangle superimposed.

Using similar techniques, another program has been written to transform isosceles Euclidean triangles to isosceles hyperbolic triangles, and thus Euclidean Escher patterns (of which there are many) can be transformed to hyperbolic patterns.

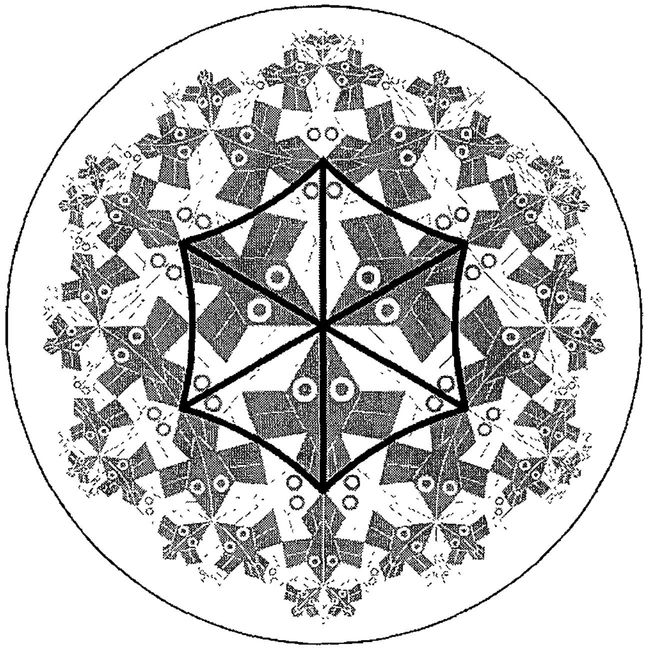


**Figure 3:** The isosceles triangles superimposed on *Escher's Circle Limit I* pattern.

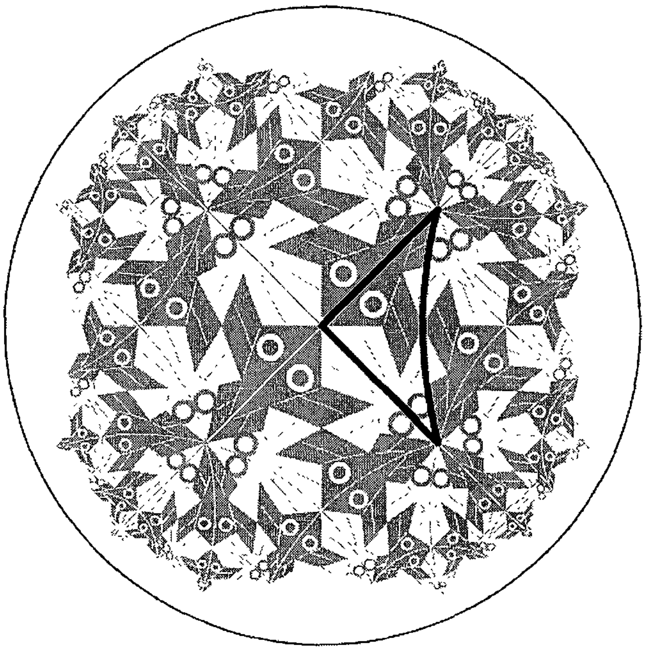


**Figure 4:** A morphed Circle Limit I based on {4; 6} showing an isosceles triangle.

## 5 FUTURE WORK

Directions of future research include: (1) finding different morphing transformations, (2) allowing the fundamental region to be a non-isosceles triangle or quadrilateral, and (3) transforming between any of the three classical geometries. The morphing transformation described above is not conformal. Theoretically, the Riemann Mapping Theorem says there is a holomorphic (and hence conformal) isomorphism between the Poincaré isosceles triangles of any two tessellations $\{p; q\}$ and $\{p'; q'\}$. A natural fundamental region for Escher's *Circle Limit III* is a quadrilateral divided into two triangles whose sides are two hyperbolic line segments and a segment of an equidistant curve – the above methods may extend to such triangles. Finally, transforming from spherical to Euclidean (and hence hyperbolic) patterns would only involve finding a mapping from isosceles spherical triangles to isosceles Euclidean triangles.

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