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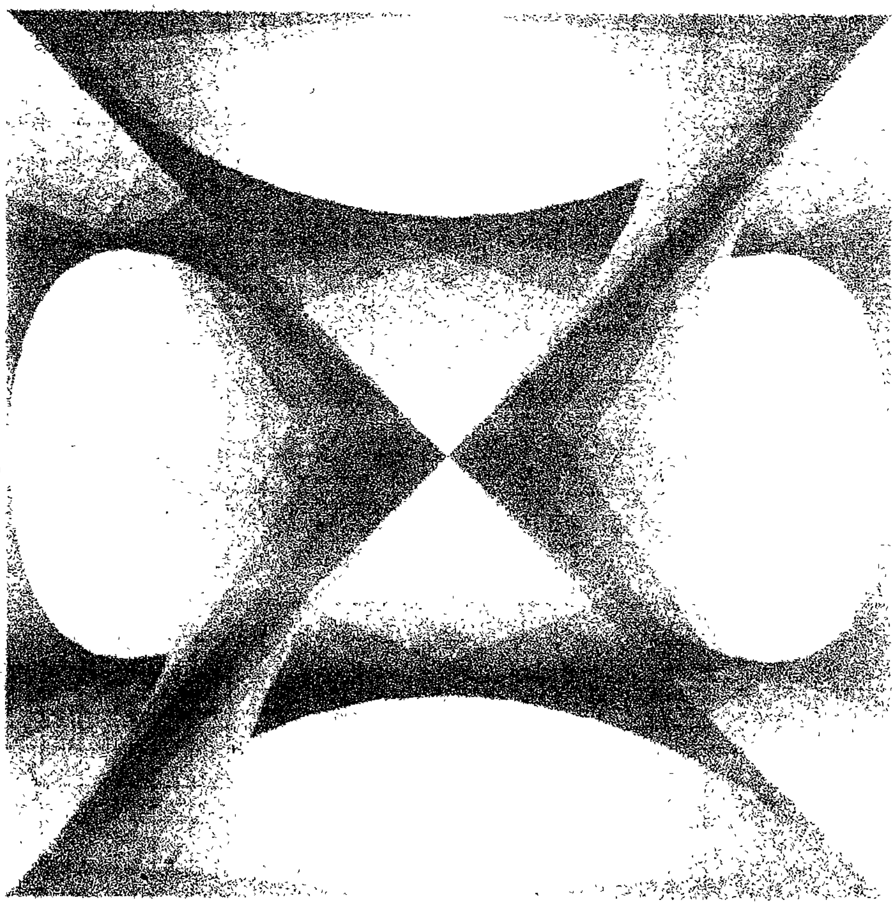

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PERFECT PRECISE COLOURINGS OF TRIANGULAR TILINGS, AND HYPERBOLIC PATCHWORK

Dedicated to the memory of Raphael M. Robinson

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Publications: Some new regular compound tessellations, *Proceedings of the Royal Society, London*, A 422, (1989) 311-318; Symmetry in geometry, *Symmetry: Culture and Science,* 1, (1990) 63-76; Compound tilings and perfect colourings, *Leonardo,* 24, (1991) 53-62; (Joint paper with Coxeter, H. S. M.) Frieze patterns, triangulated polygons and dichromatic symmetry, in: Guy, R.K. and Woodrow, R.E., eds. *The Lighter Side of Mathematics,* The Mathematical Association of America, Washington D.C., (1994) 15-27; Precise colourings of regular triangular tilings, *The Mathematical Intelligencer,* 20, no.1, (1998) 4-11.

Abstract: *We consider the problem of colouring the regular triangular tiling* $\{3,n\}$ *with n colours, in such a way that one tile of each colour occurs at each vertex. Such a tiling will be called* precise, *and the most interesting precise colourings are those that are also perfect. We shall find a method of enumerating a large class of perfect precise colourings, and shall also consider various related problems. Other aspects of the subject are discussed in Yaz (1997) and Rigby (1998), which were written after the present article was first submitted.*

1. INTRODUCTION

The following problem appeared in the American Mathematical Monthly, proposed by Raphael M. Robinson (Robinson 1993).

> *The hyperbolic plane is tiled with equilateral triangles meeting seven at each vertex. Can the tiles be colored with seven colors in such a way that no two tiles of the same color meet even at a vertex? (This problem was suggested to the proposer by David Gale.)*

The proposer mentioned to me in a letter that he had already solved the problem in 1984, but that David Gale was interested in the more general problem: *can we find a colouring of the regular tiling* $\{3,n\}$ *with equilateral triangles (on the sphere for* $n = 3, 4, 5$, *in the Euclidean plane for* $n = 6$, *and in the hyperbolic plane for* $n \geq 7$) *using* n *colours, such that one triangle of each colour occurs at each vertex?*

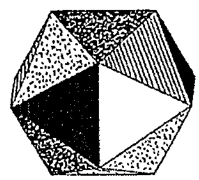

Figure 1

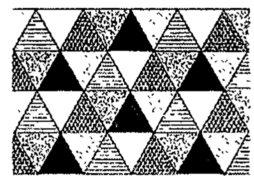

Figure 2

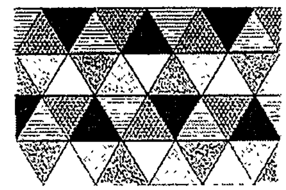

Figure 3

We shall say that such a colouring is *precise.* There is no precise colouring of the tetrahedron $\{3,3\}$. The unique precise colouring of the octahedron $\{3,4\}$ is obvious. The precise colouring of $\{3,5\}$ shown in Figure 1 is essentially unique, but it occurs in left- and right-handed versions. The most obvious precise colouring of $\{3,6\}$ is shown in Figure 2; it is fully perfect, which means that every symmetry of the tiling (every transformation that maps the tiling to itself) permutes the colours instead of jumbling them up. Another precise colouring is shown in Figure 3; here each horizontal strip contains just three colours, and the colours in any horizontal strip can be permuted, thus producing an infinity of highly imperfect precise colourings.

In Section 2 we give a solution to the problem when $n = 7$ which results in a colouring that is not only precise but also chirally perfect (a notion that will be explained later). In Sections 2 and 3 we find a way of viewing this solution that enables us to generalise it to construct solutions for all values of n. In Section 4 we find all perfect precise colourings of a standard type, and a unique notation for them. Other related problems then present themselves, notably the existence of fully perfect precise colourings (Section 5), which occur only when n is even and $n \neq 8$, and the existence of fully perfect (but not precise) colourings when n is odd (Section 7). In Sections 8 and 9 we briefly consider non-standard perfect precise colourings and semiperfect precise colourings. In Section 10 we show how an infinity of imperfect precise colourings can be produced; many of these still have a high degree of symmetry. Finally, in Section 11, we introduce the notion of *equivalent colourings,* which is useful in the investigation of the groups of permutations of colours induced in perfect colourings by the direct symmetry group of the tiling.

The concepts, constructions and proofs in this article were conceived in a very visual manner, and are here presented in the same manner. The reader is encouraged to make photocopies of the blank tilings on the final page (Figure 31), then to get a feel for the constructions to be described in the article by creating various trees and sewing them together, and by creating various partial colourings.

2. PERFECT PRECISE COLOURINGS AND TREES

Figure 4 shows the black tiles in a precise colouring of {3,7} (Rigby 1990, Fig. 16; Rigby 1991, Fig. 7). If we stand on any black tile, it is easy to find two simple rules for proceeding to the adjacent black tiles; these rules are independent of which tile we are on, and which edge we are facing. As a consequence (a) any direct symmetry of the tiling that maps one black tile to another maps the set of black tiles to itself (this is not true of the opposite symmetries, the ones that "turn the tiling over") and (b) starting with any other tile coloured with a second colour, we can use the same rules to colour all the tiles that are to have that second colour, and proceeding in this way we can complete the colouring with seven colours. The colouring is said to be *chirally perfect*, because every direct symmetry of the tiling permutes the colours, but opposite symmetries jumble up the colours. This particular colouring has fascinating group-theoretical properties; see Mackenzie (1995) written in response to Robinson's problem, and Rigby (1991, p.60). A partial colouring of black tiles such as the one we started from in Figure 4 can be called *self-consistent*.

