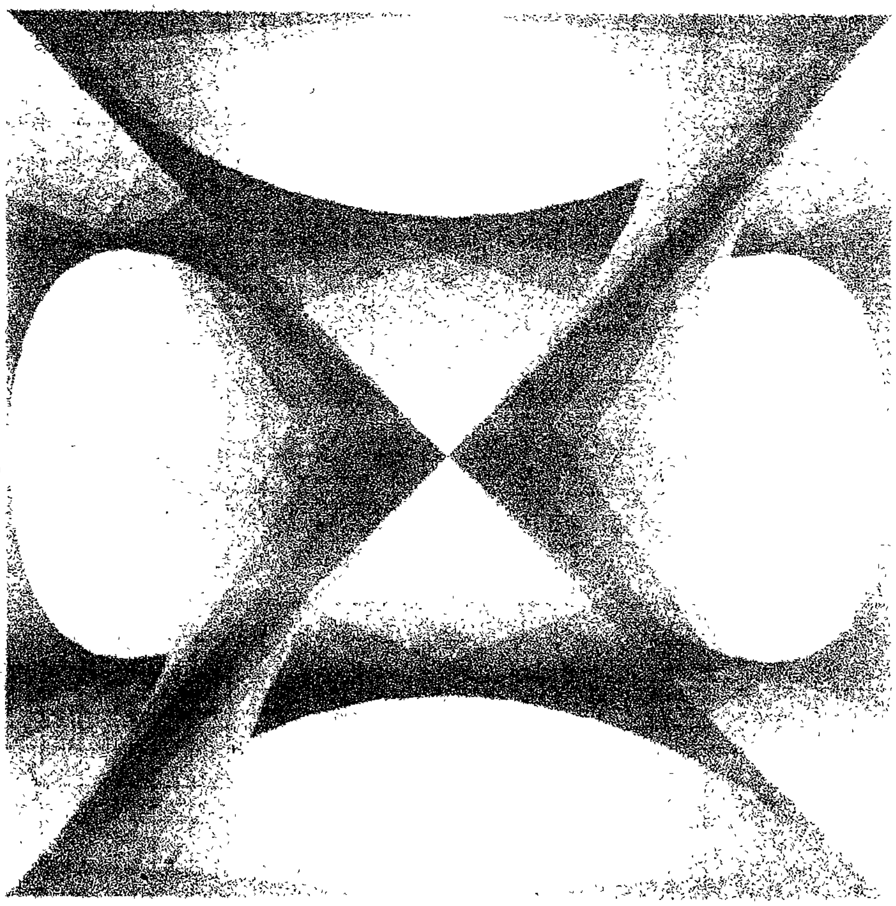
Symmetry: Culture and Science

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Volume 8, Number 3–4, 1997



Symmetry: Culture and Science
Vol. 8, Nos. 3-4, 409-429, 1997

SYMMETRY AND RECREATION

SYMMETRY IN PRACTICE – RECREATIONAL CONSTRUCTIONS

Peter Hilton[(1)] and Jean Pedersen[(2)]

Address: (1) Department of Mathematical Sciences, State University of New York, Binghamton
(2) Department of Mathematics, Santa Clara University, California

1. INTRODUCTION (RECREATIONAL CONSTRUCTIONS)

The purpose of this article is to show how fascinating geometrical ideas can be by introducing the reader to some polyhedral puzzles. Our intent is to present the material in much the order in which we ourselves discovered it. We would like you to experience some of the joy of discovery we have had, which means, of course, that you risk experiencing some frustration along the way before you finally achieve success in assembling your puzzles. However, don't despair since we will give you many hints along the way and, eventually, more complete instructions for the details involved in assembling the puzzles.

In Section 2 we will describe how to fold the tape required to make your puzzles. In Section 3 we will explain how to make the puzzle pieces for 9 models and challenge you to construct some of them without any further information. We also include in Section 3 one intriguing example of how the braided models may be used to visualise the answer to a combinatorial question in geometry. In Section 4 we give either more hints or complete instructions on how to actually assemble the remaining models. In Section 5 we suggest some variations on certain models you will already have built that, for one reason or another, seem to lack the symmetry you would expect them to have. In this last section we will challenge you to build the more perfect tetrahedron, octahedron and icosahedron without any further information other than the description, two illustrations and a picture.

Although this may be viewed as purely recreational mathematics, knowledge of the symmetry group for each model may be helpful in solving the puzzle. We ourselves are very much in favor of exploiting the mathematics connected with these fascinating models[1] and we are delighted that the editor of this journal has suggested that our article concerning some of the mathematics connected with these models should appear as a companion piece to the present article, in this same issue. That related article, entitled "Symmetry in Theory – Mathematics and Aesthetics", abbreviated in this article to [Math], contains a fairly comprehensive list of references which you may consult if you wish to build other models. Of particular relevance is [HP5] of [Math]. We also include at the end of our companion article a brief history of how some of the folding procedures evolved, and how our friendship with the great mathematician and teacher, George Pólya, and with each other, resulted in mathematical collaborations concerning these models (and other topics).

We will concentrate on the directions for constructing polyhedral puzzles (including the Platonic solids) which have regular triangles, squares, or pentagons for faces. In order to be able to carry out the instructions and build these puzzles you will need

(1) some ordinary *unreinforced* gummed tape about 2 inches wide (a minimum of 30 to 40 running feet);

(2) some large sheets of paper such as gift-wrapping paper, or brightly colored lightweight construction paper (six different designs or colors is the most that will be required for any of these models);

(3) a pair of scissors;

(4) some paper clips (maximum 30);

(5) some bobby pins (maximum 6);

(6) a sponge, a bowl and some water;

(7) some rags and a flat place to work.

[1] On one occasion we were horrified to hear a mathematics teacher answer the question, "What did you do with the models you had the students build?" with "Oh, we hung them up!"

2. HOW TO FOLD THE TAPE

We describe, in this section, two iterative folding procedures that may be used on your gummed tape to fold strips of equilateral triangles and strips which can be used to construct regular pentagons. In both of these cases the procedure is a convergent one so that the angles you produce on the tape become more and more regular as you continue to fold. You will also need to fold a strip of consecutive squares (but this is an exact construction that we feel confident you can do on your own).

To fold the equilateral triangles simply follow the instructions in the numbered frames of Figure 1.

