Symmetry: Culture and Science

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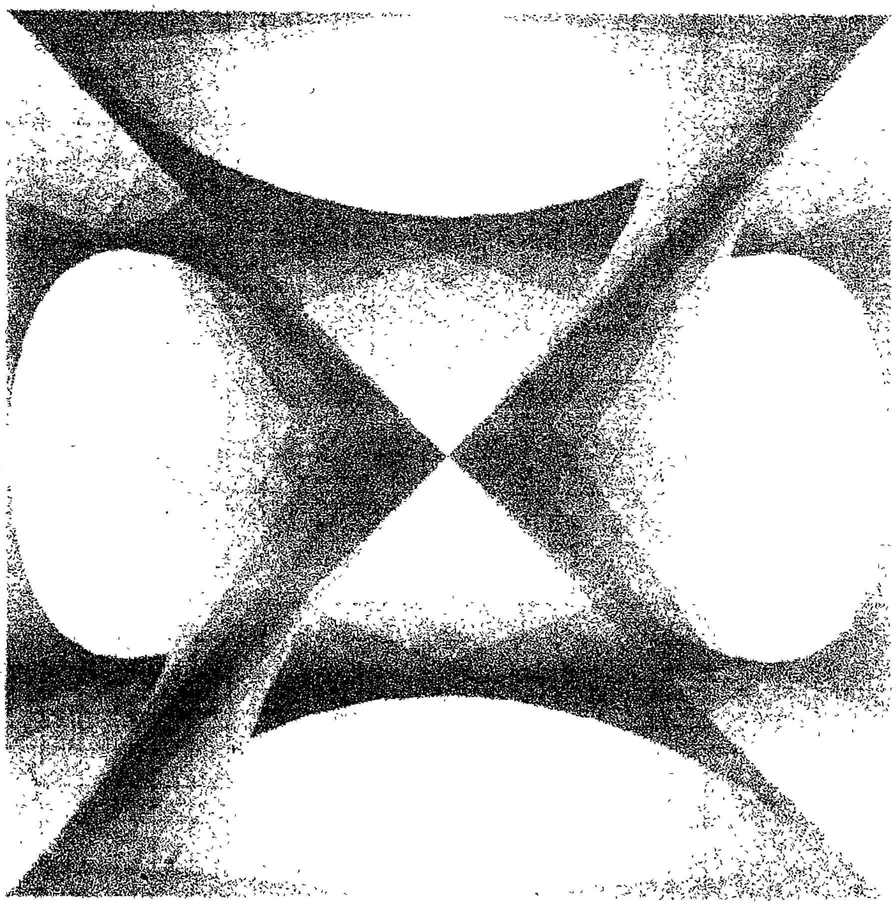

INTERNATIONAL SOCIETY FOR THE INTERDISCIPLINARY STUDY OF SYMMETRY (ISIS-SYMMETRY)

See the *Advisory Board* inside back cover

Symmetry: Culture and Science

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SYMMETROSPECTIVE: A HISTORIC VIEW

THE BIRTH OF THE TUNNEL DIODE, AND THE SEMICONDUCTOR SUPERLATTICE

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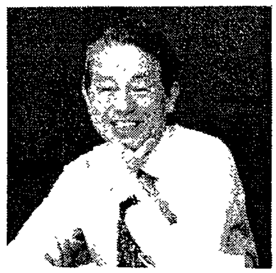

Abstract: *The paper gives an insight in the history of two discoveries by the author. Having looked back to the birth of the* tunnel diode, *the reader is made acquainted with an effect born with an idea of a special periodicity (translational symmetry) property, namely the* semiconductor superlattice, *and how did it lead to physical applications. Finally lessons are summarized.*

DYNAMIC EVOLUTION IN SCIENCE AND TECHNOLOGY: THE BIRTH OF THE TUNNEL DIODE

Recently, a friend of mine showed me a proceeding of a meeting held in Osaka, Japan some forty-one years ago. Apparently, at that meeting, I gave a talk on the subject of transistors. The year was 1953, and I was not yet thirty years of age. According to the proceeding I began my speech with the following words:

"Since the beginning of this century, there has been great progress made in telecommunications thanks to the tremendous advances in the vacuum tube. However, we appear to be approaching the limits of progress in vacuum tube design and production. Now that we have reached this stage, it can be said, quite ironically, that the vacuum tube itself has become the greatest obstacle to the next stage of dramatic progress in the field of telecommunications.

I would like to talk a little today about semiconductor amplifiers, which made their appearance with the promise to overcome the limitations of the vacuum tube. As you know, these semiconductor amplifiers came to be known as transistors.

We first became aware of the transistor in 1948. Our source of information was a paper jointly written by John Bardeen and Walter Brattain of Bell Telephone Laboratories, which appeared in Physical Review, an American journal of physics. The term 'transistor' was used for the first time in the title of that paper and was described as a semiconductor triode."

As an aside, let me read you the words which appear below the bust of Alexander Graham Bell, the inventor of the telephone, which stands in the foyer of the main entrance to Bell Laboratories. "Leave the beaten track occasionally and dive into the woods. You will be certain to find something that you have never seen before." I suppose that a number of great inventions and discoveries have been made by searching in the woods. The transistor was one such case. The inventors, the aforementioned Bardeen and Brattain, were awarded the Nobel Prize in 1956 along with William Shockley for this, the most remarkable invention of the twentieth century.

It was in 1947 that I finished my studies at the Physics Department of Tokyo University and moved on to life as a researcher. My area of research was vacuum tube materials, and I was employed by Kobe Industries, a company, which no longer exists today. Coincidentally, that was the same year the transistor was invented. Looking back on history, the decade of the 1950s was quite remarkable. It was a period of technological innovation in the field of electronics: The epoch-making evolution from vacuum tubes to transistors led to a vast improvement in the performance of all electronic products, including consumer-oriented goods, telecommunications and data processing. It is no exaggeration to say that today's information oriented society was made possible by this technological innovation. Japanese electronics industries played an important role in this evolution and thus made a significant contribution to the economic growth in this country.

I feel extremely fortunate to have started my career in the midst of such a period of technological innovation. This environment stimulated me, encouraged me, and led me to the work for which I would later be awarded the Nobel Prize. This example of technological transformation stands as an important lesson for us. The transistor is substantially different from the vacuum tube, and no amount of research and improvement of the vacuum tube could have led to the birth of the transistor. There is a tendency, especially in the stable societies, to assume that the future is simply a natural extension of the past and the present. However, during periods of great change, innovations are born which have never existed before, and it is actually these innovations which shape and form the future. Needless to say, it is the power of creativity, which plays the decisive role in this process.

The powers of the human mind can be divided into two major categories. One is *the power of judicial mind*, which allows human beings to understand fundamental principles and to make discretionary judgements. The other is *the power of the creative mind*. The former acts as our tool in analyzing, understanding, selecting, and making fair judgements. The latter involves the ability to create new ideas through perceptiveness and the creative process. It is this creativity which provides the engine for progress and which has sustained the advance of human civilization.

Of course, it is through the mutual interaction of these two powers that a far greater power is generated. If we say that creativity is individualistic and represents the challenge of the future, then the power of discretion can be said to have a non-individualistic aspect and can be essentially concerned with the body of existing knowledge. Having said that, we must be aware of an important issue. That is the academic education provided by our schools is primarily aimed at nurturing the judicial mind. There is no assurance that schooling will necessarily contribute to the development of the creative mind.

Let me again take you back to the 1950s. In 1957, I developed a semiconducting device known as the Esaki tunnel diode. This diode could operate at very high speeds. But the true novelty of this diode is that it was the first quantum electron device. Namely, the wave nature of electrons was manifested for the first time in a semiconductor. This constituted my doctoral thesis, and I was awarded the Nobel Prize in Physics in 1973 for this achievement.

This new diode was highly acclaimed in the United States for its novelty. As a result, I was given the opportunity to go and work in America. One interesting thing about the tunnel diode is the way it was described by different people. Engineers said that I had invented it, while scientists said that I had discovered it. The Nobel Prize called it a discovery, as would be normal for the description of any scientific achievement. On the other hand, the tunnel diode was an invention from the viewpoint of patents and technologies. Inventions and discoveries both represent the fruits of our creative activities, but they clearly point to two different concepts. It is unusual to find something that can be labeled both an invention and a discovery. For instance, Antarctica was discovered, while the light bulb was invented. While I am certain that my own prejudice is partly responsible for the conclusion that I draw from this fact, I nevertheless believe that the tunnel diode stands at the intersection of science and technology.

DO-IT-YOURSELF QUANTUM MECHANICS: THE BIRTH OF THE SEMICONDUCTOR SUPERLATTICE

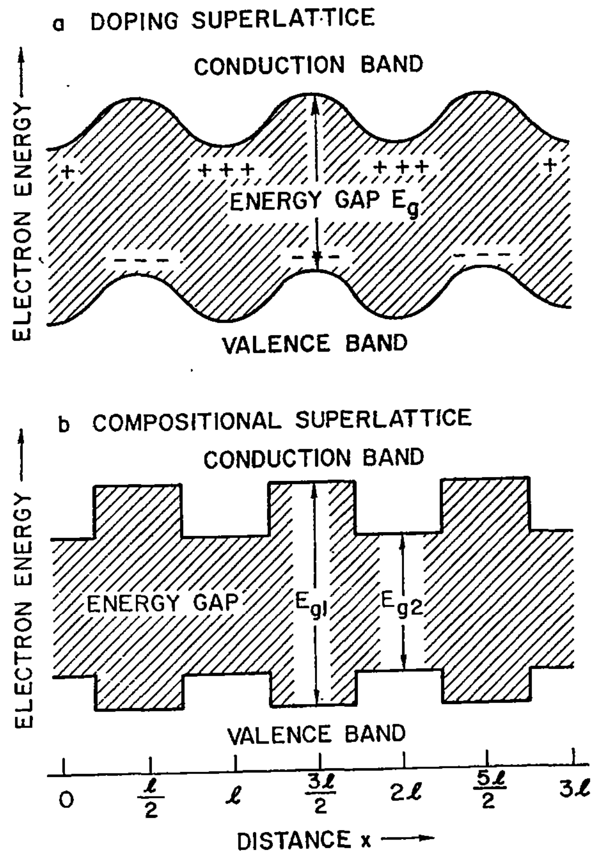

Figure 1: Spatial variations of the conduction and valence band edges in two types of superlattices: **a** doping, **b** compositional

In 1969, research on artificially structured materials was initiated with a proposal for an engineered semiconductor superlattice with a one-dimensional periodic potential by Esaki and Tsu (1969 and 1970). In anticipation of advancement in controlled epitaxy of ultra-thin layers, two types of superlattices were envisioned: Doping and compositional, as shown at the top and bottom of Fig. 1, respectively.