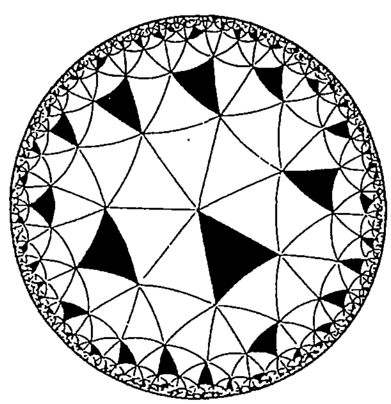

Figure 4

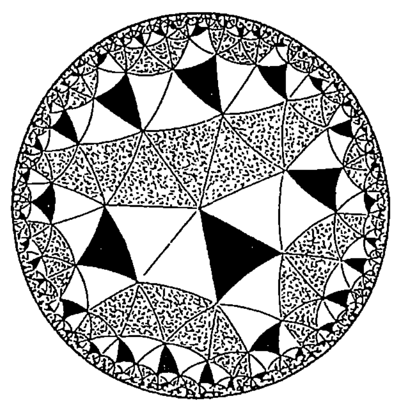

Figure 5

If we start with the mirror image of the partial colouring of black tiles in the previous paragraph, we obtain the mirror image of the chirally perfect colouring: every chirally perfect colouring occurs in left- and right-handed versions.

To solve the problem for other values of n, we need to analyse Figure 4 further, with a view to producing similar self-consistent partial colourings for other tilings. Figure 5 shows the partially coloured tiling of Figure 4 divided up into pieces; alternate pieces have been stippled in order to show the division more clearly. The stippled pieces are strips extending to infinity in both directions. The remaining pieces are all directly congruent; they contain both white and black tiles, and will be called *4-trees* because of their branching shape. The prefix 4 refers to the fact that four tiles of the 4-tree come together at each vertex of the 4-tree. Note also that no vertex of the complete tiling lies inside the 4-tree: every tile of the 4-tree has all three of its vertices on the boundary. Each 4-tree is *vertex-transitive:* there is a direct symmetry of the 4-tree mapping any vertex to any other vertex. The 4-trees are *partially coloured* (some tiles are black), and this partial colouring of each 4-tree is *precise* and *(chirally) consistent*: there is one black tile at each vertex, and every direct isometry of the tree maps the set of black tiles to itself. The stippled strips, with three tiles at each vertex, will be called 3-trees, even though no branching occurs.

Patchwork quilts are often made by first creating portions of the quilt, then sewing the portions together to make the entire quilt; hence the title of the article.

Our initial technique (to be modified later) for dealing with the general $\{3,n\}$ tiling will be to construct k-trees for all values of k ($k \neq 2$), and then to sew together partially coloured k-trees and plain $(n-k)$-trees alternately (for a particular value of k) to cover the entire plane; this will produce the black tiles in a precise chirally perfect colouring of $\{3,n\}$. We shall require these k-trees to be vertex-transitive under direct isometries, and their partial colourings to be precise and chirally consistent (as defined above for 4-trees); also all vertices of each tile in a k-tree must lie on the boundary of the tree. The k-trees must all be isomorphic, as must the $(n-k)$-trees. The next three figures show examples of this technique. Figure 6 shows partially coloured and plain 4-trees sewn together to make a self-consistent partial colouring of $\{3,8\}$. In Figure 7 the partially coloured and the plain 4-trees have opposite orientations; this produces a different partial colouring of $\{3,8\}$. In Figure 8, partially coloured 5-trees and plain 3-trees are sewn together to produce a third partial colouring of $\{3,8\}$. Note that precise consistent partial colourings of 3-trees do not exist.

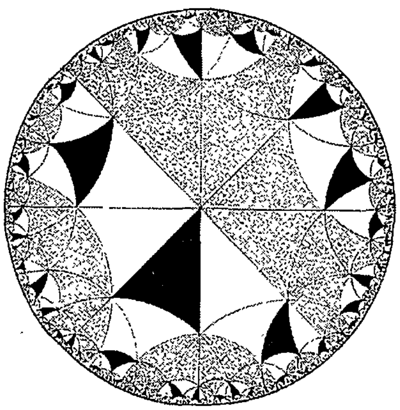

Figure 6

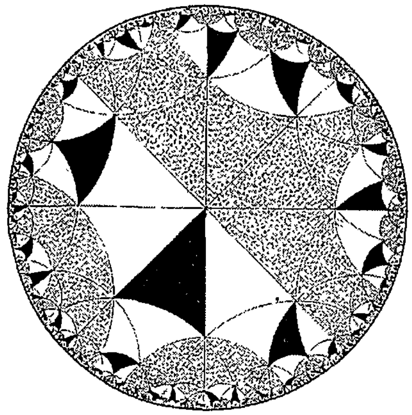

Figure 7

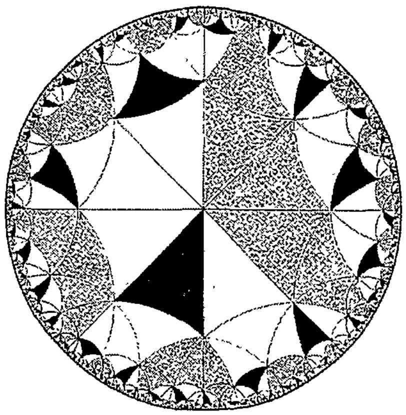

Figure 8

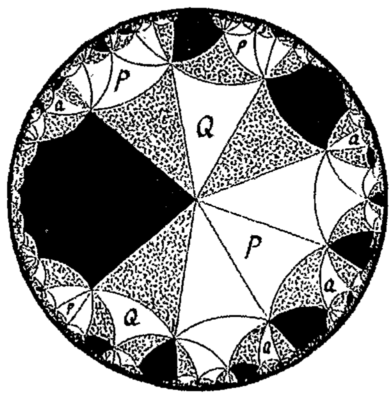

Figure 9

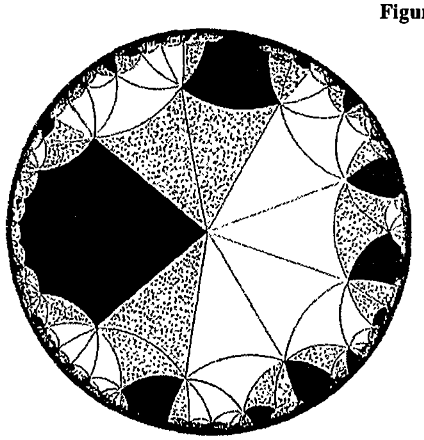

Figure 10

3. THE CONSTRUCTION OF STANDARD TREES

A single tile can be regarded as a 1-tree, and a single edge of the tiling as a 0-tree; there are no 2-trees.

Figure 9 shows a 7-tree; all tiles not belonging to the tree have been coloured black. (In this figure the underlying tiling is {3,9}, so the branches of the tree join up to form loops; but it is easy to construct the same type of tree for $n \geq 10$, so that we then have a genuine tree without loops.) All the tiles of the tree that have an edge along the boundary of the tree have been stippled. The remaining tiles (the white tiles) form 3-trees and 1-trees. Each stippled tile has its base along the boundary, its left side adjacent to a 3-tree (labelled P in the figure), and its right side adjacent to a 1-tree (labelled Q), so we shall denote this 7-tree by the symbol (31). Figure 10 shows a different type of 7-tree; if we break this down in the same way as before, we find that the left side of each stippled tile is adjacent to a 4-tree, and the right side is adjacent to a 0-tree (a single edge), so the symbol for this 7-tree is (40). But we can break down a 3-tree in the same way: all its tiles will be stippled, so the symbol for a 3-tree is (00). Similarly the 4-trees in Figure 10 have the symbol (10). Hence the symbols (31) and (40) for the 7-trees in Figures 9 and 10 can be further refined to (00)1 and (10)0. There is a third type of 7-tree with symbol (01)0, and the mirror images of these three 7-trees have symbols 1(00), 0(01) and 0(10), obtained by reversing the previous symbols.