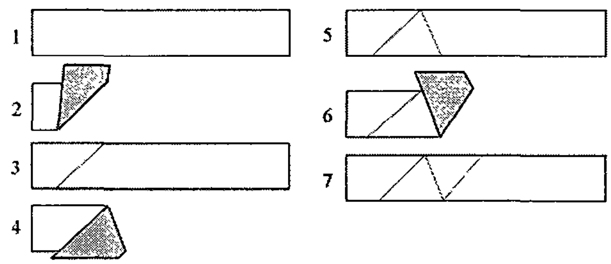


Figure 1: 1) Begin with a long strip of gummed tape; 2) Fold UP - any way will do; 3) Unfold; 4) Fold DOWN - now you must do it exactly as shown; 5) Unfold; 6) Fold UP, exactly as shown; 7) Unfold

Now go to frame 4, and keep repeating the folding in frames 4 through 7, to make a string of triangles as long as you need. Notice two things. First, the folding procedure, after your initial fold, goes DOWN, UP, DOWN, UP,... We will abbreviate this folding procedure by D^1U^1 and call the tape produced the D^1U^1-tape.[2] Second, you can readily see that, as we claimed, the triangles become more and more regular as you fold. Thus, if you wish to use this tape to construct models requiring equilateral triangles, all that you need to do is compare successive triangles, beginning with the first one formed, until it is not possible to detect any difference between them - and then throw away the defective ones and continue to fold, in the prescribed manner, to obtain the equilateral triangles you need for the constructions in Section 3. (We show in [Math] how to prove that all angles on this tape do, in fact, approach $\pi/3$.)

[2] Of course, one could adopt the 'systematic' folding procedure in which 'DOWN' and 'UP' are interchanged. The procedure would then be written U^1D^1.

To fold tape from which you can construct regular pentagons simply follow the instructions in the numbered frames of Figure 2.

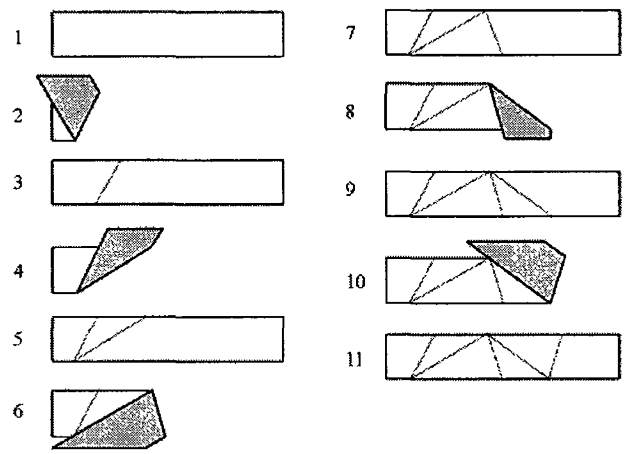


Figure 2: 1) Begin with a long strip of gummed tape; 2) Fold UP - any way will do; 3) Unfold; 4) Fold DOWN - now you must do it exactly as shown; 5) Unfold; 6) Fold UP, exactly as shown; 7) Unfold; 8) Fold DOWN, exactly as shown; 9) Unfold; 10) Fold UP, exactly as shown; 11) Unfold.

Now go to frame 4, and keep repeating the folding in frames 4 through 11, to make a string of tape from which you will be able to construct regular pentagons. First, notice that the folding process, after the first two initial folds, goes DOWN, DOWN, UP, UP, DOWN, DOWN, UP, UP,... We will abbreviate this folding procedure by D^2U^2 and call the tape produced the D^2U^2-tape. Second, notice that this tape has two kinds of crease lines, which we will refer to as *short* and *long* crease lines. Third, it is evident that the configuration formed by these crease lines is becoming more and more regular, reproducing the same angles at each edge of the tape. (We show in [Math] how to prove that the smallest angles on this tape do, in fact, approach $\pi/5$.)

This is the tape that you will use to make regular pentagons and models with regular pentagonal faces. To see how this works throw away the first few triangles you have folded (10 will be very safe) and continue to fold, in the prescribed manner, to obtain the tape you need to produce the constructions in Section 3. Just to practise now, cut off a piece of tape and make the pentagon shown in Figure 3. Notice that when you constructed this pentagon you cut the D^2U^2-tape along a short crease line, and folded on short crease lines.

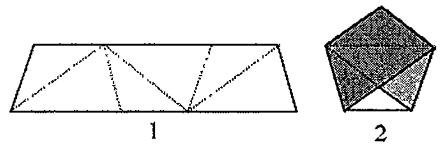


Figure 3: Make a section of tape that looks like this (1), into a pentagon that looks like this (2) (Shading indicates the other side of the tape).

What about the long crease lines? Try cutting along a long crease line and folding on successive long crease lines to construct the pentagon shown in Figure 4 (of course you will also have finally to cut along another long crease line to complete the model as it is shown).

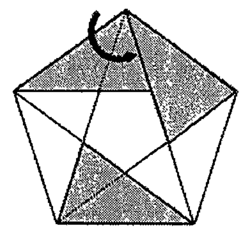


Figure 4: The end of the tape must be tucked in here (arrow).

3. HOW TO MAKE THE PUZZLE PIECES OR STRIPS

This section describes how to make the puzzle pieces for the following 9 models which naturally divide themselves into three types.

A pentagonal dipyramid with a single strip of 31 equilateral triangles.
A triangular dipyramid with a single strip of 19 equilateral triangles.
The Platonic[3] Puzzles:

- A tetrahedron with 2 strips
- A hexahedron (cube) with 3 strips
- An octahedron with 4 strips
- An icosahedron with 5 strips
- A dodecahedron with 6 strips

(Do you notice any interesting pattern here? Do you see why we have written the dodecahedron last, although it's usually written before the icosahedron?)

- A diagonal cube with 4 strips.
- A golden dodecahedron with 6 strips.

There are some general comments that apply to each of these 9 models. In each case you should first make the required pattern pieces (or strips), by folding the gummed tape. Then glue the strips onto colored paper (when more than one strip is involved glue each strip onto paper of a different color).

---

[3] As any Greek scholar will tell you, the names of the Platonic solids are designed to show that they have 4, 6, 8, 20, 12 faces, respectively.

When gluing the strips onto the colored paper, make certain the paper you plan to use for each strip is large enough. Then place a sponge (or washcloth) in a bowl and add water to the bowl so that the top of the sponge is very moist (squishy). Next moisten one end of the strip by pressing it onto the sponge and then, holding that end, pull the rest of the strip across the sponge (This part of the process is often messy!)[4] Make certain the entire strip gets moistened and then place it on the colored paper. Use a hand towel (or rag) to wipe up the excess moisture and to press the tape into contact with the colored paper.

Put some books on top of the tape so that it will dry flat. When the tape is dry, cut out the pattern piece, trimming off a small amount of the gummed tape (about 1/16 to 1/8 of an inch) from the edge as you do this. This trimming procedure serves to make the model look neater and, more importantly, it allows for the increased thickness produced by gluing the strip to another piece of paper. Refold the piece (only along the lines you need for your particular model) so that the raised (mountain) folds are on the colored side of the paper. You will then have your puzzle pieces and can proceed to construct your model.

We begin the actual puzzles with the two dipyramids to let you get a feel for the way your materials behave. We will be fairly explicit here, to get you started, but we'll be less detailed when we come to the Platonic Puzzles.

