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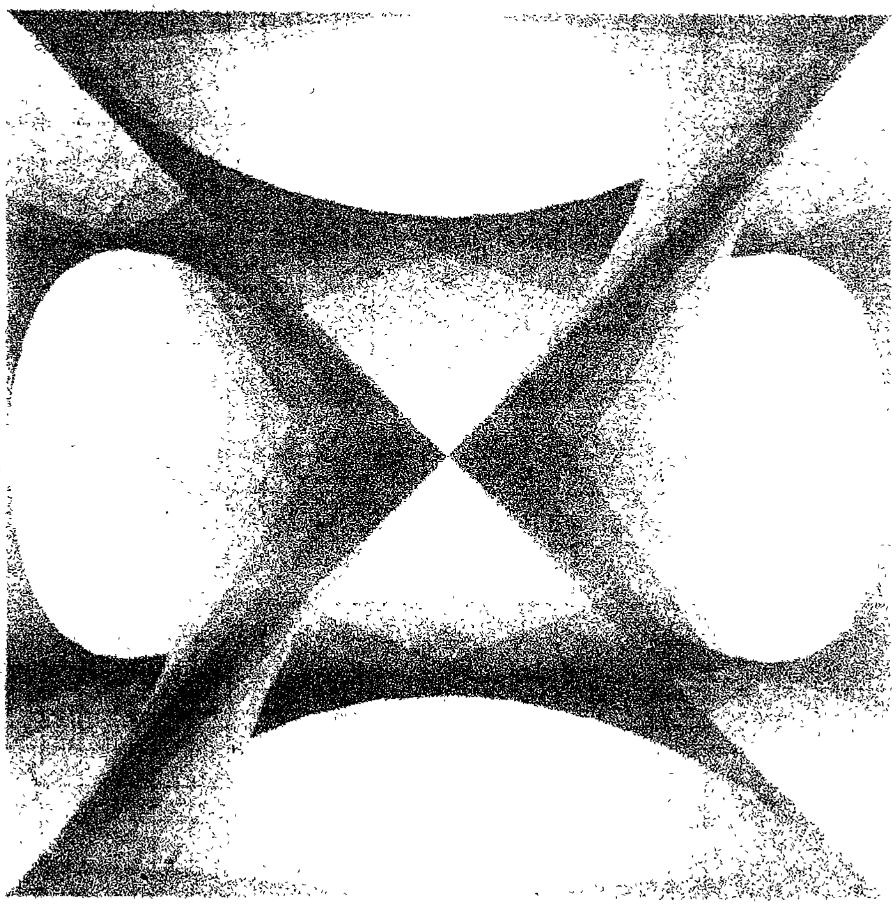

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SYMMETRY: SCIENCE & ART

SYMMETRY IN THEORY – MATHEMATICS AND AESTHETICS

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1. INTRODUCTION

In the course of our very pleasant correspondence with Professor Dénes Nagy about our contribution to the Proceedings of The Third International Conference on Symmetry, held in Washington D. C. in August 1995, the idea took shape of our writing two articles about the symmetry of geometrical figures, one of a practical nature, the other of a more theoretical nature. Thus this article is a companion to the article *Symmetry in Practice* (in this issue), which describes very practical ways of constructing regular polygons and polyhedra. We subtitle that article *Recreational Constructions* – and refer to it henceforth as [Rec] - because the constructions, involving the use of colored paper, have an undoubted recreational flavor. However, it is our conviction, based on many years' experience, that the execution of such model constructions can play a vital role in enlivening and enriching the study of geometry, especially if the mathematical theory underlying the constructions features prominently. Thus it is our strong hope that readers of [Rec] will be encouraged to move on to this more theoretical sequel, to learn why the constructions work and better to understand the nature of symmetry. We also set the mathematical development in its historical context and show explicitly how the geometry is related to other parts of mathematics – real analysis, number theory, group theory, combinatorics. Such connections should, in our view, form an integral part of the teaching and learning of any part of mathematics. We will refer to the present article, briefly, as [Math].

In Section 2 we link the practical instructions of [Rec] to a mathematical discussion of the parameters of the polygons constructed. Thus we answer two questions which stand in a converse relation to each other, namely, (i) given the folding instructions for our tape, when will we be able to produce a regular convex polygon and how many sides will it have, and (ii) given a number p, what folding instructions will produce a regular p-sided polygon (or p-gon)?

Having learnt in Section 2 how to construct certain regular figures, we turn in Section 3 to the question of just what we should understand by the symmetry of a geometrical figure, and how it should be measured. From a mathematical point of view it makes very little sense to say that a given figure A is symmetrical,[1] but we have a precise idea of its *group of symmetries*, that is, of the subgroup of the group of Euclidean movements of the ambient space of A under which A is invariant. Based on this idea, we can give meaning to the statement that figure A is more symmetrical than figure A'. However, we need to bear in mind that the symmetry group of A depends on our convention as to what *is* the ambient space of A. Thus if A is a circle, then its symmetry group as a subset of the plane depends on whether we allow *reflexions* of the plane or not (note that a reflexion of the plane *cannot* be achieved by a movement *in* the plane, but only by a movement in 3-dimensional space).

Another important aspect of symmetry arises when one considers actual physical models of geometrical configurations. Suppose we have constructed a model M of the figure A by braiding together colored strips; A may be a regular dodecahedron, say. Our model cannot have more symmetry than A itself – but it may well have less. For to every symmetry g of A we have a movement of the model M which may create an image Mg recognizably different from M because of the arrangement of colors. Thus the symmetry group of M may only be a subgroup of the symmetry group of A; and aesthetics come into the story here by requiring the symmetry group of M to be as large as possible. Thus can mathematics contribute to the study of aesthetics!

It turns out (not surprisingly!) that, if B is a subset of A and if G_A is the symmetry group of A, then the set of images of B under the action of elements of G_A is the set of *homologues* of B in the sense of George Pólya; we explain this in Section 4. Actually, Pólya never wrote down his work on homologues (which, so far as we know, he only discussed in the case where A is a Platonic solid), but, when he was a very old man, he

[1] We might perhaps say that A is symmetrical if there is a non-trivial Euclidean movement sending A to itself. A classification of symmetricality due to Kepler is to be found in [C2].

asked us to write it down for him, and we are proud and happy to have this opportunity to do so (see Figure 9 of for the only extant copy of his original notes on the subject).

In Section 5 we explain Pólya's famous *Enumeration Theorem*, one of the most important theorems of that branch of mathematics known as *combinatorics*. We apply it to the symmetries of geometrical figures, where parts of the figures are colored in prescribed ways, and again recover the notion of homologue from the formulation of the theorem.

The final section is an informal epilogue, describing our relationship with George Pólya. We are grateful to Dénes Nagy for inviting us to write these two articles, [Rec] and [Math], and for persuading us to include some personal reflections on our good fortune in knowing that remarkable man so well.

2. 2-PERIOD FOLDING PROCEDURES FOR CONSTRUCTING REGULAR POLYGONS AND A GENERALIZATION

We agreed in [Rec] that, however symmetry is defined, the most symmetric polygons are the regular polygons, both the regular convex polygons and the regular star polygons (see [C1]). This is our justification for devoting this section to a particularly easy way of constructing examples of such polygons; in fact, we will confine ourselves, in this article, to the construction of regular *convex* polygons. The *practical* problems of such constructions are discussed in our companion article [Rec].