In Rigby (1996) I asserted that any tree, of type X say, can be broken down in the way just described, using stippled tiles, into trees of types P and Q say, so that every stippled tile has its base on the boundary of X, its left side adjacent to a tree of type P, and its right side adjacent to a tree of type Q. This is incorrect, as I discovered by finding a counterexample whilst drawing various figures to illustrate Section 7 (see Section 8). But the converse *is* true: let P denote any type of p-tree and let Q denote any type of q-tree; then we can put together trees of types P and Q in the manner just described, with the help of stippled tiles, to form a $(p+q+3)$-tree denoted by (PQ). As long as $p+q+3 \leq n-3$, the tree (PQ) will be a genuine tree, with no loops, and the construction of (PQ) will not run into any snags such as the possible overlapping of different branches of (PQ).

We shall now define *standard trees*, using an inductive definition: 0-trees and 1-trees are standard trees, and if P and Q are standard trees, then (PQ) is a standard tree. The symbol for any standard tree is a sequence of 0s and 1s, correctly bracketed as in the previous examples. Any correctly bracketed sequence will represent a type of standard tree, which will always exist if n is large enough, and because of the uniqueness of the breakdown of a standard tree into the form (*PQ*) different sequences represent different types of tree: to enumerate all types of standard tree we need only enumerate all correctly bracketed sequences. In the interests of clarity and legibility the outermost pair of brackets will frequently be omitted: for instance 0(10) is preferred to (0(10)).

When *P* and *Q* are of the same type it is important that each tree used in the construction of (*PQ*) should be labelled either *P* or *Q*, to prevent any tree from being used as both *P* and *Q*. The importance of this remark will become clearer in Section 8.

We shall discuss non-standard trees in Section 8; non-standard *k*-trees exist only when $k \geq 9$. We have already enumerated all the 7-trees. As a second example, the standard 9-trees are (00)(00) which has mirror symmetry, together with ((00)0)0, (0(00))0, (11)1 and their reversals or mirror images.

A reduction formula for s_k, the number of standard *k*-trees, is

$$s_k = \sum_{r=0}^{k-3} s_r s_{k-3-r} \ (k \geq 3),\ s_0 = s_1 = 1,\ s_2 = 0.$$

When $k = 2m + 1$, the number of standard *k*-trees with mirror symmetry is s_{m-1}.

How do we obtain partially coloured standard trees? If the partially coloured standard tree *X* is broken down into $X = (PQ)$, it is easy to see that the stippled tiles are not coloured, and hence either the *P*-trees or the *Q*-trees must be partially coloured. Continuing the breaking down, we eventually find that at some stage 1-trees occurring in the breakdown must be coloured. We can denote coloured 1-trees by the symbol $\overline{1}$. Thus the partially coloured 4-trees in Figure 6 and 7 are denoted by $(0\overline{1})$, and the 5-trees in Figure 8 by $(1\overline{1})$. The other possible way of partially colouring the 5-tree (11) is $(\overline{1}1)$; this is just the mirror image of $(1\overline{1})$. In contrast, the 9-tree (11)1 has three essentially different partial colourings, namely $(\overline{1}1)1$, $(1\overline{1})1$ and $(11)\overline{1}$.

When partially coloured *p*-trees of type *P* and plain *q*-trees of type *Q* are sewn together alternately to cover the entire tiling and to form a self-consistent partial colouring of $\{3, n\}$, where $n = p + q$, we can denote this partial colouring by *P.Q;* then the colourings of Figures 6, 7 and 8 are $(0\overline{1}).(01)$, $(0\overline{1}).(10)$ and $(1\overline{1}).(00)$.

Note the distinction between "sewing together trees of types P and Q alternately" to cover the entire tiling, giving a patchwork or a partial colouring denoted by $P.Q$, and "putting together trees of types P and Q with the aid of stippled tiles" to form a tree denoted by (PQ). Note also that $P.Q = Q.P$, but (PQ) and (QP) are in general distinct.

4. THE ENUMERATION OF ALL STANDARD PERFECT PRECISE COLOURINGS

When $n \leq 6$, the only way of obtaining self-consistent partial colourings of $\{3,n\}$ is to use trees whose branches join up to form loops, as indicated below.

$n = 4$. There is only one self-consistent partial colouring. We can think of the six white tiles as forming a 3-tree (00), so the colouring is $(00).\bar{1}$. Alternatively, we can regard it as $(0\bar{1}).0$ or $(\bar{1}0).0$, where all eight tiles form a 4-tree $(0\bar{1})$ or $(\bar{1}0)$ whose bounding edges are joined up to each other.

$n = 5$. The four black tiles of the icosahedron shown in Figure 1 give a self-consistent partial colouring. The remaining tiles form a 4-tree (01), so the partial colouring is $(01).\bar{1}$; but it can also be regarded as $(\bar{1}0).1$ or $(1\bar{1}).0$.

$n = 6$. The partial colouring given by the black tiles in Figure 2 is $(11).\bar{1}$, $(\bar{1}1).1$ or $(1\bar{1}).1$.

These examples show that the same partial colouring can be arrived at in different ways by sewing trees together; but they also provide two clues as to how we can obtain a *unique* symbol for each *standard* partial colouring (a concept to be defined below), and therefore for each standard perfect precise colouring. First, we must make use of trees with loops; secondly, each self-consistent partial colouring is uniquely determined by the $(n-1)$-tree (with loops) formed by the uncoloured tiles. If this $(n-1)$-tree is standard, we shall say that the partial colouring and the associated precise colouring are standard. Thus there is a one-one correspondence between standard partial colourings of $\{3,n\}$ and standard $(n-1)$-trees, *as long as we can be sure that every possible standard $(n-1)$-tree (with loops) that can be constructed according to the method described above actually exists within the tiling* $\{3,n\}$, i.e. as long as the branches of every standard $(n-1)$-tree join up correctly to form loops and do not overlap in an irregular way. Let us see why no irregular overlapping occurs.

Let P, Q and R denote types of p-, q- and r-trees respectively, and consider the patchwork formed when $(p+q+3)$-trees of type (PQ) and trees of type R are sewn together alternately to form the partial colouring $(PQ).R$ that completely covers the tiling $\{3,n\}$, where $n = p+q+r+3$. This patchwork contains stippled tiles, each stippled tile surrounded by trees of types P, Q and R (R along its base, P on its left, Q on its right), and at each vertex trees of types P, Q and R alternate with stippled tiles. Figure 9 can be used to illustrate the general situation if the black regions are labelled R, but it is not a "genuine" example, because the regions labelled R are not trees: each is a 9-gon with a vertex of the tiling at its centre. (The Q-tiles in Figure 9 provide a partial colouring leading to a perfect colouring of $\{3,9\}$ in ten colours. But that is another story; it leads to a way of obtaining perfect colourings of $\{3,n\}$ in $n+1$ colours except when $n = 4$, 5 or 8.) Thus we see that the patchwork initially denoted by (PQ). R can also be denoted by $R.(PQ)$, $(QR).P$, $P.(QR)$, $(RP).Q$ or $Q.(RP)$. (This is reminiscent of the scalar triple product $(a\times b).c = c.(a\times b) = (b\times c).a = \ldots$ in vector algebra.).