Before arriving at the superlattice concept, we were examining the feasibility of structural formation of potential barriers ad wells that were thin enough to exhibit resonant tunneling (Bohm, 1951). A resonant tunnel diode (Iogansen, 1963) (Tsu and Esaki, 1973) appeared to have more spectacular characteristics than the Esaki tunnel diode (Esaki, 1958): the first quantum electron device consisting of only a single tunnel barrier. It was thought that advanced technologies with semiconductors might be ready for demonstration of de Broglie electron waves. Resonant tunneling can be compared to the transmission of an electromagnetic wave through a Fabry-Perot resonator. The equivalent of a Fabry-Perot resonant cavity is formed by the semiconductor potential well sandwiched between the two potential barriers.

The idea of the superlattice occurred to us as a natural extension of double-, triple- and multiple-barrier structures: The superlattice consists of a series of potential wells coupled by resonant tunneling. An important parameter for the observation of quantum effects in the structure is the phase-coherent length, which approximates the electron mean free path. This depends on bulk as well as on the interface quality of crystals, and also on the temperatures and values of the effective mass. As schematically illustrated in Fig. 2, if characteristic dimensions such as superlattice periods or well widths are reduced to less than the phase-coherent length, the entire electron system will enter a mesoscopic quantum regime of low dimensionality, being in a scale between the macroscopic and the microscopic. Our proposal was indeed to explore quantum effects in the mesoscopic regime.

The introduction of the one-dimensional superlattice potential perturbs the band structure of the host materials, yielding a series of narrow subbands and forbidden gaps, which arise from the subdivision of the Brillouin zone into a series of minizones. Thus, the superlattice was expected to exhibit unprecedented electronic properties. At the inception of the superlattice idea, it was recognized that the long, tailored lattice period provided a unique opportunity to exploit electric field-induced effects. The electron dynamics in the superlattice direction was analyzed for conduction electrons in a narrow subband of a highly perturbed energy-wave vector relationship. The result led to the prediction of the occurrence of a negative differential resistance at a modestly high electric field, which could be a precursor of the Bloch oscillation. The superlattice, apparently, allows us to enter the regime of electric field-induced quantization: The formation of Stark ladders (James, 1949), (Wanniers, 1959 and 1960), for example, can be proved in a (one-dimensional) superlattice (Shokley, 1972), whereas, in natural (three-dimensional) crystals, the existence and real nature of these localized states in a high electric field have been controversial (Zak, 1968 and 1991), (Rabinovitch and Zak, 1971).

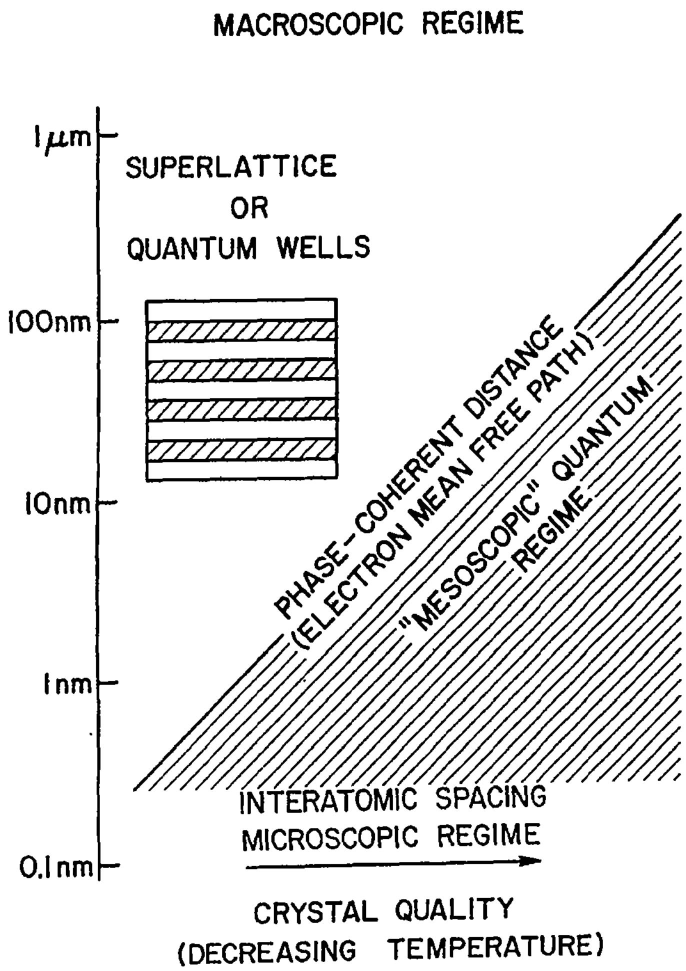

Figure 2: Schematic illustration of a 'mesoscopic' quantum regime (*hatched*) with a superlattice or quantum wells in the *inset*.

This was, perhaps, the first proposal, which advocated to engineer, with advanced thin-film growth techniques, a new semiconductor material designed by applying the principles of the quantum theory. The proposal was indeed made to the US Army Research Office (ARO), a funding agency, in 1969, having daringly stated, with little confidence in a successful outcome at the time, "the study of superlattices and observations of quantum mechanical effects on a new physical scale may provide a valuable area of investigation in the field of semiconductors".

Although this proposal was favorable received by ARO, the original version of the paper (Esaki and Tsu, 1969) was rejected for publication by Physical Review on the referee's unimaginative assertion that it was 'too speculative' and involved 'no new physics'. The shortened version published in *IBM J. Res. Develop.* (Esaki and Tsu, 1970), was selected as a Citation Classic by the Institute for Scientific Information (ISI) in July 1987. Our 1969 proposal was cited as one of the most innovative ideas at the ARO 40th Anniversary Symposium, Durham, North Carolina, 1991.

At any rate, with the proposal, we initiated such a formidable task as to make a '*gedanken*-experiment' a reality. In some circles the proposal was criticized as close to impossible. One of the objections was that a man-made structure with compositional variations in the order of several nanometers could not be thermodynamically stable because of interdiffusion effects. Fortunately, however, it turned out that interdiffusion was negligible at the temperatures involved.

In 1970, Esaki, Chang and Tsu (1970) studied a GaAs-$GaAs_{0.5}P_{0.5}$ superlattice with a period of 20 nm synthesizes with CVD (chemical vapor deposition) by Blakeslee and Aliotta (1970). Although transport measurements failed to reveal any predicted effect, the specimen probably constituted the first strained-layer superlattice having a relatively large lattice mismatch. Early efforts in our group to obtain epitaxial growth of $Ge_{1-x}Si_x$ and $Cd_{1-x}Hg_xTe$ superlattices were soon abandoned because of rather serious technological problems at that time. Instead we focused our research effort on compositional GaAs-$Ga_{1-x}AI_xAs$ superlattices growth with MBE (molecular beam epitaxy). In 1972, Esaki et al. (1972) found a negative resistance in such superlattices, which was, for the first time, interpreted in terms of superlattice effect.

Following the derivation of the voltage dependence of resonant tunnel currents (Esaki, 1958), Chang, Esaki and Tsu (1974) observed the current-voltage characteristics with a negative resistance. Subsequently, Esaki and Chang (1974) measured quantum transport properties in a superlattice with a narrow bandwidth, which exhibited an oscillatory behavior. Tsu et al. (1975) performed photocurrent measurements on superlattices subject to an electric field perpendicular to the plane layers with the use of a semitransparent Schottky contact, which revealed their miniband configurations.

Heteroepitaxy is of great interest for the growth of compositional superlattices. Innovations and improvements in epitaxial techniques such as MBE and MOCVD (metallo organic chemical vapor deposition) have made it possible to prepare high-quality heterostructures. Such structures possess predesigned potential profiles and impurity distributions with dimensional control close to interatomic spacing. This great precision has cleared access to the mesoscopic quantum regime (Esaki, 1986), (Esaki, 1991).

Since a one-dimensional potential can be introduced along with the growth direction, famous examples in the history of one-dimensional mathematical physics, including the above-mentioned resonant tunneling (Bohm, 1951), Kronig-Penney bands (1931), Tamm surface states (1932), Zener band-to-band tunneling (1934), and Stark ladders including Bloch oscillations (James, 1949), (Wannier, 1959 and 1960), (Shockley, 1972), all of which had remained more or less textbook exercises, could, for the first time, be practiced in a laboratory. Thus, do-it-yourself quantum mechanics is now possible, since its principles dictate the details of semiconductor structures (Esaki, 1992).

RETROSPECTING TO THE TWO DISCOVERIES

We have witnessed remarkable progress in semiconductor research of superlattices and quantum wells over the last two decades. Our original proposal (Esaki and Tsu, 1969) and pioneering experiments apparently triggered a wide spectrum of experimental and theoretical investigations on this subject. A variety of engineered structures exhibited extraordinary transport and optical properties, which may not even exist in any natural crystal. Thus, this new degree of freedom offered in semiconductor research through advanced material engineering has inspired many ingenious experiments, resulting in observations of not only predicted effects but also totally unknown phenomena have been found. As a measure of the growth of the field, Fig. 3 shows the number of papers related to the subject and the percentage of the total presented at the biennial International Conference on the Physics of Semiconductors. After 1972, when the first paper (Esaki *et al.*, 1972) came on the scene, the field went through a short period of incubation and then experienced a phenomenal expansion in the 1980s. It appears that nearly half of semiconductor physicists in the world are working in this area. Activity in this new frontier of semiconductor physics has in turn given immeasurable stimulus to device physics, provoking new ideas for applications. Thus, a new class of transport and opto-electronic devices has emerged. In this interdisciplinary research, there have been numerous beneficial cross-fertilizations. I hope this article provides some flavor of the excitement associated with the birth of the semiconductor superlattice.