To set our problem in its historical context, we should really begin with the Greeks and their fascination with the challenge of constructing regular convex polygons. We will refer to such p-sided polygons as regular convex p-gons, and we may even suppress the word *convex* if no confusion would result. The Greeks, working on these problems about 350 B.C., restricted themselves to constructions using only what we call *Euclidean tools*, namely an unmarked straightedge and a compass. No doubt the Greeks would have liked to be able to describe Euclidean constructions whenever possible. However, they were only able to provide such constructions for regular convex polygons having p sides, where

$$p = 2^c p_0, \text{ with } p_0 = 1, 3, 5, \text{ or } 15.$$

About 2,000 years later Gauss (1777 – 1855) showed that Euclidean constructions were possible only rarely. He proved that a Euclidean construction is possible *if* and *only if* the number of sides p is of the form $p = 2^c\prod\rho_i$, where the ρ_i are distinct Fermat primes – that is, primes of the form $F_n = 2^{2^n}+1$.

Gauss's discovery was remarkable – it tells us precisely which regular p-gons admit a Euclidean construction, provided, of course, that we know which Fermat numbers F_n are prime. In fact, not all Fermat numbers are prime. Euler (1707 – 1783) showed that $F_5 = 2^{2^5}+1$ is not prime, and although many composite Fermat numbers have been identified, to this day the only known prime Fermat numbers are

$$F_0 = 3,\ F_1 = 5,\ F_2 = 17,\ F_3 = 257 \text{ and } F_4 = 65537.$$

Thus, even with Gauss's contribution, there exists a Euclidean construction of a regular p-gon for very few values of p, and even for these p we do not in all cases know an explicit construction. For example, in *The World of Mathematics* [N] we read:

> Simple Euclidean constructions for the regular polygons of 17 and 257 sides are available, and an industrious algebraist expended the better part of his years and a mass of paper in attempting to construct the F_4 regular polygon of 65,537 sides. The unfinished outcome of all this grueling labor was piously deposited in the library of a German university.

Despite our knowledge of Gauss's work we still would like to be able to construct (somehow) *all* regular p-gons. Our approach is to redefine the question so that, instead of exact constructions, we will ask *for which $p \geq 3$ is it possible, systematically and explicitly, to construct an arbitrarily good approximation to a regular p-gon*? We take it as obvious that we can construct a regular p-gon *exactly* if p is a power of 2. What we will show is that it is possible, simply and algorithmically, to construct an *approximation* (to any degree of accuracy) to a convex p-gon for any value of $p \geq 3$. In fact, we will give explicit (and uncomplicated) instructions involving only the folding of a straight strip of paper tape in a prescribed periodic manner.

Although the construction of regular convex p-gons would be a perfectly legitimate goal by itself, the mathematics we encounter is generous and we achieve much more. In the process of making what we call the *primary crease lines* used to construct regular convex p-gons we obtain tape which can be used to fold certain (but not all) regular star polygons. It is not difficult to add *secondary crease lines* in order to obtain tape that may be used to construct the remaining regular star polygons.

As it turns out, the mathematics we encounter, in validating our folding procedures, leads quickly and naturally to questions, and hence to new results, in number theory.

Those interested may consult [HP3]. In the interests of mathematical simplicity, as we have said, we will confine attention, in this article, to convex polygons. The more ambitious reader, interested in star polygons, may consult [C1, HP2, 3].

Let us begin by explaining a precise and fundamental folding procedure, involving a straight strip of paper with parallel edges (adding-machine tape or ordinary *unreinforced* packaging gummed tape work well), designed to produce a regular convex p-gon. For the moment assume that we have a straight strip of paper that has *creases* or *folds* along straight lines emanating from vertices, which are equally spaced, at the top edge of the tape. Further assume that the creases at those vertices, labeled A_{nk}, on the top edge, form identical angles of π/p with the top edge, as shown in Figure 1(a). If we fold this strip on $A_{nk}A_{nk+2}$, as shown in Figure 1(b), and then twist the tape so that it folds on $A_{nk}A_{nk+1}$, as shown in Figure 1(c), the direction of the top edge of the tape will be rotated through an angle of $2\pi/p$. We call this process of *folding and twisting* the *FAT-algorithm.*

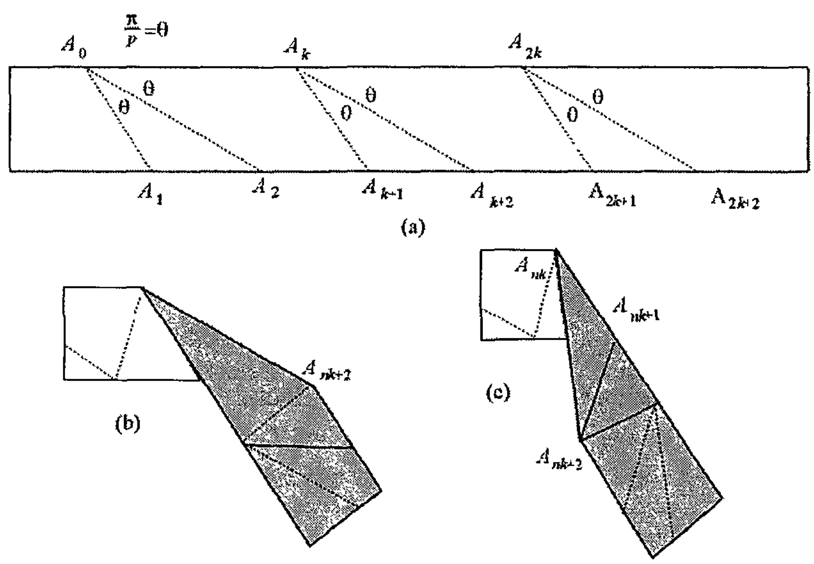


Figure 1

Now observe that if the FAT-algorithm is performed on a sequence of angles, each of which measures π/p, at the first p of a number of equally spaced locations along the top of the tape, in our case at A_{nk} for $n = 0, 1, 2, \ldots, p\text{-}1$, then the top edge of the tape will have turned through an angle of 2π, so that the point A_{pk} will then be coincident with A_0. Thus the top edge of the tape visits every vertex of a regular convex p-gon, and thus itself describes a regular p-gon. A picture of the tape with its crease lines, and the resulting start of the construction of the regular p-gon, is given in Figure 2. Notice that we have *not* adhered there to our systematic enumeration of the vertices on the two edges of the tape that play a role in the construction. (The enumeration has served its purpose!)

Notice, too, that if we had the strip of paper shown in Figure 2(a), with its crease lines, we could then introduce *secondary crease lines* bisecting each of the angles nearest the top edge of the tape and this tape could then be used to construct a regular $2p$-gon with the *FAT*-algorithm. We could then, in principle, repeat this secondary procedure, as often as we wished, to construct regular $4p$-gons, $8p$-gons,... It is for this reason that we only need to concern ourselves with devising primary folding procedures for regular polygons having an *odd* number of sides in order to be able to assure ourselves we can, indeed, fold *all* regular polygons.