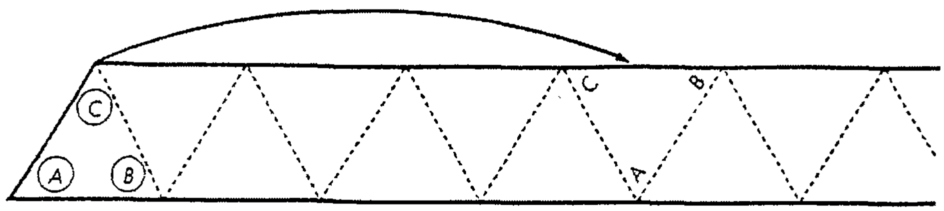


Figure 5: (a) Left-hand end of pattern piece

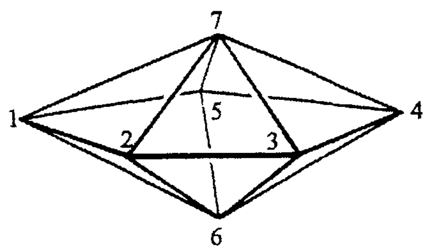
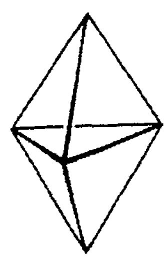


Figure 5: (b) Pentagonal dipyramid



(c) Triangular dipyramid

---

[4] Perhaps we should have included some very old clothes as optional (or even essential) materials.

Figure 5(a) shows the left-hand end of the 31-triangle strip used to construct the pentagonal dipyramid. You should mark the first and eighth triangles *exactly* as shown (note the orientation of each of the letters within their respective triangles). To assemble the model place the first triangle *over* the eighth triangle so that the circled letters *A*, *B*, *C* are over the uncircled letters *A*, *B*, *C*, respectively. Holding those two triangles together in that position, you will notice that you have the frame of a double pyramid for which there will be five triangles above and five triangles below the horizontal plane of symmetry (the plane containing the vertices 1, 2, 3, 4, 5 in Figure 5(b)). Now hold this configuration up and turn it so that the long strip of triangles falls around this frame. If the creases are folded well, the remaining triangles will fall into place. When you get to the last triangle there will be a crossing of a strip that the last triangle can tuck into, and your model will be complete and stable.

If you have trouble because the strip doesn't fall into place there are two frequent explanations. The first (and most likely) reason is that you have not folded the crease lines firmly enough. In that case all you need to do is crease them again with more gusto! The second possible reason is that the strip seems too short to reach around the model and tuck in. This may be remedied by trimming a tiny amount from each edge of the strip.

An analogous construction may be made for the triangular dipyramid shown in Figure 5(c). This model can be made from a strip of 19 equilateral triangles. Knowing what the finished model should look like and that you should begin by forming the *top* three faces with one end of the strip should be sufficient hints.

You may discover that you can construct each of these dipyramids with fewer triangles, but we chose the construction that produces the most *balanced* model. You will note that both of these constructions place *precisely* three thicknesses of paper on each face, except where the last triangle tucks in (producing four thicknesses). In both cases, you could remedy this small defect by cutting off half of the first and last triangle on the strip.

Now let us turn to the Platonic Puzzles.

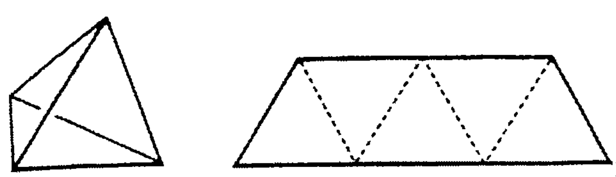
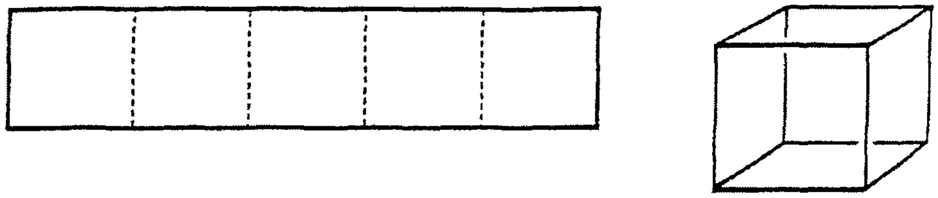
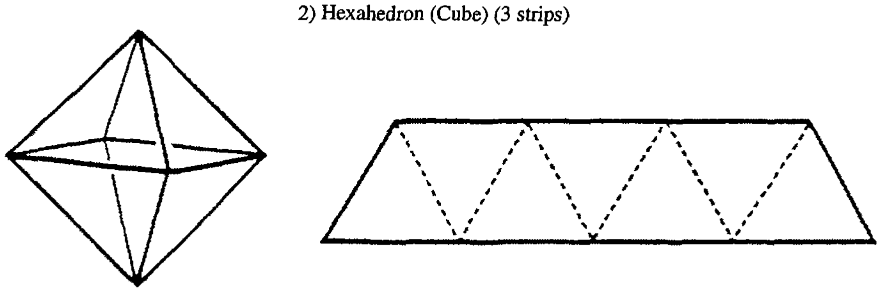


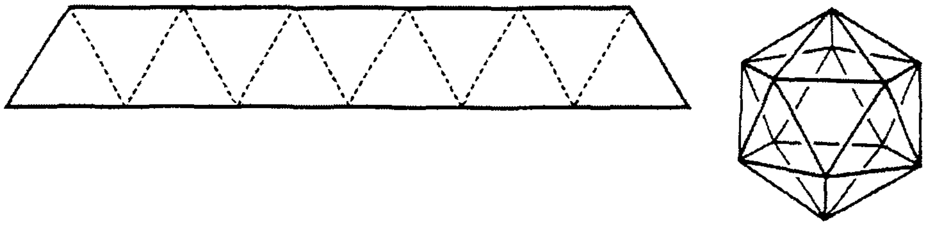
Figure 6: 1) Tetrahedron (2 strips)



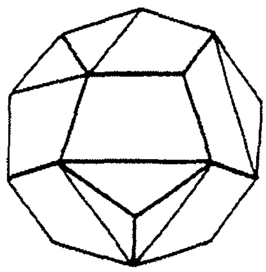
2) Hexahedron (Cube) (3 strips)



3) Octahedron (4 strips)



4) Icosahedron (5 strips)



5) Dodecahedron (6 strips, 3 of each kind).

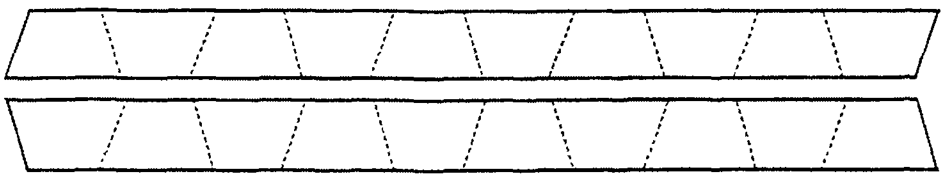


Figure 6 shows a typical puzzle piece (or strip) next to each solid, and tells you how many are needed. In each case the puzzle is this: take the required strips and braid them together to from the required solid in such a manner that

(a) the same area is visible on each strip, and
(b) all ends are tucked in.