We are concerned here with whether the $(p+q+3)$-tree (PQ) (with loops) exists when $n = p+q+4$, or equivalently whether the patchwork $(PQ).1$ exists when $n = p+q+4$. Now either $p \geq 3$ or $q \geq 3$ unless $n \leq 6$, and we know all about the cases $n \leq 6$. Hence without loss of generality $p \geq 3$, and $(PQ).1$ can be written as $P.(Q1)$ which certainly exists since P and $(Q1)$ are genuine types of tree without loops which can be sewn together alternately to cover the tiling $\{3,n\}$.

The number of standard perfect precise colourings of $\{3,n\}$ is therefore s_{n-1}, in the notation of Section 3. The number of colourings increases rapidly with n; for instance, when $n = 20$ there are 6 standard fully perfect precise colourings (see Section 5 below) and 1548 standard chirally perfect precise colourings occurring in 774 left- and right-handed pairs.

5. FULLY PERFECT PRECISE COLOURINGS

The perfect precise colourings of $\{3,4\}$ and $\{3,6\}$ are *fully perfect* rather than chirally perfect: *every* symmetry (direct or opposite) of the tiling permutes the colours. But, as we shall see, there is no fully perfect precise colouring of $\{3,8\}$. We shall show that a fully perfect precise colouring of $\{3,n\}$ exists if and only if n is even and $n \neq 8$.

A fully perfect precise colouring of $\{3,n\}$ occurs if and only if the underlying self-consistent partial colouring has mirror symmetry. Suppose that in Figure 11 we try to construct a fully perfect precise colouring when n is odd. Each of the n colours occurs once at the central vertex, and the rotation α through an angle $2\pi/n$ about the central vertex permutes the n colours cyclically. Since α induces this same permutation of colours on the outer ring of tiles $A, B, C, \ldots$, exactly one tile in this ring must be black. But reflection in the dotted line in the figure maps black to black; hence we obtain *two* black tiles in the ring $A, B, C, \ldots$ *unless the black tile occurs at* A. This is impossible, and hence no fully perfect precise colouring occurs when n is odd. The same conclusion can be reached, for standard colourings only, by considering the (0,1)-symbol for the associated $(n-1)$-tree: for a fully perfect colouring this tree must have mirror symmetry, and hence its (0,1)-symbol must be symmetric, which cannot occur when $(n-1)$ is even.

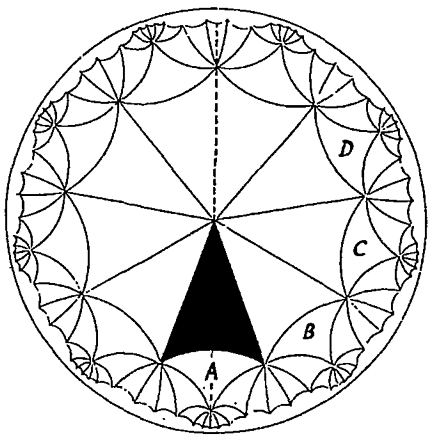

Figure 11

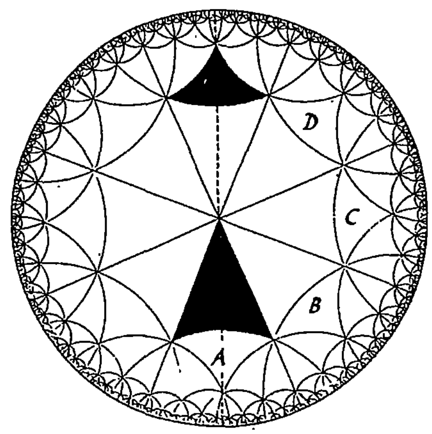

Figure 12

The corresponding situation when n is even is shown in Figure 12. We see by a similar argument that the unique black tile in the outer ring of tiles must occur in the position shown. When $n = 8$, this figure now gives us a rule for proceeding from one black tile to the neighbouring ones, and it can easily be checked that this rule leads to a perfect colouring in 10 colours rather than 8 (Rigby 1991, Fig.18; Rigby 1994, Fig.23). Hence no fully perfect precise colouring exists when $n = 8$.

Let P be any p-tree (not necessarily standard), and let P' be its mirror image. Then (PP') is a $(2p+3)$-tree with mirror symmetry, and the associated precise colouring for $n = 2p + 4$ will be fully perfect. Since p-trees exist whenever $p \geq 0$ and $p \neq 2$, fully perfect precise colourings exist whenever n is even and $n \neq 8$.

6. A SPECIAL TYPE OF COLOURING

The black tiles in Figure 13 determine the same colouring as Figure 6, and the stippled tiles are the tiles of a second colour in this same colouring. The two colours of tile are directly opposite each other at every vertex; they form branched chains of alternate colours. Does such a type of colouring exist for other even values of n? Certainly it does not when $n = 4$ or 6.

One of the patchwork symbols for the partial colouring of black tiles in Figure 13 is $(0\,\bar{1}).(01)$. We can extend our notation and use instead the symbol $(0A).(0B)$, where A denotes the black tiles and B denotes the stippled tiles; the figure is obtained by the sewing together alternately of 4-trees partially coloured with A-tiles, and directly congruent 4-trees partially coloured with B-tiles. Since partially coloured k-trees exist except when $k = 2$, 3 and 6, we can use the same method to obtain precise colourings of $\{3,n\}$ of this special type, when $n = 2k$, by piecing together alternately k-trees partially coloured with A-tiles and with B-tiles, except when $n = 4$, 6 and 12.

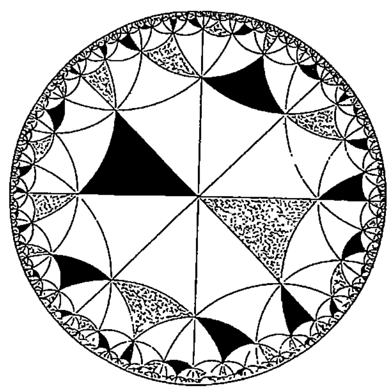

Figure 13

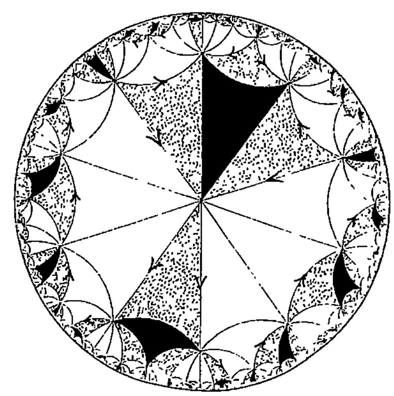

Figure 14

7. FULLY PERFECT COLOURINGS WHEN n IS ODD

The main subject of this article is precise colourings, but it is now natural to ask the question: *are there any fully perfect colourings of* $\{3,n\}$ *(not precise of course) when n is odd*? I had not come across any such before starting this investigation, apart from the obvious colourings of the icosahedron $\{3,5\}$ in ten colours, with opposite faces having the same colour. We can ignore colourings using one colour only, and colourings of polyhedra in which all faces have different colours. In the hyperbolic case, only a finite number of colours are to be used. We shall show that such colourings exist, with twenty-eight colours when $n = 7$, and $2n$ colours for all other odd values of n except $n = 3$.

We need to construct a self-consistent partial colouring with the extra requirement of *complete mirror symmetry*: every symmetry of the tiling that reflects any black tile to itself must reflect the complete set of black tiles to itself.

Let us first look again at fully perfect precise colourings when n is even, taking $n = 10$ as a convenient example. The partial colouring (00)(00). $\overline{1}$ with $n = 10$ (shown by the black tiles in Figure 14) leads to a fully perfect colouring. We shall refer to a tile with its three adjacent tiles as a *node*. Figure 14 can be regarded as black-and-stippled nodes and 3-trees sewn together. The fully perfect partial colouring when $n = 6$ can be regarded in the same manner as nodes and 1-trees sewn together. In general, p-trees and their mirror images can be sewn together with nodes to produce a partial colouring leading to a fully perfect colouring of $\{3, 2p + 4\}$, except when $p = 2$ (see the end of Section 5).