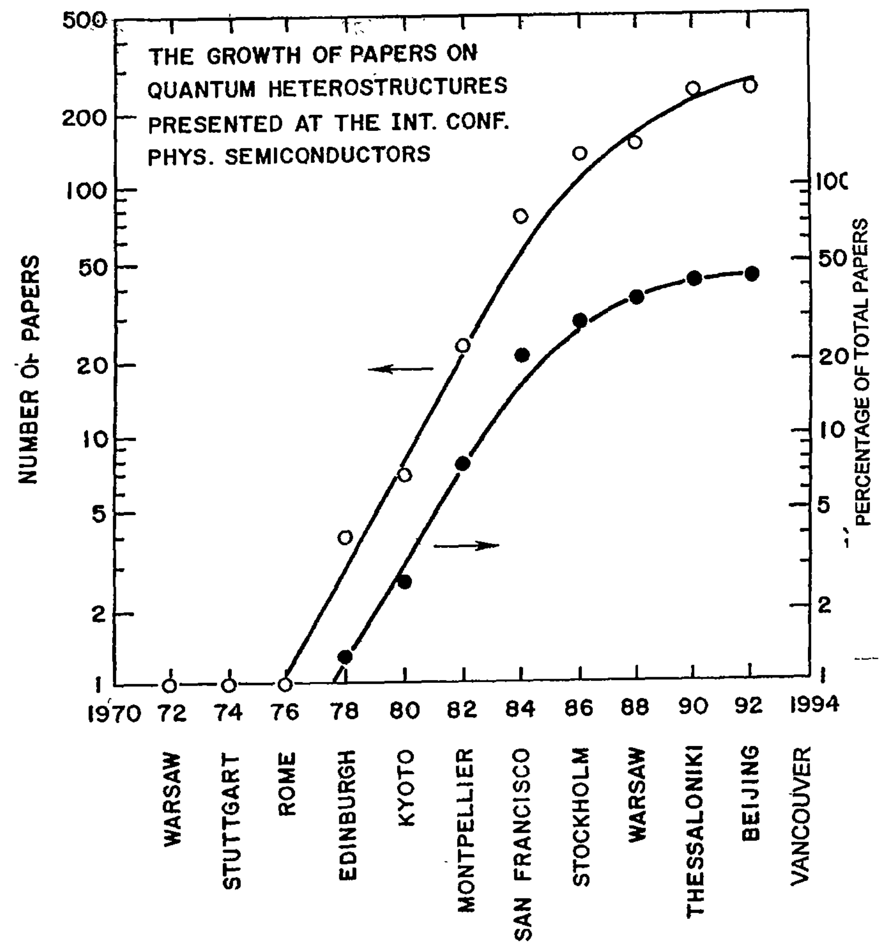

Figure 3: Growth in relevant papers in the biennial International Conference on the Physics of Semiconductors.

When I left IBM Research in New York and, after living in the United States for 32 years, I returned to Japan to take up my position as President of the University of Tsukuba, was not a move I had anticipated. The call came when a group of Tsukuba professors, who thought that the University needed 'new blood', nominated and elected me to be President. That was a minor revolution, which may upset some senior faculty members at the University. The circumstances of my election were indeed most unusual: I was the first president of any of Japan's 98 national universities to come from outside academia, and, what's more, I was working and living outside Japan. As you know, the country is rather famous for university inbreeding: the career path of choice is to obtain all degrees from the same university and then to stay there as a professor in the footsteps of the mentor. In view of this environment, my appointment was exceptional. In physics terms, it was almost a forbidden transition.

Today, I find myself doing my best to navigate through the academic woods.

The University of Tsukuba is located at the center of Tsukuba Science City, a planned community set up by the Japanese government three decades ago as a magnet for scientific and technical talents. To date more than 40 national research institutes have been drawn there - representing nearly half of the government's research facilities and 200 industrial laboratories. The High Energy Physics Laboratory (KEK), which you may know, is also at Tsukuba ten minutes drive from my office. In this rather unique environment, I am trying to increase collaboration and exchange with nearby national institutes and industrial research laboratories. Among other things, I plan to expand and strengthen Tsukuba's graduate education.

In closing, I would like to share with you a list of five 'don't'-s which anyone with an interest in realizing his or her creative potential should follow. Who knows, it may even help you win a Nobel Prize.

Rule number one: Don't allow yourself to be trapped by your past experiences. If you allow yourself to get caught up in social conventions or circumstances, you will not notice the opportunity for a dramatic leap forward when it presents itself. Looking back at the history of the Nobel Prize, you will notice that most of the laureates have received the Nobel Prize for work they had done during their thirties. In my case I was 32 years old when I developed the 'Esaki tunnel diode'. The point that I am trying to make is that younger people are able to look at things with a clear vision, one that is not clouded by social conventions and past history.

Rule number two: Don't allow yourself to become overly attached to any authority in your field - the great professor, perhaps. By becoming closely involved with the great professor, you risk losing sight of yourself and forfeiting the free spirit of youth.

Although the great professor may be awarded the Nobel Prize, it is unlikely that his subordinates will ever receive it.

Rule number three: Don't hold on to what you don't need. The information-oriented society facilitates easy access to an enormous amount of information. The brain can be compared to a personal computer with an energy consumption of about 25 watts. In terms of memory capacity or computing speed, the human brain has not really changed since ancient times. Therefore, we must constantly be inputting and deleting information, and we should save only the truly vital and relevant information. As the president of a university, I had the opportunity to meet many people and to exchange *meishi* (name card) with them. I try to discard the name cards as soon as possible, so that I always leave maximum memory space open. I'm kidding, of course.

Rule number four: Don't avoid confrontation. I myself became embroiled in some trouble with the company I was working for many years ago. At times, it is necessary to put yourself first and to defend your own position. My point is that fighting is sometimes unavoidable for the sake of self-defense.

Rule number five: Don't forget your spirit of childhood curiosity. It is the vital component for imagination.

Having listed five rules, let me say that they do not constitute the sufficient conditions for success. They are merely suggested guidelines. Good Luck!

REFERENCES

Blakeslee, A.E Aliotta, C. F. (1970) *IBM J. Res. Develop.* 14, 686.

Bohm, D. (1951) *Quantum Theory*, Englewood Cliffs, N.J.: Prentice Hall, p. 283.

Chang, L. L., Esaki, L, and Tsu, R. (1974) *Appl. Phys. Lett.* 24, 593.

Esaki, L. (1958) *Phys. Rev.* 109, 603.

Esaki, L. (1986) *IEEE J. Quantum Electron.*, QE-22, 1611.

Esaki, L. (1991) In: Esaki, L., (ed.), *Highlights in Condcensed Matter Physics and FutureProspects,* New York: Plenum Press, p. 55.

Esaki, L. (1992) *Phisica Scripta*, T42, 102.

Esaki, L. Chang, L. L., and Tsu, R. (1970) *Proc. 12th Int. Conf. Low Temp. Phys.*, Kyoto, Japan, p. 551.

Esaki, L., Chang, L. L. (1974) *Phys. Rev. Lett.* 33, 495.

Esaki, L., Chang, L. L. , Howard, E. E., and Rideout, V. L. (1972) *Proc. 11th Int. Conf. Phys. Semicondustros*, Warsaw, Poland, p. 431.

Esaki, L., Tsu, (1969) *R.. IBM Research Note* RC-2418.

Esaki, L., Tsu, R. (1970) *IBM J. Res. Develop.* 14, 61.

Iogansen, L. V. (1963) *Zh. Eksp. Teor. Fiz.* 45, 207; (1964) *Sov. Phys. JETP* 18, 146.

James, H.M. (1949) *Phys. Rev.* 76, 1611.

Kronig, R. de L., Penney, W. G. (1931) *Proc. Roy. Soc.,* London, A130, 499.

Rabinovitch, A., Zak, (1971) *J. Phys. Rev.* B4, 2358.

Shockley, W. (1972) *Phys. Rev. Lett.* 28, 349.

Tamm, I. (1932) *Phys. Z Sowjetunion*, 1, 733.

Tsu, R., Chang, L. L., Sai - Halasz, G. A., and Esaki, L. (1975) *Phys. Rev. Lett.* 34, 1509.

Tsu, R , Esaki, L. (1973) *Appl. Phys. Lett.* 22, 562.

Wannier, G. H. (1959) *Elements of Solid State Theory*, Cambridge: Cambridge University Press, p. 190; (1960) *Phys. Rev.* 117, 432.

Zak, J. (1968) *Phys. Rev. Lett.* 20, 1477; (1991) *Phys. Rev.* B43, 4519.

Zener, C. (1934) *Proc. Roy. Soc.*, London 145, 523.

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SYMMETRY: SCIENCE & ART

SYMMETRY IN THEORY – MATHEMATICS AND AESTHETICS

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1. INTRODUCTION

In the course of our very pleasant correspondence with Professor Dénes Nagy about our contribution to the Proceedings of The Third International Conference on Symmetry, held in Washington D. C. in August 1995, the idea took shape of our writing two articles about the symmetry of geometrical figures, one of a practical nature, the other of a more theoretical nature. Thus this article is a companion to the article *Symmetry in Practice* (in this issue), which describes very practical ways of constructing regular polygons and polyhedra. We subtitle that article *Recreational Constructions* – and refer to it henceforth as [Rec] - because the constructions, involving the use of colored paper, have an undoubted recreational flavor. However, it is our conviction, based on many years' experience, that the execution of such model constructions can play a vital role in enlivening and enriching the study of geometry, especially if the mathematical theory underlying the constructions features prominently. Thus it is our strong hope that readers of [Rec] will be encouraged to move on to this more theoretical sequel, to learn why the constructions work and better to understand the nature of symmetry. We also set the mathematical development in its historical context and show explicitly how the geometry is related to other parts of mathematics – real analysis, number theory, group theory, combinatorics. Such connections should, in our view, form an integral part of the teaching and learning of any part of mathematics. We will refer to the present article, briefly, as [Math].