The *tetrahedron, octahedron*, and *icosahedron* all involve strips obtained from the D^1U^1-folding procedure. All you need to do is prepare the pattern pieces as we described above and try to assemble the models. You may note that, on all of these models, if you take into account the coloring of the surface, they will have lost some of the symmetry you would expect to find on Platonic Solids; that is, not all edges will look the same. For some edges the two adjacent faces will have the same color but, for other edges, the two adjacent faces will have different colors. (We will propose another type of construction that corrects this defect in Section 5). If you manage to get all three of these together without any hints you are really an *expert*! If you have trouble getting your models together check the hints given in Section 4.

The *hexahedron (cube)* pattern pieces may be made by making exact folds on the tape. All that you need to remember is that if you fold the tape directly back on itself you will produce an angle of precisely $\pi/2$, and if you bisect that angle you will know exactly where to fold the tape back on itself to produce a square. Once you have one square on the tape you may then simply fold the tape back and forth, accordion style, on top of this square to produce the required number of squares. There are actually two ways to braid these three pieces together to satisfy the conditions for the puzzle. One of these ways produces a cube with opposite faces the same color and the other way produces a cube with certain pairs of adjacent faces the same color. From the point of view of symmetry the first is more symmetric because, on that model, all edges abut two faces of different colors. If you have trouble assembling this model check the hints given in Section 4.

The *dodecahedron* involves strips obtained from the D^2U^2-folding procedure. But notice on the final pattern piece you should only fold the pattern piece firmly along the *short* crease lines (ignoring the long lines) after you have cut out each piece. We should tell you that on this model *four* sections of each strip will overlap (for stability). It may also be helpful to let you know that the strips go together in pairs and the construction is then similar to that of the more symmetric cube you have constructed above – and, if coloring is taken into account, the completed model loses a lot of the symmetry you expect to see on a dodecahedron. You may now have enough hints, but if you have difficulty consult Section 4.

The *diagonal cube* involves four strips each containing 7 right isosceles triangles as shown in Figure 7(a). The strips for these pieces may be folded by the exact procedure similar to that described above for the cube in the Platonic Puzzles. Just remember that this time you want to emphasize those crease lines that make an angle of $\pi/4$ with the edges of the tape. To assemble the cube you begin by laying out the four pieces as shown in Figure 7(b), with the colored side of the paper not showing. You may wish to put a small piece of tape in the exact center to hold the strips in position. Now, thinking of the dotted square surrounding the center as the base of your cube, you begin to *braid* the strips to make the vertical faces, remembering that each strip should go successively over and under the strips it meets as it goes around the model. When you get to the top face you will find that all the ends will tuck in to produce a very beautiful and highly symmetric cube; indeed, *none* of the symmetry of the cube has been lost. You will notice that every face has a different arrangement of four colors and that every vertex is surrounded by a different arrangement of three colors.

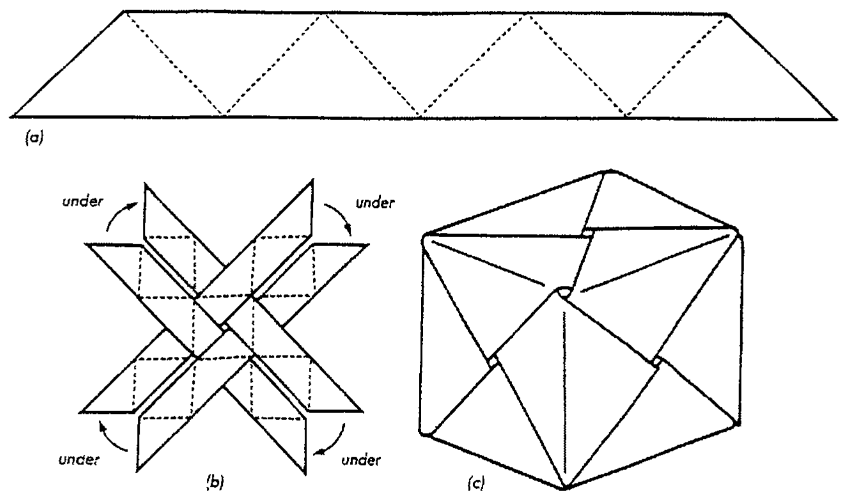


Figure 7

The *golden dodecahedron* involves strips obtained from the D^2U^2-folding procedure. But notice on the pattern pieces you should only fold the pattern piece firmly along the *long* crease lines (ignoring the short lines) after you have cut out each piece. To complete the construction of this model, begin by taking five of the strips and arranging them, with the colors showing, as shown in Figure 8(b), securing them with paper clips at the points marked with arrows. View the center of the configuration as the North Pole. Lift this arrangement and slide the even-numbered ends clockwise over the odd numbered ends to form the five edges coming south from the arctic pentagon. Secure the strips with paper clips at the points indicated by crosses. Now weave in the sixth (equatorial) strip, shown shaded in Figure 8(c), and continue braiding and clipping, where necessary, until the ends of the first five strips are tucked in securely around the South Pole. Above all, *keep calm*, you can even take a break – the model will wait for you! Just make certain that every strip goes alternately over and under each strip it meets all the way around the model. When the model is complete (with the last ends tucked in) you may remove all the paper clips and the model will remain stable. We notice that this constructed dodecahedron is aesthetically very satisfying – more so than the dodecahedron previously described. This is due to its amazing symmetry – *none* of the possible symmetry has been lost.

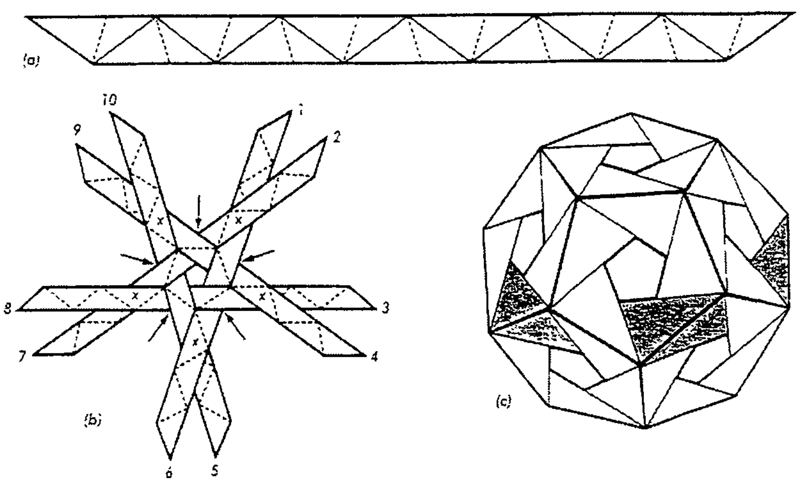


Figure 8

Before giving you the hints for the remaining models we cannot resist showing you a lovely use for the cube with three strips, the diagonal cube and the golden dodecahedron. An interesting question[5] that has been asked by geometers (see [KW]) is "How many disjoint pieces – both finite and infinite (or unbounded) – are formed by the extended face planes[6] for each Platonic solid?" As it turns out we can use our braided models to answer some of these questions in the case of the unbounded regions.