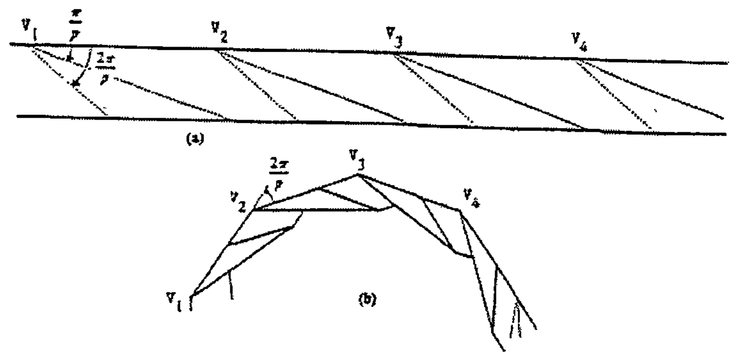


Figure 2

Now, since the regular convex 7-gon is the first polygon we encounter for which we do not have available a Euclidean construction, we are faced with a real difficulty in making available a crease line making an angle of $\pi/7$ with the top edge of the tape. We proceed by adopting a general policy, that we will eventually say more about - we call it our *optimistic strategy.* Assume that we can crease an angle of $2\pi/7$ (certainly we can come close) as shown in Figure 3(a). Given that we have the angle of $2\pi/7$, then simply folding the top edge of the strip DOWN to bisect this angle will produce two adjacent angles of $\pi/7$ at the top edge as shown in Figure 3(b). (We say that $\pi/7$ is the *putative* angle on this tape.) Then, since we are content with this arrangement, we go to the bottom of the tape, and *now we really start the folding procedure.*

We observe that the angle to the right of the last crease line is $6\pi/7$ – and our policy, as paper folders, is that we always avoid leaving even multiples of π in the numerator of any angle next to the edge of the tape, so we bisect this angle of $6\pi/7$, by bringing the bottom edge of the tape UP to coincide with the last crease line as shown in Figure 3(c). We settle for this (because we are content with an odd multiple of π in the numerator) and go to the top of the tape where we observe that the angle to the right of the last crease line is $4\pi/7$ – and, in accordance with our stated policy, we bisect this angle twice, each time bringing the top edge of the tape DOWN to coincide with the lase crease line, obtaining the arrangement of crease lines shown in Figure 3(d). But now we notice something miraculous has occurred! If we had really started with an angle of exactly $2\pi/7$, and if we now continue introducing crease lines by repeatedly folding the tape UP once at the bottom and DOWN twice at the top, we get precisely what we want; namely, pairs of adjacent angles, measuring $\pi/7$, at equally spaced intervals along the top edge of the tape. Let us call this folding procedure the U^1D^2- or D^2U^1-*folding procedure* and call the strip of creased paper it produces[2] U^1D^2- or D^2U^1-*tape*.

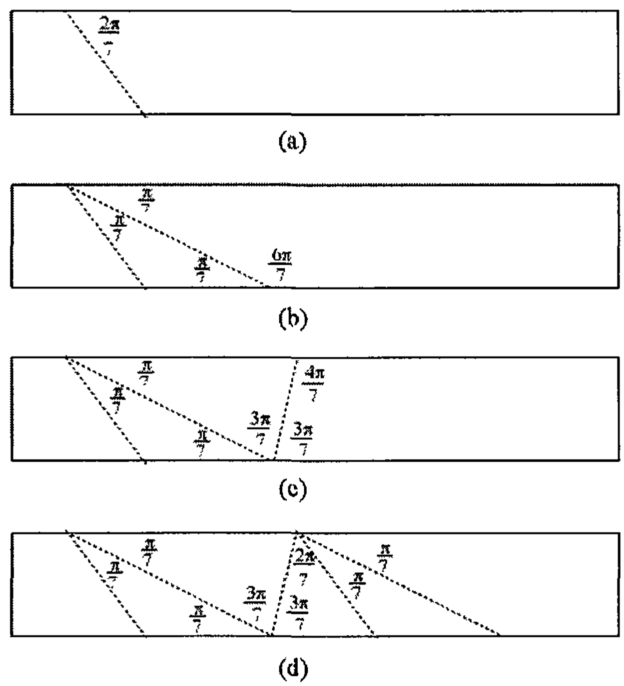


Figure 3

[2] It is our habit to refer to D^2U^1-tape, but this choice is quite arbitrary.

We suggest that before reading further you get a piece of paper and fold an acute angle that you regard as a good approximation to $2\pi/7$. Then fold about 40 triangles using the D^2U^1-folding procedure just described, throw away the first 10 triangles, and try to construct the *FAT* 7-gon shown in Figure 4(b). You will have no doubt that what you have created is, in fact, a 7-gon, but you may wonder *why* it should have worked so well. In other words, how can we *prove* that this evident convergence must take place? One approach is to admit that the first angle folded down from the top of the tape in Figure 3(a) might not have been precisely $2\pi/7$. Then the bisection forming the next crease would make two acute angles nearest the top edge in Figure 3(b) only approximately $\pi/7$; let us call them $\pi/7 + \varepsilon$ (where ε may be either positive or negative). Consequently the angle to the right of this crease, at the bottom of the tape, would measure $6\pi/7 - \varepsilon$. When this angle is bisected, by folding up, the resulting acute angles nearest the bottom of the tape, labeled $3\pi/7$ in Figure 3(c), would, in fact, measure $3\pi/7$-$\varepsilon/2$, forcing the angle to the right of this crease line at the top of the tape to have measure $4\pi/7 + \varepsilon/2$. When this last angle is bisected twice by folding the tape down, the two acute angles nearest the top edge of the tape will measure $\pi/7 + \pi/2^3$. This makes it clear that every time we repeat a D^2U^1-folding on the tape the error is reduced by a factor of 2^3.

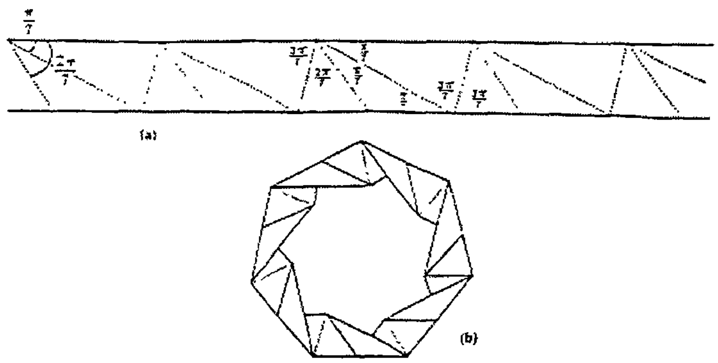


Figure 4

Now it should be clear how our *optimistic strategy* has paid off. By blandly *assuming* we have an angle of $\pi/7$ to begin with, and folding accordingly, we *get what we want* – successive angles at the top of the tape which, as we fold, rapidly get closer and closer to $\pi/7$! A truly remarkable vindication of our optimistic strategy!

In practice the approximations we obtain by folding paper are quite as accurate as the *real world* constructions with a straight edge and compass - for the latter are only perfect in the mind. In both cases the real world result is a function of human skill, but our procedure, unlike the Euclidean procedure, is very forgiving in that it tends to reduce the effects of human error – and, for many people, it is far easier to bisect an angle by folding paper than it is with a straight edge and compass.