In Section 2 we link the practical instructions of [Rec] to a mathematical discussion of the parameters of the polygons constructed. Thus we answer two questions which stand in a converse relation to each other, namely, (i) given the folding instructions for our tape, when will we be able to produce a regular convex polygon and how many sides will it have, and (ii) given a number p, what folding instructions will produce a regular p-sided polygon (or p-gon)?

Having learnt in Section 2 how to construct certain regular figures, we turn in Section 3 to the question of just what we should understand by the symmetry of a geometrical figure, and how it should be measured. From a mathematical point of view it makes very little sense to say that a given figure A is symmetrical,[1] but we have a precise idea of its *group of symmetries*, that is, of the subgroup of the group of Euclidean movements of the ambient space of A under which A is invariant. Based on this idea, we can give meaning to the statement that figure A is more symmetrical than figure A'. However, we need to bear in mind that the symmetry group of A depends on our convention as to what *is* the ambient space of A. Thus if A is a circle, then its symmetry group as a subset of the plane depends on whether we allow *reflexions* of the plane or not (note that a reflexion of the plane *cannot* be achieved by a movement *in* the plane, but only by a movement in 3-dimensional space).

Another important aspect of symmetry arises when one considers actual physical models of geometrical configurations. Suppose we have constructed a model M of the figure A by braiding together colored strips; A may be a regular dodecahedron, say. Our model cannot have more symmetry than A itself – but it may well have less. For to every symmetry g of A we have a movement of the model M which may create an image Mg recognizably different from M because of the arrangement of colors. Thus the symmetry group of M may only be a subgroup of the symmetry group of A; and aesthetics come into the story here by requiring the symmetry group of M to be as large as possible. Thus can mathematics contribute to the study of aesthetics!

It turns out (not surprisingly!) that, if B is a subset of A and if G_A is the symmetry group of A, then the set of images of B under the action of elements of G_A is the set of *homologues* of B in the sense of George Pólya; we explain this in Section 4. Actually, Pólya never wrote down his work on homologues (which, so far as we know, he only discussed in the case where A is a Platonic solid), but, when he was a very old man, he

[1] We might perhaps say that A is symmetrical if there is a non-trivial Euclidean movement sending A to itself. A classification of symmetricality due to Kepler is to be found in [C2].

asked us to write it down for him, and we are proud and happy to have this opportunity to do so (see Figure 9 of for the only extant copy of his original notes on the subject).

In Section 5 we explain Pólya's famous *Enumeration Theorem*, one of the most important theorems of that branch of mathematics known as *combinatorics*. We apply it to the symmetries of geometrical figures, where parts of the figures are colored in prescribed ways, and again recover the notion of homologue from the formulation of the theorem.

The final section is an informal epilogue, describing our relationship with George Pólya. We are grateful to Dénes Nagy for inviting us to write these two articles, [Rec] and [Math], and for persuading us to include some personal reflections on our good fortune in knowing that remarkable man so well.

2. 2-PERIOD FOLDING PROCEDURES FOR CONSTRUCTING REGULAR POLYGONS AND A GENERALIZATION

We agreed in [Rec] that, however symmetry is defined, the most symmetric polygons are the regular polygons, both the regular convex polygons and the regular star polygons (see [C1]). This is our justification for devoting this section to a particularly easy way of constructing examples of such polygons; in fact, we will confine ourselves, in this article, to the construction of regular *convex* polygons. The *practical* problems of such constructions are discussed in our companion article [Rec].

To set our problem in its historical context, we should really begin with the Greeks and their fascination with the challenge of constructing regular convex polygons. We will refer to such p-sided polygons as regular convex p-gons, and we may even suppress the word *convex* if no confusion would result. The Greeks, working on these problems about 350 B.C., restricted themselves to constructions using only what we call *Euclidean tools*, namely an unmarked straightedge and a compass. No doubt the Greeks would have liked to be able to describe Euclidean constructions whenever possible. However, they were only able to provide such constructions for regular convex polygons having p sides, where

$$p = 2^c p_0, \text{ with } p_0 = 1, 3, 5, \text{ or } 15.$$

About 2,000 years later Gauss (1777 – 1855) showed that Euclidean constructions were possible only rarely. He proved that a Euclidean construction is possible *if* and *only if* the number of sides p is of the form $p = 2^c\prod\rho_i$, where the ρ_i are distinct Fermat primes – that is, primes of the form $F_n = 2^{2^n}+1$.

Gauss's discovery was remarkable – it tells us precisely which regular p-gons admit a Euclidean construction, provided, of course, that we know which Fermat numbers F_n are prime. In fact, not all Fermat numbers are prime. Euler (1707 – 1783) showed that $F_5 = 2^{2^5}+1$ is not prime, and although many composite Fermat numbers have been identified, to this day the only known prime Fermat numbers are

$$F_0 = 3,\ F_1 = 5,\ F_2 = 17,\ F_3 = 257 \text{ and } F_4 = 65537.$$

Thus, even with Gauss's contribution, there exists a Euclidean construction of a regular p-gon for very few values of p, and even for these p we do not in all cases know an explicit construction. For example, in *The World of Mathematics* [N] we read:

> Simple Euclidean constructions for the regular polygons of 17 and 257 sides are available, and an industrious algebraist expended the better part of his years and a mass of paper in attempting to construct the F_4 regular polygon of 65,537 sides. The unfinished outcome of all this grueling labor was piously deposited in the library of a German university.

Despite our knowledge of Gauss's work we still would like to be able to construct (somehow) *all* regular p-gons. Our approach is to redefine the question so that, instead of exact constructions, we will ask *for which $p \geq 3$ is it possible, systematically and explicitly, to construct an arbitrarily good approximation to a regular p-gon*? We take it as obvious that we can construct a regular p-gon *exactly* if p is a power of 2. What we will show is that it is possible, simply and algorithmically, to construct an *approximation* (to any degree of accuracy) to a convex p-gon for any value of $p \geq 3$. In fact, we will give explicit (and uncomplicated) instructions involving only the folding of a straight strip of paper tape in a prescribed periodic manner.

Although the construction of regular convex p-gons would be a perfectly legitimate goal by itself, the mathematics we encounter is generous and we achieve much more. In the process of making what we call the *primary crease lines* used to construct regular convex p-gons we obtain tape which can be used to fold certain (but not all) regular star polygons. It is not difficult to add *secondary crease lines* in order to obtain tape that may be used to construct the remaining regular star polygons.

As it turns out, the mathematics we encounter, in validating our folding procedures, leads quickly and naturally to questions, and hence to new results, in number theory.

Those interested may consult [HP3]. In the interests of mathematical simplicity, as we have said, we will confine attention, in this article, to convex polygons. The more ambitious reader, interested in star polygons, may consult [C1, HP2, 3].

Let us begin by explaining a precise and fundamental folding procedure, involving a straight strip of paper with parallel edges (adding-machine tape or ordinary *unreinforced* packaging gummed tape work well), designed to produce a regular convex p-gon. For the moment assume that we have a straight strip of paper that has *creases* or *folds* along straight lines emanating from vertices, which are equally spaced, at the top edge of the tape. Further assume that the creases at those vertices, labeled A_{nk}, on the top edge, form identical angles of π/p with the top edge, as shown in Figure 1(a). If we fold this strip on $A_{nk}A_{nk+2}$, as shown in Figure 1(b), and then twist the tape so that it folds on $A_{nk}A_{nk+1}$, as shown in Figure 1(c), the direction of the top edge of the tape will be rotated through an angle of $2\pi/p$. We call this process of *folding and twisting* the *FAT-algorithm.*

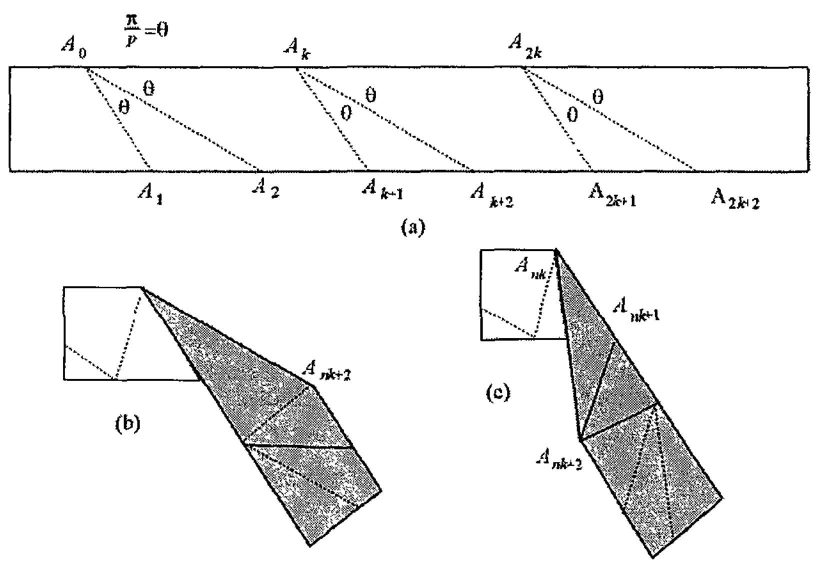


Figure 1

Now observe that if the FAT-algorithm is performed on a sequence of angles, each of which measures π/p, at the first p of a number of equally spaced locations along the top of the tape, in our case at A_{nk} for $n = 0, 1, 2, \dots, p\text{-}1$, then the top edge of the tape will have turned through an angle of 2π, so that the point A_{pk} will then be coincident with A_0. Thus the top edge of the tape visits every vertex of a regular convex p-gon, and thus itself describes a regular p-gon. A picture of the tape with its crease lines, and the resulting start of the construction of the regular p-gon, is given in Figure 2. Notice that we have *not* adhered there to our systematic enumeration of the vertices on the two edges of the tape that play a role in the construction. (The enumeration has served its purpose!)