Let us use the ordinary cube to show how the braided models are useful. Notice that the edges of the three strips used to create the braided model lie in 6 planes which interesect each other to form a cube. Figure 9, suitably interpreted, shows that the extended face planes of a cube partition space into 27 pieces. There is, of course, the cube itself, which is bounded. Then come the unbounded regions. There are

[5] A question is always interesting to mathematicians if the answer is not obvious but they can see a possible way to answer it.

[6] The planes in which the faces of a polyhedron lie are called the extended face planes of the polyhedron.

(a) 6 unbounded square prisms from its faces,
(b) 12 unbounded wedges from its edges and
(c) 8 unbounded trihedral regions from its vertices.

Now if you examine your braided cube you will see that there are
(a) 6 square regions that are covered with precisely two thicknesses of paper,
(b) 12 small slits along the edges where there is just one thickness of paper and
(c) 8 tiny triangular holes at the vertices.

This observation gives us the clue as to how braided models may be useful more generally in answering our question.

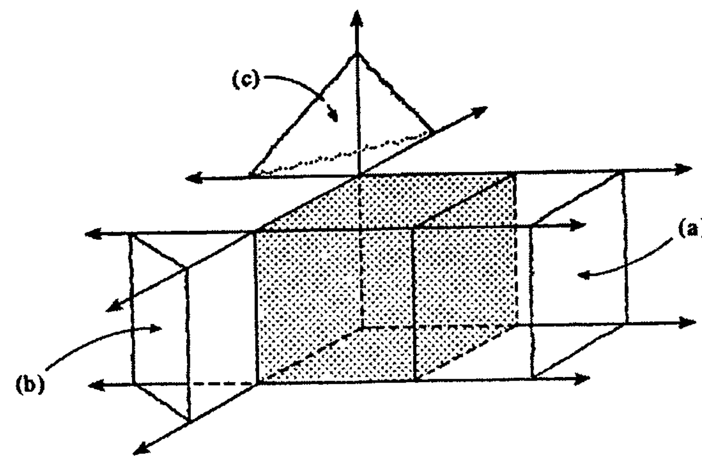


Figure 9: Using the braided cube to count the unbounded regions created by the extended face planes of the cube.

What turns out to be true is that the braided models partition the surface of the polyhedron into mutually disjoint sets of 'polygons' where each polygon is covered by 0, 1 or 2 thicknesses of paper. The polygons where there are holes (0-thickness) define unbounded polyhedral regions, the polygons which are narrow slits (1-thickness) define unbounded wedges, and the polygons where the strips actually are crossing each other (2 thicknesses) define unbounded prisms. The shapes of these unbounded regions may vary with the braided model, but these general statements always hold.

If we are to be able to answer our question for the octahedron we need a braided model with 4 strips so that their edges will define 8 face planes – and, of course, the braided model should also have the same symmetry group as the octahedron. Fortunately the diagonal cube satisfies our conditions (see [Math] concerning the duality of the cube and octahedron). Figure 10 shows the octahedron with some of its face planes extended so that you can see a typical finite region and typical unbounded regions of each type.

The braided models don't help to count the bounded regions (in this case, however, we can see from the part of Figure 10 labeled (a) that there is a tetrahedron on each face of the original octahedron). The rest of the labels in Figure 10 indicate unbounded regions and Figure 11 reproduces those regions as they are associated with the surface of the diagonal cube. Thus, using Figure 11, we may now count the unbounded regions. They are (using the labels in Figure 11):

(b) 6 unbounded tetrahedral regions from the holes in the center of the faces,
(c) 24 unbounded wedges from the 4 slits on each of the 6 faces,
(d) 8 unbounded trihedral regions from the vertices,
(e) 12 unbounded, prism-like regions from the crossings of the strips on the edges comprising a total of 50 unbounded regions.

So the golden dodecahedron must also be useful. In fact it is composed of six strips and the planes defined by the edges of those strips intersect inside this model to form a dodecahedron. Thus the surface of the golden dodecahedron can be used to see that there are 122 unbounded regions created by the extended face planes of the dodecahedron. You might like to try to count them yourself using your golden dodecahedron (or see [P] for more details).

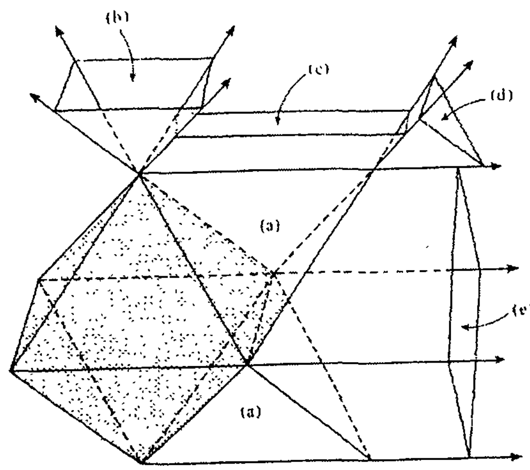


Figure 10: Extending the face planes of an octahedron.

What about the icosahedron? Figure 11 shows how a model, braided from 10 straight strips, may be made from the D^2U^2-tape that can be used to count the 362 unbounded regions created by the extended face planes of the icosahedron (see [P] for more details). We have not written down anywhere how to make the model shown in Figure 11, but we're sure the interested reader will be able to figure it out from what we have said and the illustration. You may be asking yourself why we have slighted the tetrahedron. The answer is that the tetrahedron does not have faces lying in opposite parallel planes, so our models are not useful here. However, it is not difficult to imagine extending the face planes of the tetrahedron and seeing that you have one finite region (the tetrahedron) and 14 unbounded regions (4 from vertices, 6 from edges and 4 from faces).

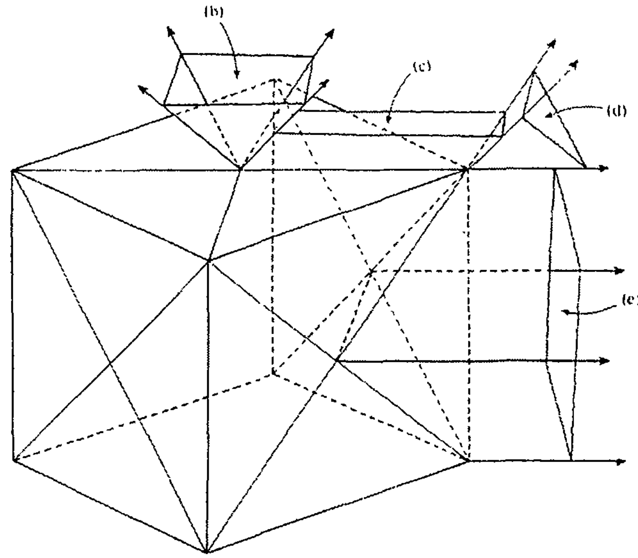


Figure 11: The use of the diagonal cube to visualise the unbounded regions formed by the extended face planes of the regular octahedron.