Observe that it is in the nature of the folding procedure that we will always be folding DOWN a certain number of times at the top and then folding UP a certain (not necessarily the same) number of times at the bottom and then folding DOWN (possibly an entirely new) number of times at the top, etc. Indeed, a typical folding procedure may be represented by a sequence of exponents attached to the letters *DU DU DU DU*... the sequence stopping to avoid simply repeating a given finite string of exponents. The length of the repeat for the exponents is called the *period* of the folding procedure. (Thus the folding that produced the 7-gon is called a 2-*period* folding procedure.) It is an important fact that, for every odd p, a regular p-gon may be folded by instructions so encoded. It is thus very natural to ask *what regular p-gons can be produced by the 2-period folding procedure*?

In the process of answering this question we make straightforward use of the following:[3]

Lemma 2.1 For any three real numbers a, b and x_0, with $a \neq 0$, let the sequence $\{x_k\}$, $k = 0,1,2,\ldots$ *be defined by the recurrence relation*

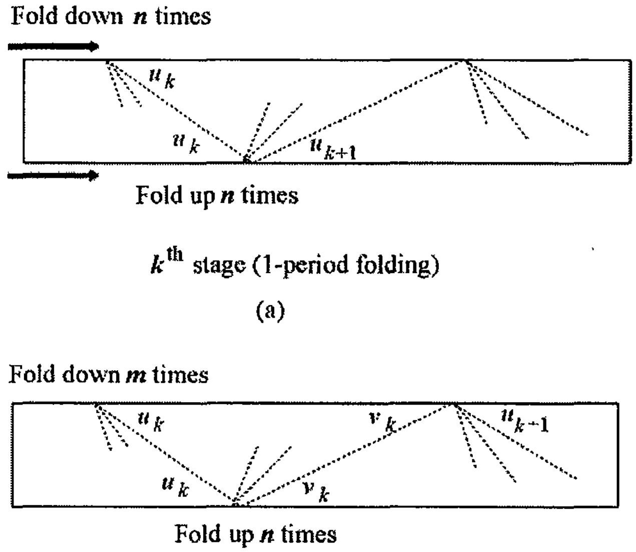
$$x_k + ax_{k+1} = b,\ k = 0, 1, 2, \ldots \tag{2.1}$$

Then if $|a| > 1$, $x_k \to b/(1+a)$ as $k \to \infty$.

Proof: Set $x_k = b/(1+a) + y_k$. Then $y_k + ay_{k+1} = 0$. It follows that $y_k = ((-1)/a)^k y_0$. If $|a| > 1$, $((-1)/a)^k \to 0$, so that $y_k \to 0$ as $k \to \infty$. Hence $x_k \to b/(1+a)$ as $k \to \infty$. Notice that y_k is the error at the k^{th} stage, and that the absolute value of y_k is equal to $|y_0|/|a|^k$.

This result is the special linear case of the Contraction Mapping Principle (see [W]). We point out that it is significant that neither the convergence nor the limit depends on the initial value x_0. This implies, in terms of the folding, that the process will converge, and to the same limit, no matter how we fold the tape to produce the first line – this is what justifies our *optimistic strategy*! And, as we have seen in our example, and as we will soon demonstrate in general, the result of the lemma tells us that the convergence of our folding procedure is rapid, since in all cases $|a|$ will be a positive power of 2.

[3] This lemma is actually applicable to folding procedures of arbitrary period.



k^{th} stage (1-period folding)

(a)

k^{th} stage (2-period folding)

(b)

Figure 5

Now we will look at the general 2-period folding procedure, D^mU^n. In this case a typical portion of the tape would appear as shown in Figure 5(b). If the folding process had been started with an arbitrary angle u_0 at the top of the tape we would have, at the k^{th} stage,

$$u_k + 2^n v_k = \pi,$$
$$v_k + 2^m u_{k+1} = \pi,$$

and hence it follows that

$$u_k - 2^{m+n} u_{k+1} = \pi(1-2^n),\ k = 0,1,2,...$$

Thus, using Lemma 2.1, we see that

$$u_k \to \frac{2^n - 1}{2^{m+n} - 1}\pi \quad \text{as} \quad k \to \infty$$

so that $\frac{2^n - 1}{2^{m+n} - 1}\pi$ is the putative angle $a\pi/b$. Thus the FAT-algorithm will produce, from this tape, a star $\{b/a\}$-gon, where the fraction b/a may turn out not to be reduced (for example when $m = 4$, $n = 2$), with $a = 2^n$-1, $b = 2^{m+n}$-1. By symmetry we infer that

$$v_k \longrightarrow \frac{2^m - 1}{2^{m+n} - 1}\pi \quad \text{as} \quad k \longrightarrow \infty$$

Furthermore, if we assume an initial error E_0 then we know that the error at the k^{th} stage (when folding D^mU^n has been done exactly k times) will be given by $E_k = E_0/2^{(m+n)k}$. Hence, we see that in the case of our D^2U^1-folding (Figure 3) any initial error E_0 is, as we already saw from our initial argument, reduced by a factor of 2^3 between consecutive states. It should now be clear why we advised throwing away the first part of the tape – but, likewise, it should also be clear that it is never necessary to throw away very much of the tape. In practice, convergence is very rapid indeed, and if one made it a rule of thumb always to throw away the first 20 crease lines on the tape for any iterative folding procedure, one would be absolutely safe.

We have seen that the D^mU^n-folding procedure, or, as we may more succinctly describe it, the (m, n)-folding procedure, produces angles π/s on the tape, where

$$s = \frac{2^{m+n} - 1}{2^n - 1}. \tag{2.2}$$

Notice that when $n = m$ the folding becomes, technically, a 1-period folding procedure which produces a regular s-gon, where $s = \dfrac{2^{m+n} - 1}{2^n - 1} = 2^n+1$. Thus we see, immediately, that the D^nU^n-folding will produce tape to which the *FAT*-algorithm can be applied to obtain regular (2^n+1)-gons. These constructions provide approximations to many (but not all) of the polygons the Greeks and Gauss were able to construct with Euclidean tools. We can certainly construct a regular polygon whose number of sides is a Fermat number, but (see [Rec]) it is never possible to construct, with a 1-period folding those regular polygons where the number of sides is the product of at least two distinct Fermat numbers (thus, 15 serves as the first example where we find trouble).

The polygons which are of most interest to us in the construction of regular polyhedra are those with 3 or 5 sides (since we have exact constructions for the square). Our companion article [Rec] of this issue contains *very explicit* instructions of the folding procedures that produce the D^1U^1- and the D^2U^2-tape (which can be used to construct 3- and 5- gons, respectively) along with equally explicit instructions for building some braided polyhedra from the tape produced. The reader is encouraged to at least peruse that part of this issue before going on, since we will be making references to some of the models whose construction is described there in the next section.

Before we begin the discussion of symmetry let us finish explaining how you might construct those regular polygons that cannot be folded by the 1- or 2-period folding procedures. For example, suppose we wanted to construct a regular 11-gon. Our arguments in [HP1, 2 or 3] show that no 2-period (or 1-period) folding procedure can possibly produce an 11-gon.

In fact, the example of constructing the regular 11-gon is sufficiently general to show the construction of any regular p-gon, with p odd. So let us demonstrate how to construct a regular 11-gon. we proceed as we did in the construction of the regular 7-gon (in Section 2) – we adopt our *optimistic strategy* (which means that we assume we've got what we want and, as we will show, we then *actually get what we want*!). Thus we assume we can fold an angle of $2\pi/11$. We bisect it by introducing a crease line, and follow the crease line to the bottom of the tape. The folding procedure now commences at the bottom of the tape. Thus

(1) Each new crease line goes in the forward (left to right) direction along the tape;

(2) Each new crease line always *bisects* the angle between the last crease line and the edge of the tape from which it emanates;

(3) The isection of angles at any vertex continues until a crease line produces an angle of the form $a\pi/11$ where a is an *odd* number; then the folding stops at that vertex and commences at the intersection point of the last crease line with the other edge of the tape.