Notice, too, that if we had the strip of paper shown in Figure 2(a), with its crease lines, we could then introduce *secondary crease lines* bisecting each of the angles nearest the top edge of the tape and this tape could then be used to construct a regular $2p$-gon with the *FAT*-algorithm. We could then, in principle, repeat this secondary procedure, as often as we wished, to construct regular $4p$-gons, $8p$-gons,... It is for this reason that we only need to concern ourselves with devising primary folding procedures for regular polygons having an *odd* number of sides in order to be able to assure ourselves we can, indeed, fold *all* regular polygons.

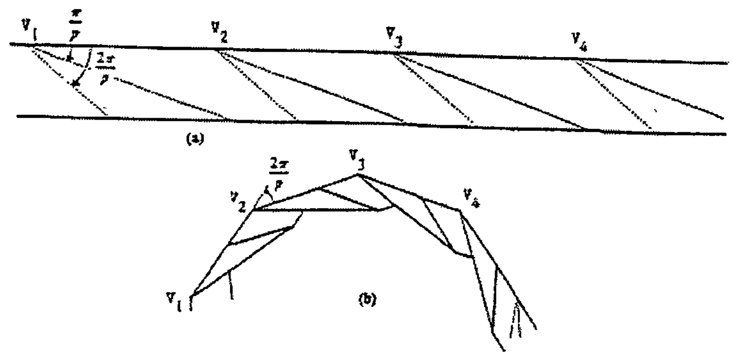


Figure 2

Now, since the regular convex 7-gon is the first polygon we encounter for which we do not have available a Euclidean construction, we are faced with a real difficulty in making available a crease line making an angle of $\pi/7$ with the top edge of the tape. We proceed by adopting a general policy, that we will eventually say more about - we call it our *optimistic strategy.* Assume that we can crease an angle of $2\pi/7$ (certainly we can come close) as shown in Figure 3(a). Given that we have the angle of $2\pi/7$, then simply folding the top edge of the strip DOWN to bisect this angle will produce two adjacent angles of $\pi/7$ at the top edge as shown in Figure 3(b). (We say that $\pi/7$ is the *putative* angle on this tape.) Then, since we are content with this arrangement, we go to the bottom of the tape, and *now we really start the folding procedure.*

We observe that the angle to the right of the last crease line is $6\pi/7$ – and our policy, as paper folders, is that we always avoid leaving even multiples of π in the numerator of any angle next to the edge of the tape, so we bisect this angle of $6\pi/7$, by bringing the bottom edge of the tape UP to coincide with the last crease line as shown in Figure 3(c). We settle for this (because we are content with an odd multiple of π in the numerator) and go to the top of the tape where we observe that the angle to the right of the last crease line is $4\pi/7$ – and, in accordance with our stated policy, we bisect this angle twice, each time bringing the top edge of the tape DOWN to coincide with the lase crease line, obtaining the arrangement of crease lines shown in Figure 3(d). But now we notice something miraculous has occurred! If we had really started with an angle of exactly $2\pi/7$, and if we now continue introducing crease lines by repeatedly folding the tape UP once at the bottom and DOWN twice at the top, we get precisely what we want; namely, pairs of adjacent angles, measuring $\pi/7$, at equally spaced intervals along the top edge of the tape. Let us call this folding procedure the U^1D^2- or D^2U^1-*folding procedure* and call the strip of creased paper it produces[2] U^1D^2- or D^2U^1-*tape*.

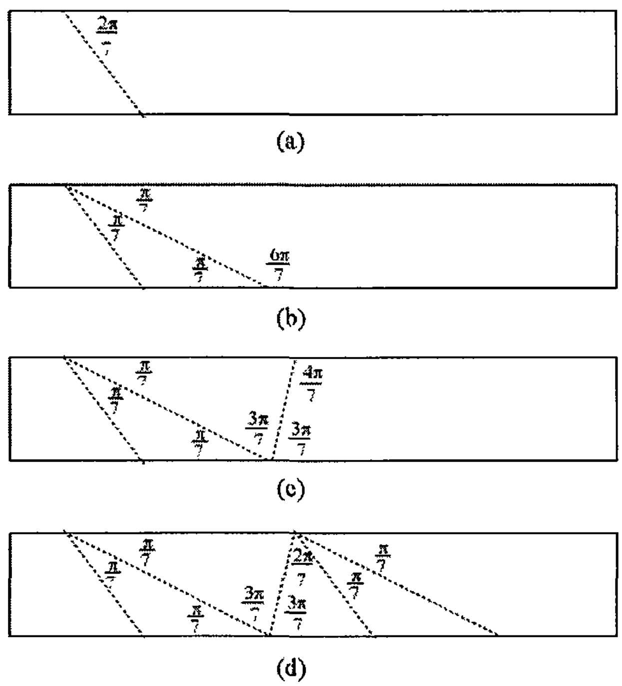


Figure 3

[2] It is our habit to refer to D^2U^1-tape, but this choice is quite arbitrary.

We suggest that before reading further you get a piece of paper and fold an acute angle that you regard as a good approximation to $2\pi/7$. Then fold about 40 triangles using the D^2U^1-folding procedure just described, throw away the first 10 triangles, and try to construct the *FAT* 7-gon shown in Figure 4(b). You will have no doubt that what you have created is, in fact, a 7-gon, but you may wonder *why* it should have worked so well. In other words, how can we *prove* that this evident convergence must take place? One approach is to admit that the first angle folded down from the top of the tape in Figure 3(a) might not have been precisely $2\pi/7$. Then the bisection forming the next crease would make two acute angles nearest the top edge in Figure 3(b) only approximately $\pi/7$; let us call them $\pi/7 + \varepsilon$ (where ε may be either positive or negative). Consequently the angle to the right of this crease, at the bottom of the tape, would measure $6\pi/7 - \varepsilon$. When this angle is bisected, by folding up, the resulting acute angles nearest the bottom of the tape, labeled $3\pi/7$ in Figure 3(c), would, in fact, measure $3\pi/7$-$\varepsilon/2$, forcing the angle to the right of this crease line at the top of the tape to have measure $4\pi/7 + \varepsilon/2$. When this last angle is bisected twice by folding the tape down, the two acute angles nearest the top edge of the tape will measure $\pi/7 + \pi/2^3$. This makes it clear that every time we repeat a D^2U^1-folding on the tape the error is reduced by a factor of 2^3.

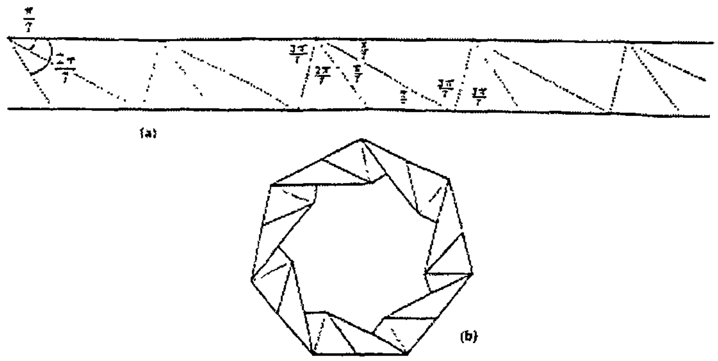


Figure 4

Now it should be clear how our *optimistic strategy* has paid off. By blandly *assuming* we have an angle of $\pi/7$ to begin with, and folding accordingly, we *get what we want* – successive angles at the top of the tape which, as we fold, rapidly get closer and closer to $\pi/7$! A truly remarkable vindication of our optimistic strategy!

In practice the approximations we obtain by folding paper are quite as accurate as the *real world* constructions with a straight edge and compass - for the latter are only perfect in the mind. In both cases the real world result is a function of human skill, but our procedure, unlike the Euclidean procedure, is very forgiving in that it tends to reduce the effects of human error – and, for many people, it is far easier to bisect an angle by folding paper than it is with a straight edge and compass.

Observe that it is in the nature of the folding procedure that we will always be folding DOWN a certain number of times at the top and then folding UP a certain (not necessarily the same) number of times at the bottom and then folding DOWN (possibly an entirely new) number of times at the top, etc. Indeed, a typical folding procedure may be represented by a sequence of exponents attached to the letters *DU DU DU DU*... the sequence stopping to avoid simply repeating a given finite string of exponents. The length of the repeat for the exponents is called the *period* of the folding procedure. (Thus the folding that produced the 7-gon is called a 2-*period* folding procedure.) It is an important fact that, for every odd p, a regular p-gon may be folded by instructions so encoded. It is thus very natural to ask *what regular p-gons can be produced by the 2-period folding procedure*?

In the process of answering this question we make straightforward use of the following:[3]

Lemma 2.1 For any three real numbers a, b and x_0, with $a \neq 0$, let the sequence $\{x_k\}$, $k =$ 0,1,2,... *be defined by the recurrence relation*

$$x_k + ax_{k+1} = b, \; k = 0, 1, 2, \ldots \tag{2.1}$$

Then if $|a| > 1$, $x_k \to b/(1+a)$ as $k \to \infty$.

Proof: Set $x_k = b/(1+a) + y_k$. Then $y_k + ay_{k+1} = 0$. It follows that $y_k = ((-1)/a)^k y_0$. If $|a| > 1$, $((-1)/a)^k \to 0$, so that $y_k \to 0$ as $k \to \infty$. Hence $x_k \to b/(1+a)$ as $k \to \infty$. Notice that y_k is the error at the k^{th} stage, and that the absolute value of y_k is equal to $|y_0|/|a|^k$.

This result is the special linear case of the Contraction Mapping Principle (see [W]). We point out that it is significant that neither the convergence nor the limit depends on the initial value x_0. This implies, in terms of the folding, that the process will converge, and to the same limit, no matter how we fold the tape to produce the first line – this is what justifies our *optimistic strategy*! And, as we have seen in our example, and as we will soon demonstrate in general, the result of the lemma tells us that the convergence of our folding procedure is rapid, since in all cases $|a|$ will be a positive power of 2.