In Figure 14, the nodes are joined together to form *branched chains*. An attempt to produce something similar when $n = 7$ results in Figure 15: we now have branched chains consisting of nodes connected by links (single edges). The chains are separated by strips of white tiles of a type that we shall denote by $s(2,3)$. These strips are not trees. They have vertices of the tiling in their interiors; at each boundary vertex of the strip there are 2 or 3 tiles of the strip, and 2- and 3-vertices occur alternately. The strips are directly transitive on 2-vertices and on 3-vertices: there is a direct symmetry of the strip mapping any 2- or 3-vertex to any other. (Each $s(2,3)$ also has mirror symmetries, but this is not relevant here.)

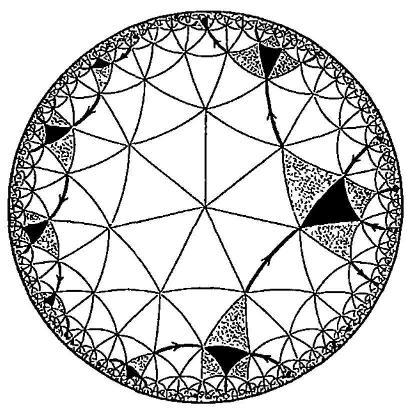

Figure 15

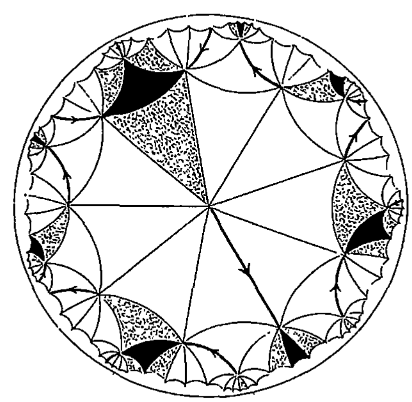

Figure 16

We can indicate an orientation for the boundary of each $s(2,3)$ by means of arrows. If, wherever we stand on the boundary facing in the direction of the arrow, the interior is on our left, the orientation is positive, if "left" is replaced by "right", the orientation is negative; we shall require that the two different disconnected parts of the boundary of an $s(2,3)$ have the same orientation. We can also indicate an orientation for the chains (the same for each chain) as shown in Figure 15. When we sew chains and $s(2,3)s$ together, we require that arrows on chains and adjacent strips must point in the same direction; this ensures that the partial colouring of Figure 15 (and of Figure 16 later) can be completed in a unique manner. The resulting partial colouring is self-consistent and has complete mirror symmetry. Note that adjacent strips have positive and negative orientation alternately. It is easy to check that one twenty-eighth of the tiles in Figure 15 are coloured black, so the complete colouring requires twenty-eight colours.

The corresponding partial colouring when $n = 9$ is shown in Figure 16. The strips are now replaced by what we may call (3,4)-trees, because no vertex of the tiling occurs in their interiors (They are called *alternating trees* in Yaz (1997, Chapter IV).) These strips will be denoted by $t(3,4)$ or its mirror image $t(3,4)'$. At each vertex of $t(3,4)$ there are either 3 or 4 tiles of $t(3,4)$, and $t(3,4)$ is directly transitive on 3-vertices and on 4-vertices; vertices with 3 or 4 tiles occur alternately along the boundary. A way of describing the construction of $t(3,4)$ will be given later. Each $t(3,4)$ in Figure 16 has positive orientation; each mirror-image $t(3,4)'$ has negative orientation. Since $t(3,4)$ does not have mirror symmetry, if we had used instead copies of $t(3,4)'$ with positive orientation and copies of $t(3,4)$ with negative orientation, whilst retaining the existing orientation for the chains, we should have obtained a partial colouring different from Figure 16.

All we need to do now is to construct, in an inductive manner, $t(k,k+1)$s for each k ($k \neq 2$), directly transitive on k-vertices and on $(k+1)$-vertices; $t(1,2)$ is just a pair of adjacent tiles, and $t(2,3)$ does not exist which is why we had to use $s(2,3)$ in Figure 15. These $t(k, k+1)$s are more complicated than the trees in Sections 3 and 4; we shall not attempt to construct all such trees, and only a brief description of the construction will be given.

Basically, we take an existing $t(k, k+1)$ of type P say, and an ordinary q-tree of type Q, and put together trees of types P and Q with the aid of stippled tiles as in Section 3, to produce a $(k+q+3, k+q+4)$-tree of type (PQ), or alternatively of type (QP). We must perform the construction carefully so that (PQ) has $(k+q+3)$-vertices and $(k+q+4)$-vertices alternately; if this is done it can be verified that (PQ) is transitive as required.

Figure 17 illustrates a $t(4,5)$ with $P = 0$ (a 0-tree) and $Q = t(1,2)$. Figure 18 shows a $t(5,6)$ with $P = t(1,2)$ and $Q = 1$.

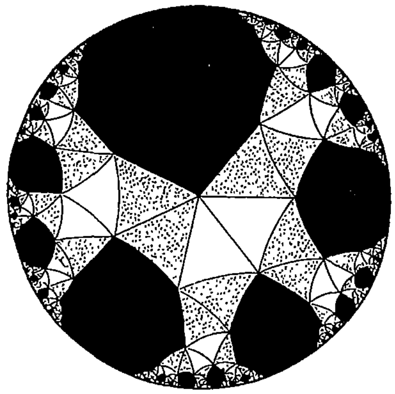

Figure 17

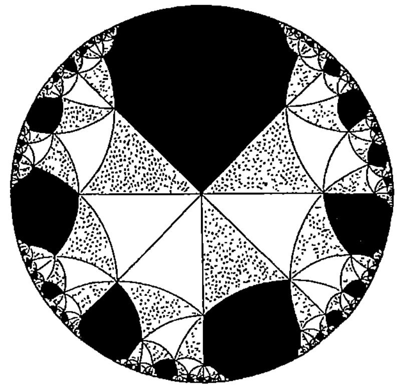

Figure 18

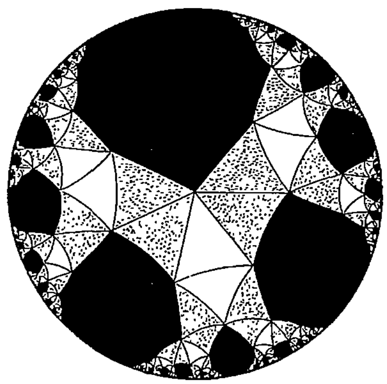

Figure 19

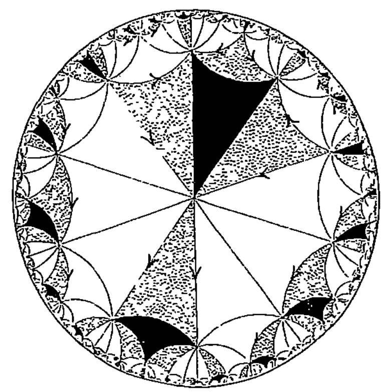

Figure 20

The tree in Figure 17 is made up of white $t(1,2)$s and stippled diamonds; every edge of each $t(1,2)$ is joined to a diamond, but only two edges of each diamond are joined to a $t(1,2)$. In Figure 19 the $t(1,2)$s and diamonds are joined differently to obtain a different $t(4,5)$. We can always join $t(k, k+1)$s and diamonds in two different ways to produce two $t(k+3, k+4)$s. The tree in Figure 18 is made up by joining white $t(1,2)$s and white-and-stippled nodes; if we try to "reverse" half of the nodes (by joining their other three edges to the $t(1,2)$s) as we did with the diamonds in Figure 18, we obtain a $t(4,7)$. This by itself is of no direct use in the present context, but if we join diamonds or nodes correctly to $t(4,7)$ we can get a new type of $t(8,9)$ or $t(9,10)$. The possibilities seem endless.