Once again the *optimistic strategy* works and our procedure results in tape whose angles converge to those shown in Figure 6(b). We could then denote this folding procedure by $U^1D^1U^3D^1\ U^1D^3$... interpreted in the obvious way on the tape – that is, the first exponent "1" refers to the one bisection (producing a line in the upward direction) at the vertices A_{6n} (for $n = 0, 1, 2,...$) on the bottom of the tape; similarly the next "1" refers to the bisection (producing a crease in the downward direction) made at the bottom of the tape through the vertices A_{6n+1}; etc. However, since the folding procedure is *duplicated* halfway through, we can abbreviate the notation and write simply $\{1,1,3\}$, with the understanding that we alternately fold from the bottom and top of the tape as described, with the *number* of bisections at each vertex running, in order, through the values 1, 1, 3, ... We call this a *primary folding procedure of period* 3 or a 3-*period folding*, for obvious reasons. The crease lines made during this procedure are called *primary crease lines.*

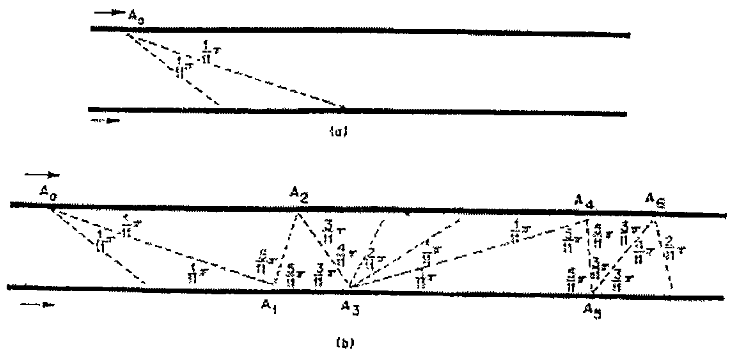


Figure 6 (Note that the indexing of the vertices is *not* the same as that in Fig. 1).

Our argument, as described for $p = 11$, may clearly be applied to any odd number p. However, our tape for the 11-gon has a special symmetry as a consequence of its *odd period*; namely that if it is "flipped" about the horizontal line halfway between its parallel edges, the result is a *translate* of the original tape. As a practical matter this special symmetry of the tape means that we can use either the top edge or the bottom edge of the tape to construct our polygons. On tapes with an *even* period the top edge and the bottom edge of the tape are not translates of each other (under the horizontal flip), which simply means that care must be taken in choosing the edge of the tape used to construct a specific polygon.

A proof for the convergence for the general folding procedure may be given that is similar to the one we gave for the primary folding procedure of period 2, using Lemma 2.1. Alternatively one could revert to an error-type proof like that given for the 7-gon. We leave the details to the reader.

For further reading, and a discussion of the construction of star polygons see [HP2, 3 and 5].

3. THE SYMMETRY GROUP OF A GEOMETRIC CONFIGURATION

We want now to take up the more mathematical aspects of symmetry. Indeed, at this stage, we lack a precise definition of symmetry – we cannot even give a meaning, in general, to the statement that one geometrical figure is more symmetric than another. Of course, a square is more symmetric than an arbitrary rectangle, and a rectangle is more symmetric than an arbitrary quadrilateral. But can we, for example, always compare the symmetries of regular polygons?

We are guided in our definitions by the approach of the great German mathematician Felix Klein (1849 – 1925) to understanding the nature of geometry. Consider, for example, the usual plane Euclidean geometry, in which we study the properties of planar figures which are invariant under certain *Euclidean motions*. These motions certainly include *translation* and *rotation*, but it is a matter of choice whether they include *reflexion*. For example the *FAT* 7-gon (Figure 4(b)) is invariant under rotations through $2\pi/7$ about its center, but not under reflexion in its plane. Thus, to define our geometry, we must decide whether we allow Euclidean motions which reverse orientation. Of course, it we allow certain Euclidean motions, we must also allow compositions and inverses of such motions, so we postulate a certain *group* G of allowed motions. If A is a planar figure, then, for any $g \in G$, Ag is again a planar figure and, *in the G-geometry of* A, we study the properties of the figure A which it shares with all the figures Ag as g varies over G; such properties are called the *G-invariants* of A, abbreviated to *invariants* if the group G may be understood.

Example 3.1 Let G be the group of motions of the plane generated by translations, rotations and reflexions (in a line); we call this the Euclidean group in 2 dimensions and may write it E_2. Then the Euclidean geometry of the plane is the study of the properties of subsets of the plane which are invariant under motions of E_2. For example, the property of being a polygon is a Euclidean property; the number of vertices and sides of a polygon is a Euclidean invariant. On the other hand, as we have hinted, orientation is not invariant with respect to this group, though it would be if we disallowed reflexions. Thus, by means of a motion in E_2 the triangle ABC may be turned over (flipped) to form the triangle $A'B'C'$ as shown in Figure 7. But the orientation of the triangle $\overrightarrow{ABC}$ is anti clockwise, while the orientation of the triangle $\overrightarrow{A'B'C'}$ is clockwise.

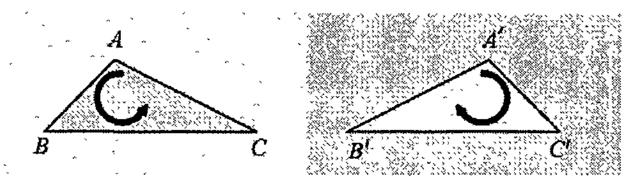


Figure 7: The triangle ABC may be transformed by a rotation in 3 dimensions into the triangle $A'B'C'$ reversing the orientation of the triangle.

Example 3.2 We may 'step up a dimension', passing to the group E_3 of Euclidean motions in 3-dimensional space. Notice that it is natural to think of reflexions in a line (of a planar figure) as a 'motion' since it can be achieved by a rotation in some suitable ambient 3-dimensional space containing the plane figure. However, it requires a greater intellectual effort to think of reflexion in a plane (of a spatial figure) as a motion in some ambient 4-dimensional space! Who would think of turning the golden dodecahedron (see Figure 8 of [Rec]) inside out? Thus it is common not to include such reflexions in defining 3-dimensional geometry. This preference is, however, a consequence of our experience of living in a 3-dimensional world and has no mathematical basis. However, since, in this article, and its companion article, we are highlighting the construction of actual physical models of geometrical configurations, it is entirely reasonable to omit 'motions' to which the models themselves cannot be subjected.

We now introduce the key idea in the precise definition of symmetry. Let a geometry be defined on the ambient space of a geometric configuration A by means of the group of motions G. Then the *symmetry group* of A, relative to the geometry defined by G, is the subgroup G_A of G consisting of those motions $g \in G$ such that $Ag = A$, that is, those motions which map A onto itself, or, as we say, under which A is *invariant*. Thus, for example, if our geometry is defined by rotations and translations in the plane, and if A is an equilateral triangle, then its symmetry group G_A consists of rotations about its center through 0°, 120°, and 240°; if, in our geometry, we also allow reflexions, then the symmetry group has 6 elements instead of 3, and is, in fact, the very well-known group S_3, called the *symmetric group on* 3 *symbols* – the symbols may be thought of as the vertices of the triangle. We must repeat for emphasis that the symmetry group G_A of the configuration A is a *relative* notion, depending on the choice of 'geometry' G.