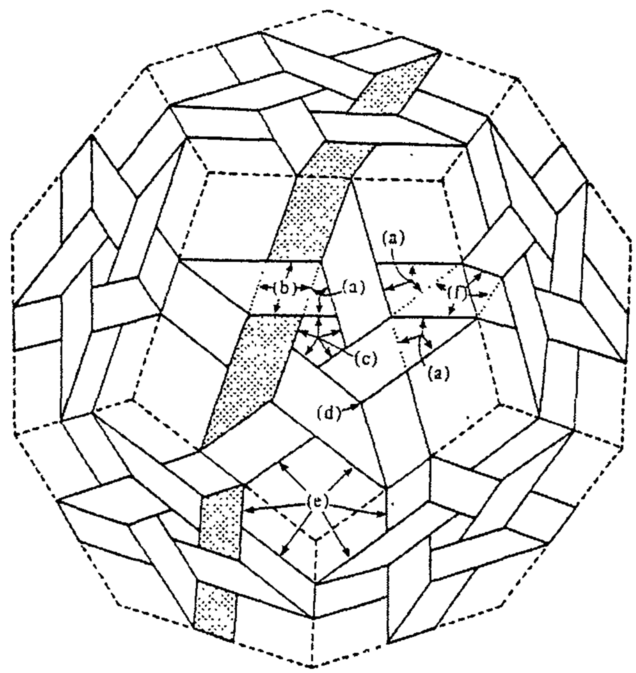


Figure 12: Strips whose edges define the extended face planes of an icosahedron lying on the surface of a phantom dodecahedron.

4. FURTHER HINTS FOR CONSTRUCTING THE PLATONIC PUZZLES

Tetrahedron: Lay one strip over the other strip (with the colors not showing) exactly as shown in Figure 13(a). Think of the triangle ABC as the base of the tetrahedron; for the moment the triangle ABC remains fixed. Then fold the bottom strip into a tetrahedron by lifting up the two triangles labeled X and overlapping them so that C' meets C, B' meets B, and D' meets D. Don't worry about what is happening to the other strip as long as it stays in contact with the bottom strip where the two triangles originally overlapped. Now you will have a tetrahedron with three triangles sticking out from one edge. Complete the model by wrapping the protruding strip around two faces of the tetrahedron (with the color showing) and tucking in the last triangle so that it looks like Figure 13(b).

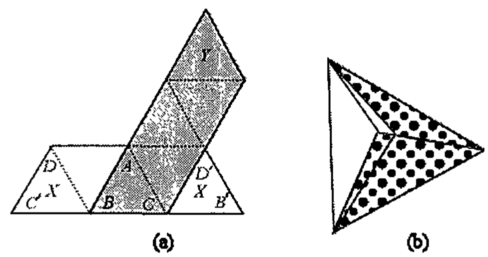


Figure 13

Hexahedron (cube): First take one strip and clip it together so that the color is outside and the end squares fit over each other. Do the same with a second strip. Slip one of these over the other so that the holes of the cube are all covered and so that the overlapping squares of the second strip do not cover any squares from the first strip, and so that the paper clip on the first strip is covered as shown in Figure 14(a). Now slide the third strip underneath the top square so that two squares from the third strip stick out on both the right and left sides of the cube, as shown in Figure 14(b). Turn the model upside down and tuck in the ends of this strip to form Figure 14(c). You may remove the paper clips before you complete the construction; but, when you become really adept, you'll find you don't need the paper clips at all.

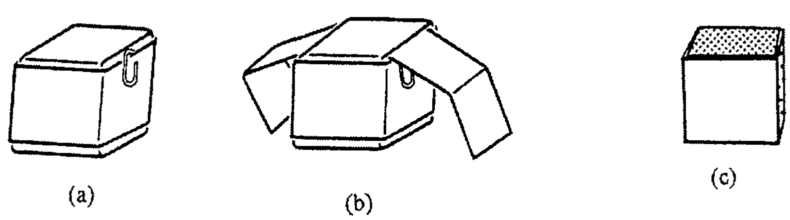
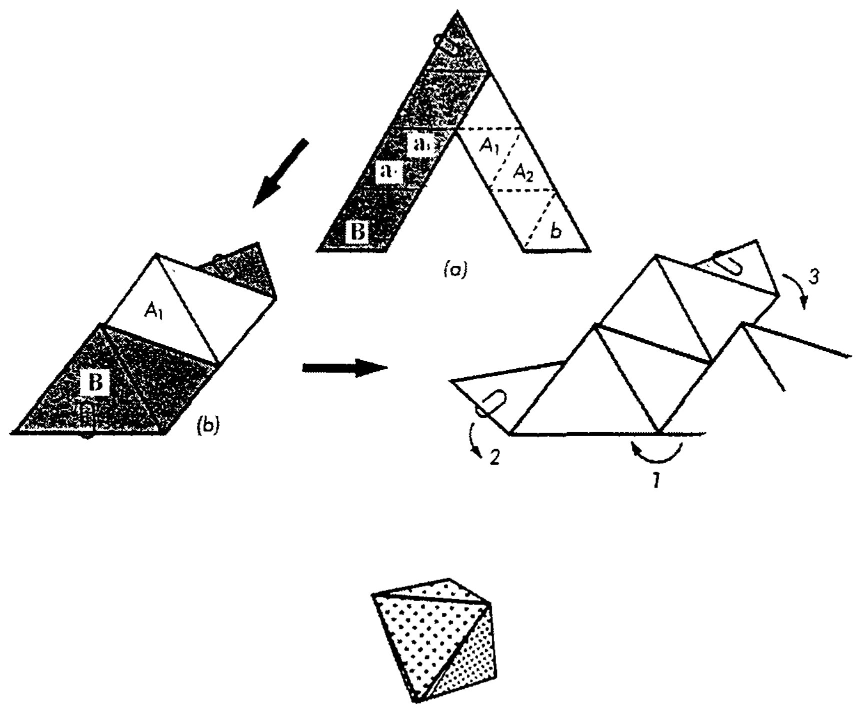


Figure 14

Octahedron: Begin with a pair of overlapping strips held together with a paper clip, as indicated in Figure 15(a) (with the color visible). Fold these two strips into a double pyramid by placing triangle a_1 under triangle A_1, triangle a_2 under triangle A_2, and triangle b under triangle B. Secure the overlapping triangles b, B with a paper clip to produce the configuration shown in Figure 15(b). Repeat this process with the other two strips. Then place the second pair of braided strips over the first pair, as shown in Figure 15(c). When doing this, make certain the flaps with the paper clips are oriented precisely as shown in the figure. Now, pick up the entire configuration and complete the octahedron by moving the pyramids together as shown by the arrow marked 1. Performing step 2 simply places the flap with the paper clip on it against a face of the octahedron. In step 3 you wrap the remaining portion around the octahedron and tuck the last flap (with a paper clip on it) *inside the model.* Again, when you become adept at this process you will be able either to do it without paper clips, or, at least, to slip the paper clips off just before you perform the last three steps. Actually this is just an aesthetic consideration, since the paper clips won't be visible on the completed model.