We can now join $t(k, k+1)$s with positive orientation, and their mirror images with negative orientation, to branched chains as in Figure 16, to create a partial colouring of $\{3, 2k+3\}$ with mirror symmetry, except when $k = 2$. Since a black tile occurs at half the vertices of this partial colouring, it leads to a fully perfect colouring in $2(2k+3)$ colours.

The colouring of the icosahedron $\{3,5\}$ in ten colours can be obtained in this way: the chain then consists of two nodes joined by six links, and the remaining tiles form six $t(1,2)$s.

Finally, when $n = 7$ (compare Figure 15) we can reverse the directions of the arrows on one of the two sides of the boundary of the strip $s(2,3)$, and the strip will still be transitive on 2-and 3-vertices, because there are glide reflections interchanging the two sides of the boundary. We can then sew together such newly oriented $s(1,2)s$ with branched chains to create a new self-consistent partial colouring with mirror symmetry, different from Figure 15. Is there a similar procedure sometimes with $t(k, k+1)s$?

8. NON-STANDARD PERFECT PRECISE COLOURINGS

In Figure 14 arrows have been inserted so that each 3-tree has either positive or negative orientation. But suppose we take 3-trees each of which is assigned one positive edge and one negative edge, and sew them to chains of nodes oriented as before, so that the directions of the arrows match. The result is shown in Figure 20; this is a partial colouring with mirror symmetry which yields a fully perfect precise colouring of $\{3,10\}$, but it is not derived from a standard 9-tree.

Before we analyse exactly how the 9-tree of non-black tiles in Figure 20 differs from a standard tree, let us make the analysis clearer by constructing another non-standard tree. First I constructed, in an *ad hoc* manner, a *pseudo-tree* in which some sections of the boundary have p tiles at each vertex, and some have q tiles at each vertex. The simplest example with $p \neq q$ seems to occur with $p = 5$; it is shown as a pseudo-tree with loops in Figure 21 (my original construction) and without loops in Figure 22. This pseudo-tree is directly transitive on 4-vetices and on 5-vertices, and any pseudo-tree of this type will be called a pseudo-tree of type ps(4,5). Let us now put together copies of $ps(4,5)$ with the aid of stippled tiles to form a tree of type X, in such a way that each stippled tile has its base along the boundary of X, its left side adjacent to a 4-boundary of a ps(4,5), and its right side adjacent to a 5-boundary of a ps(4,5); and every edge of every ps(4,5) must be adjacent to a stippled tile. The result can be seen to be a 12-tree X, satisfying the conditions for a tree given in Section 2.

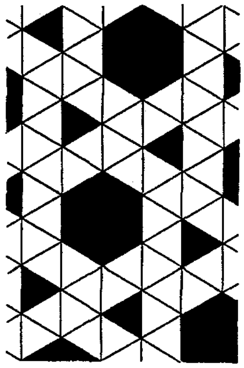

Figure 21

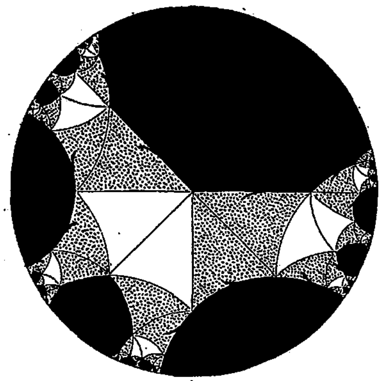

Figure 22

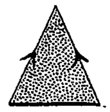

Figure 23

We can describe this process differently. Assign a positive orientation to each 4-boundary of each *ps*(4,5) and a negative orientation to each 5-boundary, as shown in Figure 22, and assign orientations to the left- and right-hand edges of each stippled tile as shown in Figure 23. Then put together *ps*(4,5)s and stippled tiles so that the orientations match. This is just what we did in Figure 20, where the 3-trees must now be regarded as (3,3) pseudo-trees. In contrast, when we put together trees of types *P* and *Q* with stippled tiles in the standard way to produce the tree (*PQ*), we can regard every section of the boundary of each *P* as having positive orientation, with negative orientation for the boundary of each *Q*.

Various questions remain. Do pseudo-trees provide the only method of constructing non-standard trees? How do we construct pseudo-trees? Is there an algorithm for constructing all pseudo-trees? Without doing further research, I can give a sketch of an answer to the second question only.

If we break down the *ps*(4,5) in the usual way with the aid of stippled tiles, as shown in Figure 22, we see that it is obtained by putting together 0-trees, *t*(1,2)s and stippled tiles in a suitable way (but compare Figure 17 where the same elements are put together differently). I conjecture that we can put together copies of $t(p,q)$, $t(r,s)$ and stippled tiles to produce $ps(p+r+3,\ q+s+3)$; but then we need a method of constructing all types of $t(p,q)$. See Yaz (1997, Chapter IV).

9. SEMIPERFECT PRECISE COLOURINGS

A colouring in which half the direct symmetries and half the opposite symmetries of the tiling permute the colours, but the remaining symmetries jumble the colours, is called a *semiperfect* colouring. When n is even, we can label the tiles of $\{3,n\}$ positive and negative alternately, like a generalised chessboard. If we can find a partial colouring with black tiles such that *there is a set of rules for proceeding from any positive black tile to the neighbouring black tiles, and the mirror images of those rules lead from any negative black tile to the neighbouring black tiles,* then this partial colouring leads to a semiperfect colouring. There is a semiperfect colouring of $\{3,8\}$ in seven colours; two colours of this colouring are shown in Figure 24, reproduced from Rigby (1990, Figure 18), but let us now look for semiperfect *precise* colourings. *All trees in this section are standard trees.*

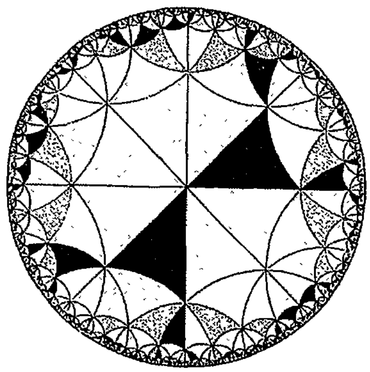

Figure 24

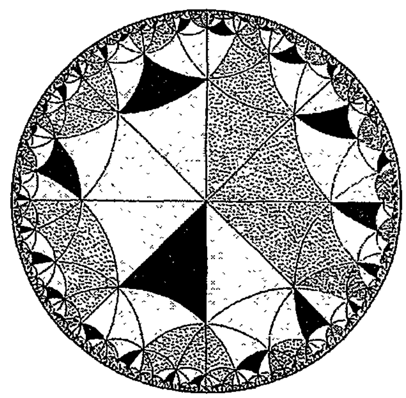

Figure 25

Consider the patchwork $(1\bar{1}).(00)$ (Figure 8), and label the tiles positive and negative alternately, as just described. The two bounding edges of a (00) can be described as positive and negative, since all the tiles adjacent to one boundary are positive, and all those adjacent to the other are negative. But the bounding edges of a $(1\bar{1})$ are either all positive or all negative, so a $(1\bar{1})$ can be called positive or negative. The black tiles of a positive $(1\bar{1})$ are all negative, and vice versa. Now, change each positive $(1\bar{1})$, in the patchwork $(1\bar{1}).(00)$, to $(\bar{1}1)$; the resulting partial colouring satisfies our requirements. This partial colouring is shown in Figure 25; note that it does not have mirror symmetry like the partial colouring of black tiles in Figure 24, but there are opposite symmetries (glide reflections with axis running down the middle of one of the 3-trees) interchanging positive and negative tiles and mapping the set of black tiles to itself.