It is plain that no compact (bounded) configuration can possibly be invariant under a translation. Thus when we are considering the symmetry group of such a figure we may suppose G to be generated by rotations and, perhaps, reflexions. Moreover, any such motion in the plane is determined by its effect on 3 independent points and any such motion in 3-dimensional space is determined by its effect on 4 independent points. Since a (plane) polygon has at least 3 vertices and a polyhedron has at least 4 vertices, and since any element of the symmetry group of a polygon or a polyhedron must map vertices to vertices, it follows that the symmetry group of a polygon or a polyhedron is *finite* (compare the symmetry groups of a circle or a sphere).

The symmetry group of any polygon with n sides is, by the argument above, a subgroup of S_n, the group of permutations of n symbols, also called the *symmetric group* on n symbols. If G is generated by rotations alone, and the polygon is regular, this group is the cyclic group of order n, often written C_n, generated by a rotation through an angle of $2\pi/n$ radians about the center of the polygonal region. If G also includes reflexions, this group has $2n$ elements and includes n reflexions; this group is called a *dihedral* group and is often written D_n.

In discussing the symmetry groups of polyhedra, we will, as indicated above, always assume that the geometry is given by the group G generated by rotations in 3-dimensional space. Then the symmetry group of the regular tetrahedron is the so-called *alternating* group A_4. In general, A_n is the subgroup of S_n consisting of the *even* permutations of n symbols; it is of index 2 in S_n, that is, its order is half that of S_n, or $n!/2$. Thus the order of A_4 is 12. The cube and the regular octahedron have the same symmetry group, namely S_4. It is easy to see why the symmetry groups are the same; for the centers of the faces of a cube are the vertices of a regular octahedron, and the centers of the faces of a regular octahedron are the vertices of a cube. Likewise, and for the same reason, the regular dodecahedron and the regular isocahedron have the same symmetry group, which is A_5. It is a matter of great interest and relevance here that the symmetries of the Diagonal Cube and the special braided octahedron of Figure 7 and Figure 16, respectively (of [Rec]) each permute the four braided strips from which the models are made. This provides a beautiful explanation of why their symmetry group is the symmetry group S_4.

We are now in a position to give at least one precise meaning to the statement "Figure A is more symmetric than Figure B". If it happens that the symmetry group G_A of A strictly contains the symmetry group G_B of B, then we are surely entitled to say that A is more symmetric than B. Notice that the situation described may, in fact, occur because B is obtained from A by adding features which destroy some of the symmetry of A. For example, the coloring of the strips used to construct the braided Platonic solids of Figure 6 of [Rec] will reduce the symmetry in all cases but that of the cube.

However, the definition above is really too restrictive. For we would like to be able to say that the regular n-gon becomes more symmetric as n increases. We are thus led to a weaker notion which will be useful provided we are dealing with figures with finite symmetry groups (e.g., polygons and polyhedra). We could then say – and do say – that A is more symmetric than B if G_A has more elements than G_B. Thus we have, in fact, two notions whereby we may compare symmetry – and they have the merit of being consistent. Indeed, if A is more symmetric than B in the first sense, it is more symmetric than B in the second sense – but not conversely.

Notice that we deliberately avoid the statement – often to be found in popular writing – "A is a symmetric figure". We regard this statement as having no precise meaning!

4. HOMOLOGUES

George Pólya, who made great contributions not only to mathematics itself, but also to the understanding of how and why we do mathematics – or perhaps one should say 'how and why we should do mathematics' – was particularly fascinated by the Platonic solids and first introduced his notion of *homologues* in connection with the study of their symmetry; they later played an important role in one of his most important contributions to the branch of mathematics know as *combinatorics*, namely, the *Pólya Enumeration Theorem* (see [P1] for an intuitive account). Let us describe this notion of homologues in terms of symmetry groups. We believe that we are thereby increasing the scope of the notion and entirely maintaining the spirit.

Let A be a geometrical configuration with symmetry group G_A, and let B be a subset of A. Thus, for example, A may be a polyhedron and B a face of that polyhedron. We consider the subgroup G_{AB} of G_A consisting of those motions in the symmetry group G_A of A which map B to itself. Now subgroups partition a group into *cosets*: If K is a subgroup of H, we define a *(right) coset* of K in H as a collection of elements kh, with *h fixed* and *k varying over K*. We call h a *representative* of this coset, which we write Kh. Any two cosets Kh, Kh' are either disjoint or identical (this is easy to prove), so we may imagine that we have picked a set of coset representatives, one for each coset. In the case in which we are interested the group H is finite so we may write, for some m

$$H = \bigcup_{i=1}^{m} Kh_i, \tag{4.1}$$

where it is understood that the union is disjoint. Notice that m, which appears in (4.1) and which we call the *index* of K in H, is just the ratio of the order of H to the order of K. An example was provided earlier with $H = S_n$, the symmetric group, and $K = A_n$, the alternating group. Then $m = 2$.

Reverting to our geometrical situation, we consider a coset of G_{AB} in G_A, that is, a set $G_{AB}g$, $g \in G_A$. Every element in $G_{AB}g$ sends B to the same subset Bg of A. The collection of these subsets is what Pólya called the collection of *homologues* of B in A. We see that the set of homologues of B is in one-one correspondence with the set of cosets of G_{AB} in G_A.

Example 4.1 Consider the pentagonal dipyramid A of Figure 5(b) of [Rec]. We may specify any motion in the symmetry group of A by the resulting permutation of its vertices 1, 2, 3, 4, 5, 6, 7. In fact, G_A is the dihedral group D_5, with 10 elements, given by the following permutations sending (1 2 3 4 5 6 7), resepectively, to

$$\begin{cases} \to (1\ 2\ 3\ 4\ 5\ 6\ 7)\ \text{(Identity)} \\ \to (2\ 3\ 4\ 5\ 6\ 7\ 1)\ \text{(rotation through } 2\pi/5 \text{ about axis 67)} \\ \to (3\ 4\ 5\ 1\ 2\ 6\ 7) \\ \to (4\ 5\ 1\ 2\ 3\ 6\ 7) \\ \to (5\ 1\ 2\ 3\ 4\ 6\ 7) \\ \to (5\ 4\ 3\ 2\ 1\ 7\ 6)\ \text{(interchanging top and bottom)} \\ \to (4\ 3\ 2\ 1\ 5\ 6\ 7)\ \text{(interchange plus rotation through } 2\pi/5) \\ \to (3\ 2\ 1\ 5\ 4\ 6\ 7) \\ \to (2\ 1\ 5\ 4\ 3\ 7\ 6) \\ \to (1\ 5\ 4\ 3\ 2\ 7\ 6) \end{cases}$$

First, let be the edge 16. Then $G_{AB} = \{Id\}$, since only the identity sends the subset (1, 6) to itself. Thus the index of G_{AB} in G_A is 10, and there are 10 homologues of the edge 16; these are the 10 'spines' of the dipyramid (i.e., we exclude the edges around the equator). Second, let B be the edge 12. Then G_{AB} has 2 elements, since there are two elements of G_A, namely the identity and permutation $(1\ 2\ 3\ 4\ 5\ 6\ 7) \to (2\ 1\ 5\ 4\ 3\ 7\ 6)$, which send the subset (1, 2) to itself. Thus the index of G_{AB} in G_A is 5, and there are 5 homologues of the 12; these are the 5 edges around the equator.