(d)

Figure 15

Icosahedron: Label each of triangles on one of the strips with a 1 on the uncolored side of the tape. Then label the next strip with a 2 on each of its triangles, the next with a 3 on each of its triangles, the next with a 4 on each of its triangles, and, finally, the last with a 5 on each of its triangles.

Now lay the 5 strips out so that they overlap each other *precisely* as shown in Figure 16(a), making sure that the center 5 triangles form a shallow cup that points *away* from you. You may wish to use some transparent tape to hold the strips in this position.

Now study the situation carefully before making your next move. You must bring the 10 ends up so that the part of the strip at the tail of the arrow goes *under* the part of the strip at the head of the arrow (this means "under" as you look down on the diagram, because we are looking at what will become the inside of the finished model). Half the ends wrap in a clockwise direction, and the other end of each strip wraps in a counterclockwise direction. What finally happens is that each strip overlaps itself at the top of the model. In the intermediate stage it will look like Figure 16(b). At this point it may be useful to put a rubber band (not too tight) around the emerging polyhedron just below the flaps that are sticking out from the pentagon. Then lift the flaps as indicated by the arrows in Figure 16(b) and bring them toward the center so that they tuck in as shown in Figure 16(c).

The model is completed by first lifting flap 1 and smoothing it into position. Then you should do the same with flaps 2, 3 and 4. Finally, flap 5 will tuck into the obvious slot and you will have produced the model shown in Figure 16(d).

This model is, in the view of the authors, the most difficult of the 9 puzzles to construct and it is not very stable. You might want to put a couple of lightweight rubber bands around it to prevent it from falling apart when it is handled.

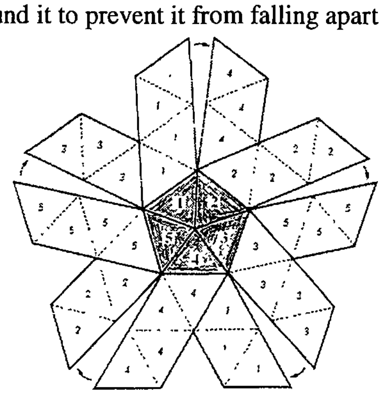
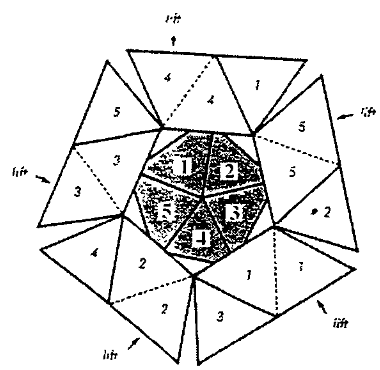


Figure 16 (a)



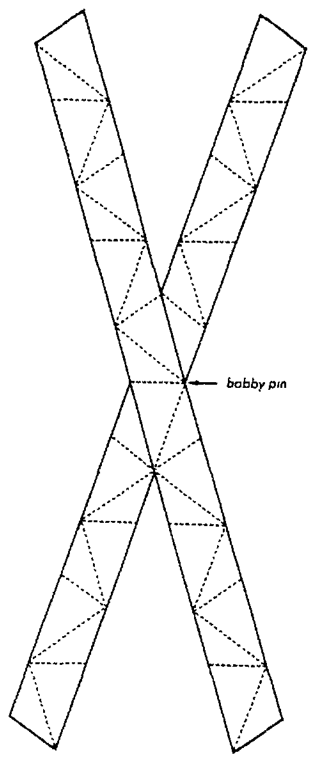
(b)

Dodecahedron: Take two of the strips and secure them with a bobby pin as shown in Figure 17(a) (with the colored side visible)[7]. Then make a bracelet out of each of the strips in such a way that

(a) four sections of each strip overlap, and
(b) the strip that is *under* on one side of the bracelet is *over* on the other side. (This will be true for both strips.)

Use another bobby pin to hold all four thicknesses of tape together on the edge that is opposite the one already secured with a bobby pin.

Repeat the steps above with another pair of strips. You will then have two identical bracelet-like arrangements. Slip one inside the other one as illustrated in Figure 17(b), so that it looks like a dodecahedron with triangular holes on four of its faces.

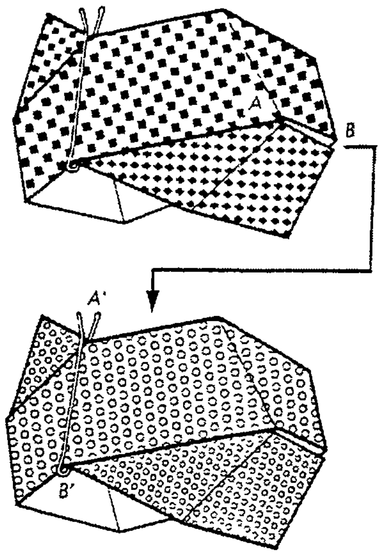


Next take the last two strips and cross them precisely as you did in Figure 17(a) (to do otherwise would destroy some of the symmetry); then secure them with a bobby pin. Carefully put two of the loose ends (either the top two or the bottom two) through the top hole and pull them out the other side so that the bobby pin lands on *CD*. Then put the other two ends through the bottom hole and pull them out the other side. Now you can tuck in the loose flaps, but make certain to reverse the order of the strips – that is, whichever one was on the bottom at *CD* should be on the top when you do the final tucking.

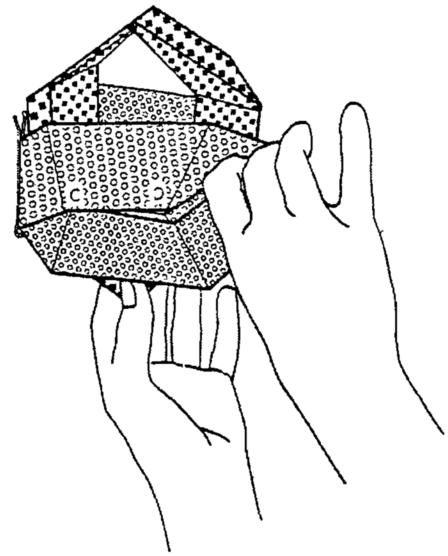
After you have mastered this construction you may wish to try to construct the model with tricolored faces, shown in Figure 17(e), which illustrates, rather vividly, exactly how to inscribe the cube symmetrically inside the dodecahedron. You may also note a similarity between this construction and the cube of Figure 10.