[3] This lemma is actually applicable to folding procedures of arbitrary period.

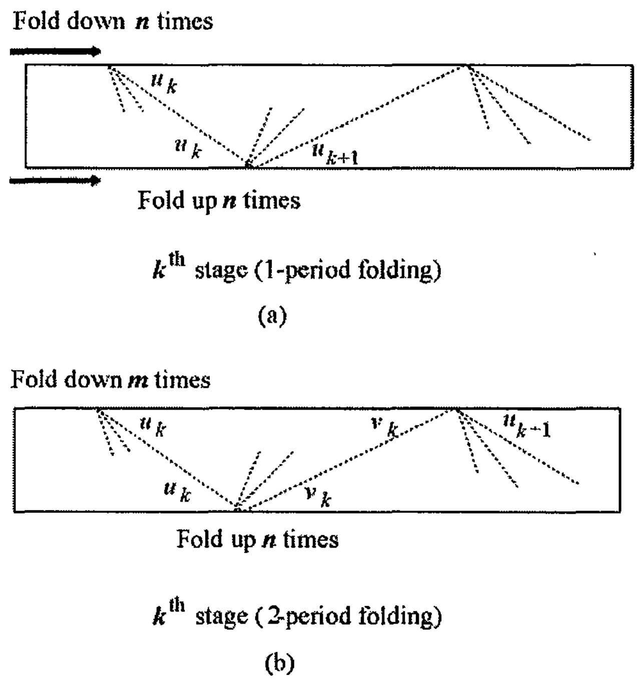


Figure 5

Now we will look at the general 2-period folding procedure, D^mU^n. In this case a typical portion of the tape would appear as shown in Figure 5(b). If the folding process had been started with an arbitrary angle u_0 at the top of the tape we would have, at the k^{th} stage,

$$u_k + 2^n v_k = \pi,$$
$$v_k + 2^m u_{k+1} = \pi,$$

and hence it follows that

$$u_k - 2^{m+n} u_{k+1} = \pi(1-2^n),\ k = 0,1,2,\ldots$$

Thus, using Lemma 2.1, we see that

$$u_k \longrightarrow \frac{2^n - 1}{2^{m+n} - 1}\pi \quad \text{as} \quad k \longrightarrow \infty$$

so that $\dfrac{2^n - 1}{2^{m+n} - 1}\pi$ is the putative angle $a\pi/b$. Thus the FAT-algorithm will produce, from this tape, a star $\{b/a\}$-gon, where the fraction b/a may turn out not to be reduced (for example when $m = 4$, $n = 2$), with $a = 2^n$-1, $b = 2^{m+n}$-1. By symmetry we infer that

$$v_k \longrightarrow \frac{2^m - 1}{2^{m+n} - 1}\pi \quad \text{as } k \longrightarrow \infty$$

Furthermore, if we assume an initial error E_0 then we know that the error at the k^{th} stage (when folding $D^m U^n$ has been done exactly k times) will be given by $E_k = E_0/2^{(m+n)k}$. Hence, we see that in the case of our D^2U^1-folding (Figure 3) any initial error E_0 is, as we already saw from our initial argument, reduced by a factor of 2^3 between consecutive states. It should now be clear why we advised throwing away the first part of the tape – but, likewise, it should also be clear that it is never necessary to throw away very much of the tape. In practice, convergence is very rapid indeed, and if one made it a rule of thumb always to throw away the first 20 crease lines on the tape for any iterative folding procedure, one would be absolutely safe.

We have seen that the $D^m U^n$-folding procedure, or, as we may more succinctly describe it, the (m, n)-folding procedure, produces angles π/s on the tape, where

$$s = \frac{2^{m+n} - 1}{2^n - 1}. \tag{2.2}$$

Notice that when $n = m$ the folding becomes, technically, a 1-period folding procedure which produces a regular s-gon, where $s = \dfrac{2^{m+n} - 1}{2^n - 1} = 2^n + 1$. Thus we see, immediately, that the $D^n U^n$-folding will produce tape to which the *FAT*-algorithm can be applied to obtain regular (2^n+1)-gons. These constructions provide approximations to many (but not all) of the polygons the Greeks and Gauss were able to construct with Euclidean tools. We can certainly construct a regular polygon whose number of sides is a Fermat number, but (see [Rec]) it is never possible to construct, with a 1-period folding those regular polygons where the number of sides is the product of at least two distinct Fermat numbers (thus, 15 serves as the first example where we find trouble).

The polygons which are of most interest to us in the construction of regular polyhedra are those with 3 or 5 sides (since we have exact constructions for the square). Our companion article [Rec] of this issue contains *very explicit* instructions of the folding procedures that produce the D^1U^1- and the D^2U^2-tape (which can be used to construct 3- and 5- gons, respectively) along with equally explicit instructions for building some braided polyhedra from the tape produced. The reader is encouraged to at least peruse that part of this issue before going on, since we will be making references to some of the models whose construction is described there in the next section.

Before we begin the discussion of symmetry let us finish explaining how you might construct those regular polygons that cannot be folded by the 1- or 2-period folding procedures. For example, suppose we wanted to construct a regular 11-gon. Our arguments in [HP1, 2 or 3] show that no 2-period (or 1-period) folding procedure can possibly produce an 11-gon.

In fact, the example of constructing the regular 11-gon is sufficiently general to show the construction of any regular p-gon, with p odd. So let us demonstrate how to construct a regular 11-gon. we proceed as we did in the construction of the regular 7-gon (in Section 2) – we adopt our *optimistic strategy* (which means that we assume we've got what we want and, as we will show, we then *actually get what we want*!). Thus we assume we can fold an angle of $2\pi/11$. We bisect it by introducing a crease line, and follow the crease line to the bottom of the tape. The folding procedure now commences at the bottom of the tape. Thus

(1) Each new crease line goes in the forward (left to right) direction along the tape;

(2) Each new crease line always *bisects* the angle between the last crease line and the edge of the tape from which it emanates;

(3) The isection of angles at any vertex continues until a crease line produces an angle of the form $a\pi/11$ where a is an *odd* number; then the folding stops at that vertex and commences at the intersection point of the last crease line with the other edge of the tape.

Once again the *optimistic strategy* works and our procedure results in tape whose angles converge to those shown in Figure 6(b). We could then denote this folding procedure by $U^1D^1U^3D^1\ U^1D^3$... interpreted in the obvious way on the tape – that is, the first exponent "1" refers to the one bisection (producing a line in the upward direction) at the vertices A_{6n} (for n = 0, 1, 2,...) on the bottom of the tape; similarly the next "1" refers to the bisection (producing a crease in the downward direction) made at the bottom of the tape through the vertices A_{6n+1}; etc. However, since the folding procedure is *duplicated* halfway through, we can abbreviate the notation and write simply {1,1,3}, with the understanding that we alternately fold from the bottom and top of the tape as described, with the *number* of bisections at each vertex running, in order, through the values 1, 1, 3, ... We call this a *primary folding procedure of period* 3 or a 3-*period folding*, for obvious reasons. The crease lines made during this procedure are called *primary crease lines.*

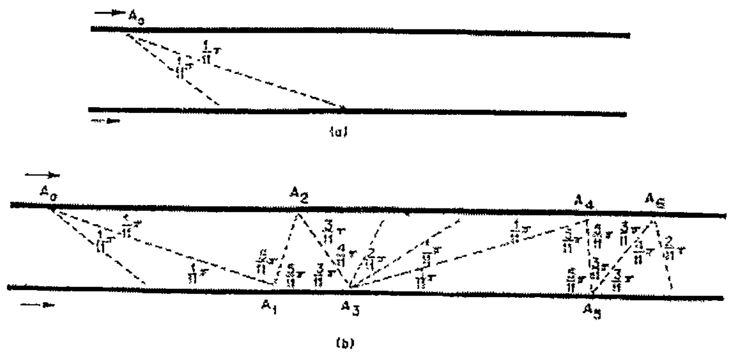


Figure 6 (Note that the indexing of the vertices is *not* the same as that in Fig. 1).

Our argument, as described for $p = 11$, may clearly be applied to any odd number p. However, our tape for the 11-gon has a special symmetry as a consequence of its *odd period*; namely that if it is "flipped" about the horizontal line halfway between its parallel edges, the result is a *translate* of the original tape. As a practical matter this special symmetry of the tape means that we can use either the top edge or the bottom edge of the tape to construct our polygons. On tapes with an *even* period the top edge and the bottom edge of the tape are not translates of each other (under the horizontal flip), which simply means that care must be taken in choosing the edge of the tape used to construct a specific polygon.

A proof for the convergence for the general folding procedure may be given that is similar to the one we gave for the primary folding procedure of period 2, using Lemma 2.1. Alternatively one could revert to an error-type proof like that given for the 7-gon. We leave the details to the reader.

For further reading, and a discussion of the construction of star polygons see [HP2, 3 and 5].

3. THE SYMMETRY GROUP OF A GEOMETRIC CONFIGURATION

We want now to take up the more mathematical aspects of symmetry. Indeed, at this stage, we lack a precise definition of symmetry – we cannot even give a meaning, in general, to the statement that one geometrical figure is more symmetric than another. Of course, a square is more symmetric than an arbitrary rectangle, and a rectangle is more symmetric than an arbitrary quadrilateral. But can we, for example, always compare the symmetries of regular polygons?

We are guided in our definitions by the approach of the great German mathematician Felix Klein (1849 – 1925) to understanding the nature of geometry. Consider, for example, the usual plane Euclidean geometry, in which we study the properties of planar figures which are invariant under certain *Euclidean motions*. These motions certainly include *translation* and *rotation*, but it is a matter of choice whether they include *reflexion*. For example the *FAT* 7-gon (Figure 4(b)) is invariant under rotations through $2\pi/7$ about its center, but not under reflexion in its plane. Thus, to define our geometry, we must decide whether we allow Euclidean motions which reverse orientation. Of course, it we allow certain Euclidean motions, we must also allow compositions and inverses of such motions, so we postulate a certain *group* G of allowed motions. If A is a planar figure, then, for any $g \in G$, Ag is again a planar figure and, *in the G-geometry of* A, we study the properties of the figure A which it shares with all the figures Ag as g varies over G; such properties are called the *G-invariants* of A, abbreviated to *invariants* if the group G may be understood.