Third, let B be the face (126). Then $G_{AB} = \{Id\}$, so that, as in the first case, there are 10 homologues of the face (126); in other words all the (triangular) faces are homologues.

Let us now explain the Pólya Enumeration Theorem - actually, there are *two* theorems - and see how the notion of homologue fits into the story.

5. THE PÓLYA ENUMERATION THEOREM

Let X be a finite set; the reader might like to keep in mind the set of vertices (or edges, or faces) of a polygon or polyhedron; and let G be a finite symmetry group acting on X. Suppose X has n elements, and that G has m elements; we write $|X| = n$, $|G| = m$. We may represent the elements of the set X by the integers $1, 2, \dots, n$. If $g \in G$, then g acts as a *permutation* of $\{1, 2, \dots, n\}$. Now every permutation is uniquely expressible as a composition of cyclic permutations on mutually exclusive subsets of the elements of X. For example, the permutation

$$\begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11 \\ 2\ 4\ 7\ 1\ 3\ 11\ 5\ 6\ 8\ 10\ 9 \end{pmatrix} \tag{5.1}$$

is the composite (1 2 4)(3 7 5)(9 8 6 11)(10), where, e.g., (1 2 4) denotes the cyclic permutation

$$\begin{pmatrix} 1\ 2\ 4 \\ 2\ 4\ 1 \end{pmatrix}$$

Thus the permutation (5.1) is the composite of one cyclic permutation of length 1, two cyclic permutations of length 3, and one cyclic permutation of length 4, the cyclic permutations acting on disjoint subsets of the set X. In general a permutation of X has the *type* $(a_1, a_2, \dots, a_n\}$ if it consists of a_1 permutations of length 1, a_2 permutations of length 2, ... , a_n permutations of length n, the permutations having disjoint domains of action; notice that $\sum_{i=1}^{n} a_i = n$. For example the permutation (5.1) has the type $\{1,0,2,1,0,0, 0, 0, 0, 0, 0\}$. If g has the type $\{a_1, a_2, \dots, a_n\}$, we define the *cycle index* of g to be the monomial

$$Z(g) = Z(g; x_1, x_2, \dots, x_n) = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}.$$

The *cycle index of G* is $Z(g) = \frac{1}{m} \sum_{g \in G} Z(g)$.

We give an example which we will revisit periodically throughout this section.

Example 5.1 We consider the symmetries of the square as shown in Figure 8.

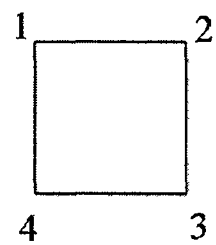


Figure 8: We denote this labeling of the square by 1234

The group G of symmetries is a group of order 8, which we describe by permutations of the set of vertices $\{1, 2, 3, 4\}$. Thus

g_1 (*Id*entity)	$(1\,2\,3\,4) \to (1\,2\,3\,4)$	cycle index	x_1^4
g_2	$(1\,2\,3\,4) \to (2\,3\,4\,1)$	cycle index	x_4
g_3	$(1\,2\,3\,4) \to (3\,4\,1\,2)$	cycle index	x_2^2
g_4	$(1\,2\,3\,4) \to (4\,1\,2\,3)$	cycle index	x_4
g_5	$(1\,2\,3\,4) \to (3\,2\,1\,4)$	cycle index	$x_1^2x_2$
g_6	$(1\,2\,3\,4) \to (1\,4\,3\,2)$	cycle index	$x_1^2x_2$
g_7	$(1\,2\,3\,4) \to (2\,1\,4\,3)$	cycle index	x_2^2
g_8	$(1\,2\,3\,4) \to (4\,3\,2\,1)$	cycle index	x_2^2

Thus the cycle index of G is $(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4)/8$.

Now suppose we want to color the elements of X; that is, we have a finite set Y of colors, $|Y| = r$, and a coloring of X is a function[4] $f : X \to Y$. For any $g \in G$, we regard the colorings f and fg as indistinguishable or equivalent; and a pattern is an equivalence class of colorings. Then Pólya's first theorem is as follows.

Theorem 5.1 The number of patterns is $Z(G; r, r, \dots, r)$.

Example 5.1 (continued) Suppose the vertices are to be colored red or blue. Then $r = 2$, and the number of patterns is $(16 + 16 + 12 + 4)/8 = 6$. In fact, the patterns are represented by the 6 colorings: RRRR, BRRR, BBRR, BRBR, RBBB, BBBB.

R■R	B■R	B■B	B■R	R■B	B■B
R■R	R■R	R■R	R■B	B■B	B■B

We now describe Pólya's second theorem. This is really the 'big' theorem and the first theorem is, in fact, deducible from it. Let us enumerate the elements of Y (the 'colors') as $y_1, y_2, \dots, y_r$.

Theorem 5.2 Evaluate $Z(G; x_1, x_2, \dots, x_n)$ at $x_i = \sum_{j=1}^{r} y'_j$. *Then the coefficient of* $y_1^{n_1} y_2^{n_2} \dots y_r^{n_r}$ *is the number of patterns assigning the color* y_j *to* n_j *elements*[5] *of* X.

Example 5.1 (continued) For the symmetries of the square we know that

$$Z(G) = (x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)/8.$$

Thus if $Y = \{R, B\}$, then the evaluation of $Z(G)$ at $x_i = R' + B'$ yields

$$((R+B)^4 + 2(R+B)^2(R^2 + B^2) + 3(R^2+B^2)^2 + 2(R^4+B^4))/8 = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4.$$

(It is, of course, no coincidence that this polynomial is homogeneous (of degree $|X|$ and symmetric. Thus the Pólya Enumeration Theorem tells us that there is one pattern with 4 red vertices (obvious); one pattern with 3 red vertices and 1 blue vertex, represented by the coloring *RRRB*; 2 patterns with 2 red vertices and 2 blue vertices, represented by the colorings *BRRB* and *RBRB*, and the remaining possibilities are analyzed by considerations of symmetry.)

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[4] We speak of a *coloring* of X; this may be literally true or it may merely be a metaphor for a rule for dividing the elements of X into disjoint classes

[5] Of course $\sum_{j=1}^{r} n_j = n.$

Now, given a pattern, there are the various functions $fg : X \to Y$, where f is a fixed coloring and $g \in G$, in that equivalence class of colorings. These are the *homologues*, or, more precisely, the *homologues of f*. Let us revert to our example.

Example 5.1 (continued) As we have seen, there is one coloring in which all vertices are colored red. There is only one homologue, namely RRRR.

There is one coloring in which 3 vertices are colored red and one blue. There are 4 homologues, namely BRRR, RBRR, RRBR, RRRB.

There are two colorings in which 2 vertices are colored red and 2 blue. In the first there are 4 homologues, namely BBRR, RBBR, RRBB, BRRB.

In the second there are 2 homologues, namely BRBR, RBRB.

The analysis is completed by considerations of symmetry.