Figure 17 (a)

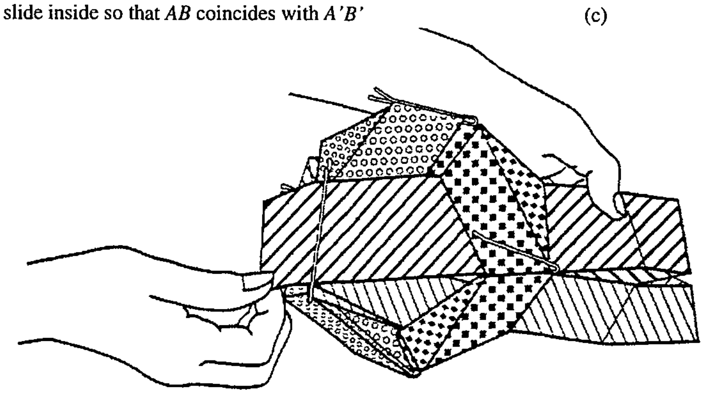
[7] Notice that the long lines are shown in this figure but, as we said earlier, your strips should only be creased on the short lines.



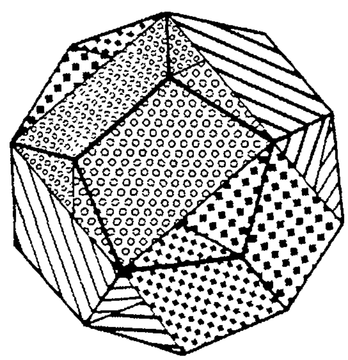
(b) Turn and slide inside so that AB coincides with $A'B'$



(c)



(d)



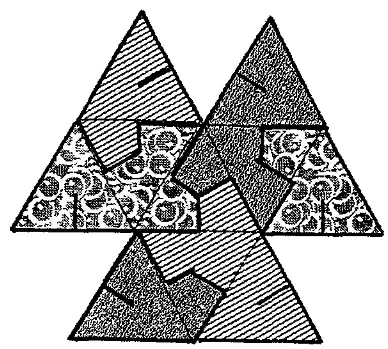
(e)

5. CONSTRUCTING THE PERFECT TETRAHEDRON, OCTAHEDRON AND ICOSAHEDRON

Recall how much pleasure we took in the fact that the diagonal cube and the golden dodecahedron retained all their inherent symmetry. Generally speaking, braided models lose some of the symmetry of the underlying geometric figure; indeed, our braided tetrahedron, octahedron and icosahedron all lost some of the underlying geometric symmetry. Thus it is natural to ask "Is it possible to braid the tetrahedron, octahedron and icosahedron in such a way as to retain all the symmetry of the original polyhedron?" We have recently discovered a way to do this. The problem was to design strips so that three strips cross over each other to form each (triangular face) in a symmetric way.

Figure 18(b) shows a typical straight strip of 5 equilateral triangles with a slit in each triangle from the top (or bottom) edge to (just past) the center[8]. The perfect tetrahedron is constructed out of Figure 18(a) where you will see how the 3 strips are interlaced initially. We leave the completion of the model as a challenge to you.

Figure 19(a) shows the layout of 3 strips for the beginning of the construction of the perfect octahedron. We'll give you one more hint. When you use Figure 19(a) remember that the strip shown below it in Figure 19(b) has to be braided into the figure above it.



(a)

(b)

Figure 18

[8] Theoretically the slit could go just to the center, but the model is then impossible to assemble. You need to have some leeway for the pieces to be free to move during the process of construction – although they will finally land in a symmetric position so that it looks as though the slit need not have gone past the center.

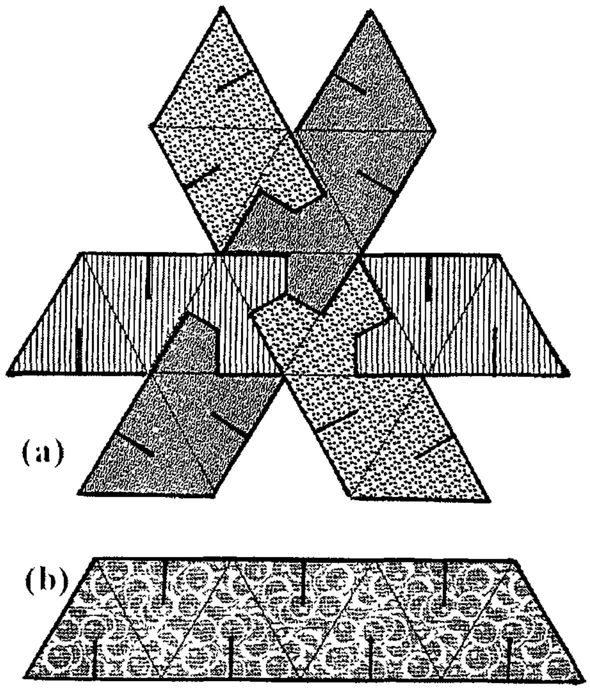
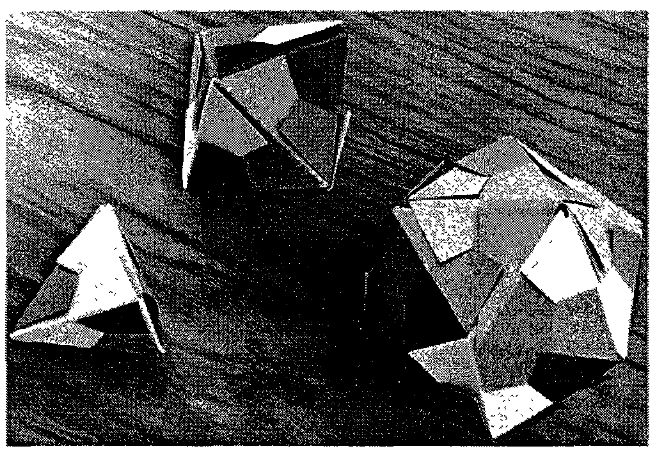


Figure 19



A perfect icosahedron may be constructed from 6 strips of this type having 11 triangles on each strip. Over to you! But take heart - these models take several hours to construct. Just to prove that they really do exist we show the photo of them in Figure 20.

Figure 20

REFERENCES

[Math] Hilton, Peter, and Jean Pedersen, Symmetry in Theory - Mathematics and Aesthetics.

[KW] Kerr, Jeanne W., and John E. Wetzel, Platonic divisions of space, *Mathematics Magazine*, 51 (1978), 229 - 234.

[P] Pedersen, Jean, Visualising parallel divisions of space, *Math. Gaz* 62 (1978), 250 - 262.
[P] Pedersen, Jean, Visualising parallel divisions of space, *Math. Gaz.* 62 (1978), 250 - 262.