Example 3.1 Let G be the group of motions of the plane generated by translations, rotations and reflexions (in a line); we call this the Euclidean group in 2 dimensions and may write it E_2. Then the Euclidean geometry of the plane is the study of the properties of subsets of the plane which are invariant under motions of E_2. For example, the property of being a polygon is a Euclidean property; the number of vertices and sides of a polygon is a Euclidean invariant. On the other hand, as we have hinted, orientation is not invariant with respect to this group, though it would be if we disallowed reflexions. Thus, by means of a motion in E_2 the triangle ABC may be turned over (flipped) to form the triangle $A'B'C'$ as shown in Figure 7. But the orientation of the triangle $\overrightarrow{ABC}$ is anti clockwise, while the orientation of the triangle $\overrightarrow{A'B'C'}$ is clockwise.

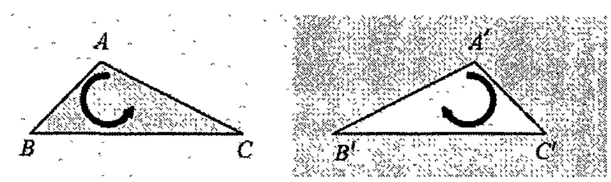


Figure 7: The triangle *ABC* may be transformed by a rotation in 3 dimensions into the triangle *A'B'C'* reversing the orientation of the triangle.

Example 3.2 We may 'step up a dimension', passing to the group E_3 of Euclidean motions in 3-dimensional space. Notice that it is natural to think of reflexions in a line (of a planar figure) as a 'motion' since it can be achieved by a rotation in some suitable ambient 3-dimensional space containing the plane figure. However, it requires a greater intellectual effort to think of reflexion in a plane (of a spatial figure) as a motion in some ambient 4-dimensional space! Who would think of turning the golden dodecahedron (see Figure 8 of [Rec]) inside out? Thus it is common not to include such reflexions in defining 3-dimensional geometry. This preference is, however, a consequence of our experience of living in a 3-dimensional world and has no mathematical basis. However, since, in this article, and its companion article, we are highlighting the construction of actual physical models of geometrical configurations, it is entirely reasonable to omit 'motions' to which the models themselves cannot be subjected.

We now introduce the key idea in the precise definition of symmetry. Let a geometry be defined on the ambient space of a geometric configuration A by means of the group of motions G. Then the *symmetry group* of A, relative to the geometry defined by G, is the subgroup G_A of G consisting of those motions $g \in G$ such that $Ag = A$, that is, those motions which map A onto itself, or, as we say, under which A is *invariant*. Thus, for example, if our geometry is defined by rotations and translations in the plane, and if A is an equilateral triangle, then its symmetry group G_A consists of rotations about its center through 0°, 120°, and 240°; if, in our geometry, we also allow reflexions, then the symmetry group has 6 elements instead of 3, and is, in fact, the very well-known group S_3, called the *symmetric group on* 3 *symbols* – the symbols may be thought of as the vertices of the triangle. We must repeat for emphasis that the symmetry group G_A of the configuration A is a *relative* notion, depending on the choice of 'geometry' G.

It is plain that no compact (bounded) configuration can possibly be invariant under a translation. Thus when we are considering the symmetry group of such a figure we may suppose G to be generated by rotations and, perhaps, reflexions. Moreover, any such motion in the plane is determined by its effect on 3 independent points and any such motion in 3-dimensional space is determined by its effect on 4 independent points. Since a (plane) polygon has at least 3 vertices and a polyhedron has at least 4 vertices, and since any element of the symmetry group of a polygon or a polyhedron must map vertices to vertices, it follows that the symmetry group of a polygon or a polyhedron is *finite* (compare the symmetry groups of a circle or a sphere).

The symmetry group of any polygon with n sides is, by the argument above, a subgroup of S_n, the group of permutations of n symbols, also called the *symmetric group* on n symbols. If G is generated by rotations alone, and the polygon is regular, this group is the cyclic group of order n, often written C_n, generated by a rotation through an angle of $2\pi/n$ radians about the center of the polygonal region. If G also includes reflexions, this group has $2n$ elements and includes n reflexions; this group is called a *dihedral* group and is often written D_n.

In discussing the symmetry groups of polyhedra, we will, as indicated above, always assume that the geometry is given by the group G generated by rotations in 3-dimensional space. Then the symmetry group of the regular tetrahedron is the so-called *alternating* group A_4. In general, A_n is the subgroup of S_n consisting of the *even* permutations of n symbols; it is of index 2 in S_n, that is, its order is half that of S_n, or $n!/2$. Thus the order of A_4 is 12. The cube and the regular octahedron have the same symmetry group, namely S_4. It is easy to see why the symmetry groups are the same; for the centers of the faces of a cube are the vertices of a regular octahedron, and the centers of the faces of a regular octahedron are the vertices of a cube. Likewise, and for the same reason, the regular dodecahedron and the regular isocahedron have the same symmetry group, which is A_5. It is a matter of great interest and relevance here that the symmetries of the Diagonal Cube and the special braided octahedron of Figure 7 and Figure 16, respectively (of [Rec]) each permute the four braided strips from which the models are made. This provides a beautiful explanation of why their symmetry group is the symmetry group S_4.

We are now in a position to give at least one precise meaning to the statement "Figure A is more symmetric than Figure B". If it happens that the symmetry group G_A of A strictly contains the symmetry group G_B of B, then we are surely entitled to say that A is more symmetric than B. Notice that the situation described may, in fact, occur because B is obtained from A by adding features which destroy some of the symmetry of A. For example, the coloring of the strips used to construct the braided Platonic solids of Figure 6 of [Rec] will reduce the symmetry in all cases but that of the cube.

However, the definition above is really too restrictive. For we would like to be able to say that the regular n-gon becomes more symmetric as n increases. We are thus led to a weaker notion which will be useful provided we are dealing with figures with finite symmetry groups (e.g., polygons and polyhedra). We could then say – and do say – that A is more symmetric than B if G_A has more elements than G_B. Thus we have, in fact, two notions whereby we may compare symmetry – and they have the merit of being consistent. Indeed, if A is more symmetric than B in the first sense, it is more symmetric than B in the second sense – but not conversely.

Notice that we deliberately avoid the statement – often to be found in popular writing – "A is a symmetric figure". We regard this statement as having no precise meaning!

4. HOMOLOGUES

George Pólya, who made great contributions not only to mathematics itself, but also to the understanding of how and why we do mathematics – or perhaps one should say 'how and why we should do mathematics' – was particularly fascinated by the Platonic solids and first introduced his notion of *homologues* in connection with the study of their symmetry; they later played an important role in one of his most important contributions to the branch of mathematics know as *combinatorics*, namely, the *Pólya Enumeration Theorem* (see [P1] for an intuitive account). Let us describe this notion of homologues in terms of symmetry groups. We believe that we are thereby increasing the scope of the notion and entirely maintaining the spirit.

Let A be a geometrical configuration with symmetry group G_A, and let B be a subset of A. Thus, for example, A may be a polyhedron and B a face of that polyhedron. We consider the subgroup G_{AB} of G_A consisting of those motions in the symmetry group G_A of A which map B to itself. Now subgroups partition a group into *cosets*: If K is a subgroup of H, we define a *(right) coset* of K in H as a collection of elements kh, with h *fixed* and k *varying over* K. We call h a *representative* of this coset, which we write Kh. Any two cosets Kh, Kh' are either disjoint or identical (this is easy to prove), so we may imagine that we have picked a set of coset representatives, one for each coset. In the case in which we are interested the group H is finite so we may write, for some m

$$H = \bigcup_{i=1}^{m} Kh_i, \tag{4.1}$$

where it is understood that the union is disjoint. Notice that m, which appears in (4.1) and which we call the *index* of K in H, is just the ratio of the order of H to the order of K. An example was provided earlier with $H = S_n$, the symmetric group, and $K = A_n$, the alternating group. Then $m = 2$.

Reverting to our geometrical situation, we consider a coset of G_{AB} in G_A, that is, a set $G_{AB}g$, $g \in G_A$. Every element in $G_{AB}g$ sends B to the same subset Bg of A. The collection of these subsets is what Pólya called the collection of *homologues* of B in A. We see that the set of homologues of B is in one-one correspondence with the set of cosets of G_{AB} in G_A.

Example 4.1 Consider the pentagonal dipyramid A of Figure 5(b) of [Rec]. We may specify any motion in the symmetry group of A by the resulting permutation of its vertices 1, 2, 3, 4, 5, 6, 7. In fact, G_A is the dihedral group D_5, with 10 elements, given by the following permutations sending (1 2 3 4 5 6 7), resepectively, to

$$\left\{\begin{array}{l}\to (1\,2\,3\,4\,5\,6\,7) \text{ (Identity)}\\ \to (2\,3\,4\,5\,6\,7\,1) \text{ (rotation through } 2\pi/5 \text{ about axis 67)}\\ \to (3\,4\,5\,1\,2\,6\,7)\\ \to (4\,5\,1\,2\,3\,6\,7)\\ \to (5\,1\,2\,3\,4\,6\,7)\\ \to (5\,4\,3\,2\,1\,7\,6) \text{ (interchanging top and bottom)}\\ \to (4\,3\,2\,1\,5\,6\,7) \text{ (interchange plus rotation through } 2\pi/5)\\ \to (3\,2\,1\,5\,4\,6\,7)\\ \to (2\,1\,5\,4\,3\,7\,6)\\ \to (1\,5\,4\,3\,2\,7\,6)\end{array}\right.$$

First, let be the edge 16. Then $G_{AB} = \{Id\}$, since only the identity sends the subset (1, 6) to itself. Thus the index of G_{AB} in G_A is 10, and there are 10 homologues of the edge 16; these are the 10 'spines' of the dipyramid (i.e., we exclude the edges around the equator). Second, let B be the edge 12. Then G_{AB} has 2 elements, since there are two elements of G_A, namely the identity and permutation $(1\ 2\ 3\ 4\ 5\ 6\ 7) \to (2\ 1\ 5\ 4\ 3\ 7\ 6)$, which send the subset (1, 2) to itself. Thus the index of G_{AB} in G_A is 5, and there are 5 homologues of the 12; these are the 5 edges around the equator.