Let us show how this conception of homologues agrees with our earlier definition. We are given the group G of permutations of X. Given a coloring $f: X \to Y$, we consider the subset G_0 of G consisting of those g such that $fg = f$, that is, those movements of X which preserve the coloring. It is easy to see (just as easy as in our earlier, simpler situation) that G_0 is a *subgroup* of G. Corresponding to each coset G_0g of G_0 in G we have a coloring fg of X and these colorings run through the pattern determined by f. We have described the set of colorings $\{fg\}$ as the set of *homologues* of the coloring f; as indicated earlier, they are in one-one correspondence with the cosets of G_0 in G.

6. EPILOGUE: PÓLYA AND OURSELVES – MATHEMATICS, TEA AND CAKES[6]

Professor George Pólya (1887 – 1985) emigrated to the United States in 1940 and joined the Mathematics Department at Stanford University in 1942. Although the rest of his professional life was spent at Stanford, he made many trips abroad to accept visiting appointments for short periods of time. During Pólya's visit to the ETH (Zürich) in 1966 he shared an office with Peter Hilton (and PH was a guest at his 80th birthday party, held in Zürich, in 1967).

[6] We present this more personal epilogue to our article at the express request of Dénes Nagy.

In 1969 Pólya was invited by Gerald Alexanderson (Mathematics Department Chairman at Santa Clara University – then and now) to give a colloquium talk at SCU. While there Pólya met Jean Pedersen and was fascinated by (a) the models in her office (some of which are described in [Rec]) and (b) by her lack of knowledge about their symmetry and their usefulness in exemplifying some of the mathematics of polyhedral geometry. After this initial meeting Pedersen visited George and Stella Pólya at their Palo Alto home once a week until his death in 1985[7]. Pedersen and her husband Kent (who shares Pólya's birthday – except for the year!) were guests at Pólya's 90th birthday party, held at Stanford, in 1977 and the Pólyas were guests at the Pedersen's home for Thanksgiving dinner for many successive years.

A typical visit, for Pedersen, included a discussion with Pólya about mathematics. After an hour or so Stella would appear with tea and cakes, or cookies, and the three of them would turn their attention to current events and politics. It was during this time that Pedersen learned about proper rotation groups (knowledge that Pólya acquired from Felix Klein himself) and the Pólya Enumeration Theorem, about Euler's famous formula connecting vertices, edges and faces of a polyhedron, and about the formula Descartes discovered concerning the total angular deficiency of a polyhedron. Pedersen found herself studying very hard[8] and looking forward to discussing the new-found aspects of her own models. Pólya and Pedersen also discussed pedagogy and, in fact, Pedersen was Pólya's last co-author (see [PP]).

In 1978 Pedersen was asked to try to get George Pólya and Peter Hilton together[9] in Seattle at the joint annual meeting of the American Mathematical Society and the Mathematical Association of America, to discuss "How to and How *Not* to Teach Mathematics". The suggestion was that Hilton should discuss "How Not to Teach Mathematics" and this would be followed by Pólya giving "some Rules of Thumb for Good Teaching". Pólya agreed to participate on the condition that Pedersen would handle the travel details of getting him to and from Seattle. Hilton also gave only conditional approval for the plan. Hilton's idea was that it would be much more interesting, and effective, if he were to *demonstrate* a thoroughly bad mathematics lecture (instead of simply talking about it). Hilton also suggested that Pedersen should be the moderator for the program.

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[7] After George Pólya's death, Pedersen continued to visit Stella Pólya at least once a week until her death in 1989, just before her 94th birthday.

[8] Figure 9 is an example, in Pólya's own handwriting of a page he once gave Pedersen saying "see if you can figure out what it means". It is connected with what we've been writing about in this article, so we leave the reader to do Pólya's homework assignment for the week! The only hint Pólya gave was to say that H = *H*exahedron (cube).

[9] This was how Hilton and Pedersen met and began a collaboration that has resulted in over 70 papers and four books – to date!

Groups S_n symmetric, $n!$
A_n alternating, $\frac{n!}{2}$

C_n	D_n	I	H	O	D	T	
cyclic	dihed.	$T = A_4$		$O = S_4$	$I = A_5$		group
n	$2n$	$12 = \frac{4!}{2}$		$24 = 4!$	$60 = \frac{5!}{2}$		order

F face D_i diagonal$_i$ | Df face Diag. | * rect. coord.
V vertex $A_{i/n}$ axis $(C_n, \frac{2\pi}{n})$ of symmetry (rot.)
E edge P_i plane of symmetry$_i$
$\measuredangle$ face angle

	whole	why not?	$A_{1/2}$	
T	I		D_2	(next index, opposite order)
	1	2	3	
	12	6	4	
	C_1	C_2	C_3	
	$\measuredangle$	E, P	F, V	

	whole	inser T	$A_{1/4}$, P_4	$A_{1/3}$
H | O	O	T	D_4	D_3
	1	2	3	4
	24	12	8	6
	C_1	C_2	C_3	D_2; C_4
	$\measuredangle$	E	V | F	(P_2) F | V

	whole	why not?			inser H	$A_{1/5}$
D | T	I				T	D_5
	1	2	3	4	5	6
	60	30	20	15	12	10
	C_1	C_2	C_3	D_2	C_5	D_3
	$\measuredangle$	E	V | F	$P_3 A_{1/2}$	F | V	$A_{1/3}$

Figure 9

All conditions were met and the Seattle presentation duly took place. It was a tremendous success. Hilton's part was hilarious and some said it nearly ruined the rest of the meeting as participants saw many of Hilton's intentional errors *unintentionally* repeated by some of the other speakers. Pólya's contribution was, as you might expect, superb and had the unmistakable mark of a master teacher. A month or so later Pedersen was asked by the National Council of Teachers of Mathematics to arrange that the Hilton-Pólya performance be repeated at their San Diego meeting in the fall of 1978 so that it could be videotaped. After a few more meetings with tea and cakes, and some long distance calls, this was done.

At the San Diego meeting Pedersen invited Hilton to visit SCU in October to give a colloquium talk. He did, and when he saw the models in Pedersen's office they again sparked long discussions, but this time the discussions centered on the differences between the ways geometers and topologists classify surfaces.

In 1982, while Peter Hilton was on sabbatical leave as a visiting professor at the ETH and Pedersen was visiting there for a quarter, they began looking seriously at the paper-folding. Hilton suggested to Pedersen that she should try to devise a really *systematic* way of constructing the polygons from the folded strips (since the 2^n+1-gons seemed to have very special features that didn't generalize). The first result of Hilton's suggestion was the *FAT*-algorithm. This innocent-looking algorithm, in fact, opened the flood gates for both the development of the general folding procedures and the number theory that grew out of the paper-folding.

After 1978 whenever Hilton visited SCU he went with Pedersen to visit the Pólyas and together they continued the tradition of mathematics, tea and cakes. In 1981 Hilton and Pedersen, along with Alexanderson, cooperated with Pólya to bring out the Combined Edition of *Mathematical Discovery* (see [P2])[10]. During many of the tea parties at Pólya's home, Pólya talked about his idea of homologues, and on one occasion told us that he had never written about them and that someday he would like us to write about them – in fact, he extracted a promise from us that we would do so. Thus we are very grateful to Dénes Nagy for giving us such a splendid opportunity to fulfill our promise to our dear friend and teacher George Pólya, and to convey the flavor, and a few of the details, of our friendly relationship with him.

[10] Alexanderson updated the references, Hilton wrote a foreword, and Pedersen provided an expanded (and less esoteric) index.

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