When n is even, suppose we can find (a) a k-tree $t(k)$ where k is odd, such that there exists a glide reflection mapping $t(k)$ to itself and interchanging positive and negative edges, and (b) a partially coloured $(n-k)$-tree $T(n-k)$ all of whose edges have the same sign. Suppose that $T(n\text{-}k)$ has positive edges, and let $T(n-k)'$ be a mirror image of $\mathrm{T}(n-k)$ with negative edges. We can cover the whole plane by piecing together directly congruent copies of $t(k)$, $T(n-k)$ and $T(n-k)'$, always joining a positive edge to a negative edge, and negative to positive. This will produce the black tiles of a semiperfect precise colouring.

To show that $t(k)$ and $T(\mathrm{n}-k)$ exist with the required properties, we need to consider the symmetries of trees. Consider a particular example. Figure 26 is a redrawing of Figure 10, and shows a 7-tree (10)0. The white tiles form 4-trees, corresponding to the (10) component of the symbol (10)0. In the construction of these 4-trees (as described in Section 3), the unlabelled white tiles correspond to the symbol 1, the dotted edges correspond to the symbol 0, and the tiles labelled s are stippled. The heavily drawn edges in the figure correspond to the second 0 in the symbol (10)0. It should be clear from the method of construction of standard trees that, *in any standard tree, the centre of any tile corresponding to a* 1 *in the* (0,1)-*symbol is a centre of threefold rotational symmetry of the tree, and the mid-point of any edge corresponding to a* 0 *in the* (0,1)-*symbol is a centre of twofold rotational symmetry (or "centre of symmetry" for short) of the tree. These remarks still hold if the tree is partially coloured, indicated in the* (0,1)-*symbol by replacing one of the* 1*s by* $\bar{1}$. We note also that in a symmetrical standard tree (one whose (0,1)-symbol is symmetrical), the perpendicular bisector of every bounding edge is a line of symmetry.

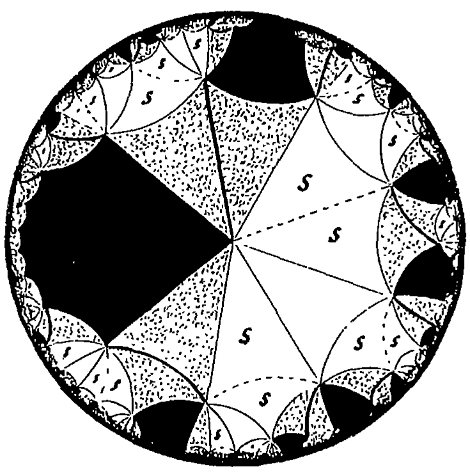

Figure 26

Since a half-turn about the centre of an edge interchanges positive and negative tiles, a partially coloured $T(n-k)$ as described above will have all its black tiles of the same sign only if there are no 0s in its tree-symbol; we can prove by induction that conversely, if there are no 0s in its tree-symbol, all the black tiles of $T(n-k)$ have the same sign, and so do all its edges. A $T(n-k)$ with these properties exists if and only if $n - k$ is of the form $4m + 1$. But $T(1)$ is of no use in the present context because it is symmetrical when partially coloured, and therefore when it is combined with a symmetrical $t(k)$ (as described in the next paragraph) it will yield a fully perfect colouring.

Let $t(k)$, where k is odd, be a symmetrical tree with a 0 in its symbol. Let α denote the reflection in m, the perpendicular bisector of a bounding edge of the tree, one of its lines of symmetry, and let β denote the half-turn about the point A, one of its centres of symmetry. Since A is the midpoint of an internal edge, A does not lie on m; also α preserves signs and β interchanges signs. Hence $\alpha\beta$ is a glide reflection that maps $t(k)$ to itself and interchanges positive and negative edges. A $t(k)$ with these properties exists except when $k = 1, 5, 7$ or 13.

We conclude that semiperfect colourings can be constructed by this method except when $n = 4, 6$ or 10.

10. IMPERFECT PRECISE COLOURINGS

In our construction of chirally perfect precise colourings of $\{3,n\}$, we have used $(n\text{-}1)$-trees to determine the tiles of a single colour; it should be noted that the trees associated with tiles of different colours are quite distinct from each other, although of the same type.

We shall now use trees in a somewhat different way. Each 4-tree in Figure 4 can be precisely coloured in four colours as in Figure 27. "Precise" here means that one tile of each colour occurs at each vertex of the tree. A different but equally logical precise colouring is shown in Figure 28, and "random" precise colourings are also possible. The 3-trees in Figure 4 can be precisely coloured in three other colours. Since the three or four colours in each tree can be permuted, we can obtain in this way an infinity of imperfect precise colourings of the entire tiling. Moreover, when joining the trees together, we can at any time use a tree of type (10) instead of (01), producing further irregularity or imperfection.

I conjecture that it is always possible to create a precise colouring of any k-tree in k colours, except possibly a k-tree with loops embedded in the tiling $\{3, k + 2\}$, and that this colouring can be carried out in a reasonably symmetrical and perhaps a perfect way (so that every direct symmetry of the tree permutes the colours).

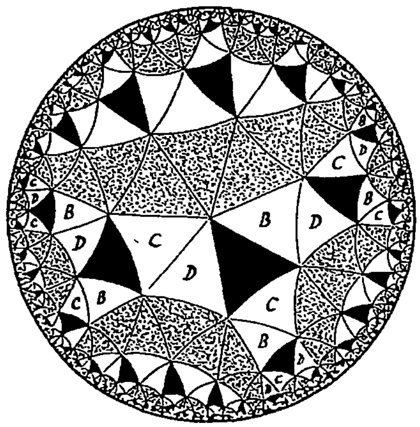

Figure 27

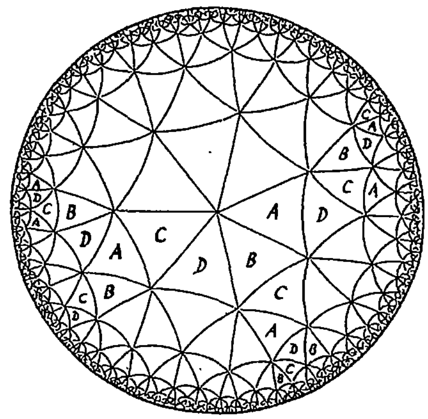

Figure 28

For larger values of n, more possibilities occur. For instance, there are three distinct types of 7-tree, each with its mirror image; when $n = 11$ we can join different types of 7-tree (precisely coloured in seven colours) and 4-trees (precisely coloured in four further colours) to produce an infinity of precise colourings of $\{3,11\}$.

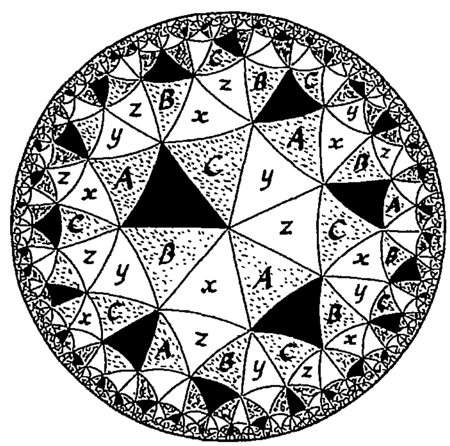

Figure 29

I found the pleasing pattern of Figure 29 when I was just experimenting with methods of making patchwork; flowers with a black centre and three petals alternate with leaves formed by two tiles. (In the notation of Section 8, the non-black tiles form a pseudo-tree with loops: *ps*(6,6).) This can be used as the basis of a subtle precise colouring: the flowers are of two types, according to the cyclic order of the three colours in the petals, and the two types of flower are surrounded in quite different ways by the x, y and z colours of the leaves.

11. EQUIVALENT COLOURINGS

We shall introduce the idea of *equivalent* precise colourings of $\{3,n\}$ by considering just one example – the simplest non-trivial example. In Figure 30, where $n = 9$, the tiles labelled A form a self-consistent partial colouring (leading to a chirally perfect precise colouring); the remaining tiles form an 8-tree whose symbol may be written $(CB)0$ rather than $(11)0$, using the notation introduced in Section 6. The complete patchwork of Figure 30 can be denoted by any of the symbols $(CB)0.A = CB.0A = C.B(0A) = B(0A).C = B.(0A)C$. This notation shows that the A-tiles lead to the precise colouring $(11)0$, the B-tiles lead to $(01)1$, and the C–tiles to $1(01)$. These are three distinct non-isomorphic precise colourings. It is important to emphasize that the patchwork is self-consistent: any direct symmetry of the tiling that maps one A-tile to another A-tile maps all A-tiles to A-tiles, all B-tiles to B-tiles, and all C-tiles to C-tiles.

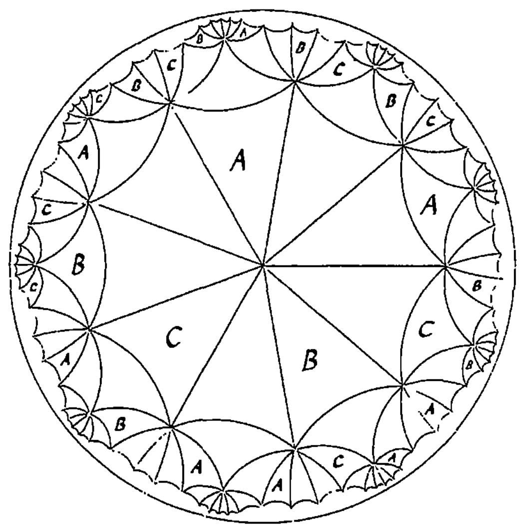

Figure 30

The direct symmetries of the tiling $\{3,n\}$ induce a group of permutations of the colours in any perfect colouring: the *(direct) colour permutation group* of the colouring. We shall show that *any direct symmetry of the tiling induces the same colour permutation in the three colourings described above,* so that the three non-isomorphic colourings have ismorphic colour permutation groups; for this reason we call them *equivalent* colourings. But note that the statement in italics makes sense only if the three tilings are suitably coloured using the same nine colours (as described in the next paragraph), and only if the three colourings are suitably overlaid so that they become three colourings of *the same tiling* (rather than of isomorphic tilings).

Let us then consider in more detail how we can produce the three colourings. Consider three copies of the tiling $\{3,9\}$ overlaid on each other; label the copies A, B and C. Think of Figure 30 as a template that can be placed over the copies of the tiling in any position (as long as edges are placed over edges). (This is of course a hyperbolic template: Figure 30 regarded as a Euclidean figure can only be placed in nine ways over the Euclidean representation of the tiling.) Now perform the following sequence of operations. (1) Place the template over the copies of the tiling. Colour the A-tiles black in Copy A, colour the B-tiles black in Copy B, and colour the C-tiles black in Copy C. (2) Place the template in a new position over the copies of the tiling, in such a way that a non-black tile in Copy A is now marked A. Because of the self-consistency of the template, no black tile in Copy A is now marked A, no black tile in Copy B is marked B, and no black tile in Copy C is marked C. Colour the A-tiles red in Copy A, colour the B-tiles red in Copy B, and colour the C-tiles red in Copy C. (3) Place the template in a new position, in such a way that a non-black and non-red tile in Copy A is now marked A, and proceed as before: use a third colour to colour the A-tiles, B-tiles and C-tiles in Copies A, B and C respectively. Continuing in this way, we eventually arrive at the three perfect precise colourings in nine colours.

Nine positions of the template are used to produce the colourings (we can refer to "the black template", "the red template", etc.), and *any direct symmetry of the tiling simply permutes the nine positions of the template, and hence produces the same permutation of the nine colours in all three colourings;* but this statement needs more explanation and discussion to make it convincing, so we shall use a different approach that leads to the same conclusion and that exhibits an extra property of the colourings.

Denote the angle $2\pi / 9$ by θ. Rotation through 4θ (clockwise) about any vertex of the template maps the A-tile at that vertex to the B-tile. Let P be any vertex of the three copies of the tiling. Rotation through 4θ about P maps the black tile at P in Copy A to the black tile at P in Copy B, and the same is true for any other colour. Hence, *rotation* *through 4θ about P maps the complete ring of colours around P in Copy A to the ring of colours around P in Copy B.* Similarly *rotation through 2θ about P maps the ring of colours around P in Copy B to the ring of colours around P in Copy C.* We emphasize that this is true for every vertex P.

Let α denote the rotation through θ about P; then α induces the same permutation of colours in all three colourings, namely the 9-cycle determined by the clockwise order of the colours around P. Let β denote the rotation through θ about an adjacent vertex Q; then β induces the same permutation of colours in all three colourings. But the direct symmetry group of the tiling is generated by α and β (Yaz, 1997, 2.1); hence any direct symmetry of the tiling induces the same colour permutation in all three colourings, and therefore the direct symmetry group of the tiling induces the same colour permutation group in all three colourings.

The "extra property" exhibited by this approach is that the colours around P in Copy B can be obtained from the colours around P in Copy A by means of a rotation *whose angle is independent of P* (namely 4θ in this instance).

The reader can easily investigate further examples of equivalent colourings. When $n = 5$, the two perfect precise colourings of the icosahedron are equivalent as well as being reflections of each other. When $n = 8$ and 10, some of the colourings are equivalent to their reflections. When $n = 11$, various non-isomorphic colourings occur in equivalent pairs. The next interesting cases, with triples of equivalent non-isomorphic colourings, occur when $n = 12$; this is because the (0,1)-symbols of all 11-trees contain two 1s.

12. FURTHER ASPECTS OF THE SUBJECT

Robinson's quite different method for finding precise colourings when n is even is described in Rigby (1998). A fuller discussion of non-standard trees, an algorithm for obtaining generators for the colour permutation group associated with the colouring determined by a specified tree, and an investigation of perfect colourings of $\{3.n\}$ using $n + 1$ colours, can be found in Yaz (1997). I am currently investigating the connection between perfect precise face colourings and perfect precise *edge colourings.*

I am grateful to Douglas Dunham for providing me with computer drawings of $\{3,10\}$. All the remaining drawing and painting was done by hand.

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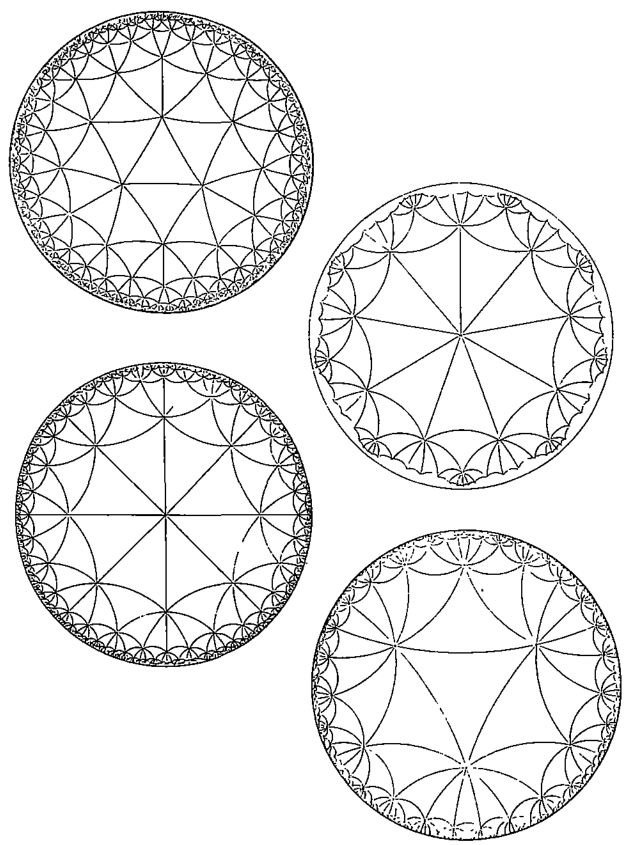

Figure 31