Third, let B be the face (126). Then $G_{AB} = \{Id\}$, so that, as in the first case, there are 10 homologues of the face (126); in other words all the (triangular) faces are homologues.

Let us now explain the Pólya Enumeration Theorem - actually, there are *two* theorems - and see how the notion of homologue fits into the story.

5. THE PÓLYA ENUMERATION THEOREM

Let X be a finite set; the reader might like to keep in mind the set of vertices (or edges, or faces) of a polygon or polyhedron; and let G be a finite symmetry group acting on X. Suppose X has n elements, and that G has m elements; we write $|X| = n$, $|G| = m$. We may represent the elements of the set X by the integers $1, 2, \dots, n$. If $g \in G$, then g acts as a *permutation* of $\{1, 2, \dots, n\}$. Now every permutation is uniquely expressible as a composition of cyclic permutations on mutually exclusive subsets of the elements of X. For example, the permutation

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11 \\ 2\,4\,7\,1\,3\,11\,5\,6\,8\,10\,9 \end{pmatrix} \tag{5.1}$$

is the composite (1 2 4)(3 7 5)(9 8 6 11)(10), where, e.g., (1 2 4) denotes the cyclic permutation

$$\begin{pmatrix} 1\,2\,4 \\ 2\,4\,1 \end{pmatrix}$$

Thus the permutation (5.1) is the composite of one cyclic permutation of length 1, two cyclic permutations of length 3, and one cyclic permutation of length 4, the cyclic permutations acting on disjoint subsets of the set X. In general a permutation of X has the *type* $(a_1, a_2, \dots, a_n\}$ if it consists of a_1 permutations of length 1, a_2 permutations of length 2, ... , a_n permutations of length n, the permutations having disjoint domains of action; notice that $\sum_{i=1}^{n} a_i = n$. For example the permutation (5.1) has the type $\{1,0,2,1,0,0, 0, 0, 0, 0, 0\}$. If g has the type $\{a_1, a_2, \dots, a_n\}$, we define the *cycle index* of g to be the monomial

$$Z(g) = Z(g; x_1,x_2, \dots , x_n) = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}.$$

The *cycle index of G* is $Z(g) = \dfrac{1}{m} \sum_{g \in G} Z(g)$.

We give an example which we will revisit periodically throughout this section.

Example 5.1 We consider the symmetries of the square as shown in Figure 8.

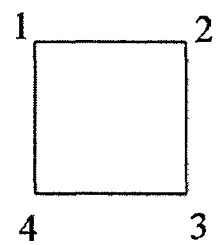


Figure 8: We denote this labeling of the square by 1234

The group G of symmetries is a group of order 8, which we describe by permutations of the set of vertices $\{1, 2, 3, 4\}$. Thus

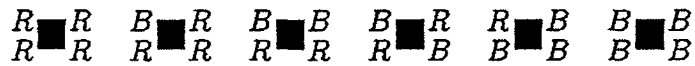
g_1 (*Id*entity)	$(1\,2\,3\,4) \to (1\,2\,3\,4)$	cycle index	x_1^4
g_2	$(1\,2\,3\,4) \to (2\,3\,4\,1)$	cycle index	x_4
g_3	$(1\,2\,3\,4) \to (3\,4\,1\,2)$	cycle index	x_2^2
g_4	$(1\,2\,3\,4) \to (4\,1\,2\,3)$	cycle index	x_4
g_5	$(1\,2\,3\,4) \to (3\,2\,1\,4)$	cycle index	$x_1^2 x_2$
g_6	$(1\,2\,3\,4) \to (1\,4\,3\,2)$	cycle index	$x_1^2 x_2$
g_7	$(1\,2\,3\,4) \to (2\,1\,4\,3)$	cycle index	x_2^2
g_8	$(1\,2\,3\,4) \to (4\,3\,2\,1)$	cycle index	x_2^2

Thus the cycle index of G is $(x_1^4 + 2x_1^{\,2} x_2 + 3x_2^2 + 2x_4)/8$.

Now suppose we want to color the elements of X; that is, we have a finite set Y of colors, $|Y| = r$, and a coloring of X is a function[4] $f: X \to Y$. For any $g \in G$, we regard the colorings f and fg as indistinguishable or equivalent; and a pattern is an equivalence class of colorings. Then Pólya's first theorem is as follows.

Theorem 5.1 The number of patterns is $Z(G; r, r, \dots, r)$.

Example 5.1 (continued) Suppose the vertices are to be colored red or blue. Then $r = 2$, and the number of patterns is $(16 + 16 + 12 + 4)/8 = 6$. In fact, the patterns are represented by the 6 colorings: RRRR, BRRR, BBRR, BRBR, RBBB, BBBB.



We now describe Pólya's second theorem. This is really the 'big' theorem and the first theorem is, in fact, deducible from it. Let us enumerate the elements of Y (the 'colors') as $y_1, y_2, \dots, y_r$.

Theorem 5.2 Evaluate $Z(G; x_1, x_2, \dots, x_n)$ at $x_i = \sum_{j=1}^{r} y^i_j$. *Then the coefficient of* $y_1^{n_1} y_2^{n_2} \dots y_r^{n_r}$ *is the number of patterns assigning the color* y_j *to* n_j *elements*[5] *of* X.

Example 5.1 (continued) For the symmetries of the square we know that

$$Z(G) = (x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)/8.$$

Thus if $Y = \{R, B\}$, then the evaluation of $Z(G)$ at $x_i = R^i + B^i$ yields

$$((R+B)^4 + 2(R+B)^2(R^2 + B^2) + 3(R^2+B^2)^2 + 2(R^4+B^4))/8 = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4.$$

(It is, of course, no coincidence that this polynomial is homogeneous (of degree $|X|$ and symmetric. Thus the Pólya Enumeration Theorem tells us that there is one pattern with 4 red vertices (obvious); one pattern with 3 red vertices and 1 blue vertex, represented by the coloring *RRRB*; 2 patterns with 2 red vertices and 2 blue vertices, represented by the colorings *BRRB* and *RBRB*, and the remaining possibilities are analyzed by considerations of symmetry.)

[4] We speak of a *coloring* of X; this may be literally true or it may merely be a metaphor for a rule for dividing the elements of X into disjoint classes

[5] Of course $\sum_{j=1}^{r} n_j = n.$

Now, given a pattern, there are the various functions $fg : X \to Y$, where f is a fixed coloring and $g \in G$, in that equivalence class of colorings. These are the *homologues*, or, more precisely, the *homologues of* f. Let us revert to our example.

Example 5.1 (continued) As we have seen, there is one coloring in which all vertices are colored red. There is only one homologue, namely RRRR.

There is one coloring in which 3 vertices are colored red and one blue. There are 4 homologues, namely BRRR, RBRR, RRBR, RRRB.

There are two colorings in which 2 vertices are colored red and 2 blue. In the first there are 4 homologues, namely BBRR, RBBR, RRBB, BRRB.

In the second there are 2 homologues, namely BRBR, RBRB.

The analysis is completed by considerations of symmetry.

Let us show how this conception of homologues agrees with our earlier definition. We are given the group G of permutations of X. Given a coloring f: $X \to Y$, we consider the subset G_0 of G consisting of those g such that $fg = f$, that is, those movements of X which preserve the coloring. It is easy to see (just as easy as in our earlier, simpler situation) that G_0 is a *subgroup* of G. Corresponding to each coset G_0g of G_0 in G we have a coloring fg of X and these colorings run through the pattern determined by f. We have described the set of colorings $\{fg\}$ as the set of *homologues* of the coloring f; as indicated earlier, they are in one-one correspondence with the cosets of G_0 in G.

6. EPILOGUE: PÓLYA AND OURSELVES – MATHEMATICS, TEA AND CAKES[6]

Professor George Pólya (1887 – 1985) emigrated to the United States in 1940 and joined the Mathematics Department at Stanford University in 1942. Although the rest of his professional life was spent at Stanford, he made many trips abroad to accept visiting appointments for short periods of time. During Pólya's visit to the ETH (Zürich) in 1966 he shared an office with Peter Hilton (and PH was a guest at his 80th birthday party, held in Zürich, in 1967).

[6] We present this more personal epilogue to our article at the express request of Dénes Nagy.

In 1969 Pólya was invited by Gerald Alexanderson (Mathematics Department Chairman at Santa Clara University – then and now) to give a colloquium talk at SCU. While there Pólya met Jean Pedersen and was fascinated by (a) the models in her office (some of which are described in [Rec]) and (b) by her lack of knowledge about their symmetry and their usefulness in exemplifying some of the mathematics of polyhedral geometry. After this initial meeting Pedersen visited George and Stella Pólya at their Palo Alto home once a week until his death in 1985[7]. Pedersen and her husband Kent (who shares Pólya's birthday – except for the year!) were guests at Pólya's 90th birthday party, held at Stanford, in 1977 and the Pólyas were guests at the Pedersen's home for Thanksgiving dinner for many successive years.

A typical visit, for Pedersen, included a discussion with Pólya about mathematics. After an hour or so Stella would appear with tea and cakes, or cookies, and the three of them would turn their attention to current events and politics. It was during this time that Pedersen learned about proper rotation groups (knowledge that Pólya acquired from Felix Klein himself) and the Pólya Enumeration Theorem, about Euler's famous formula connecting vertices, edges and faces of a polyhedron, and about the formula Descartes discovered concerning the total angular deficiency of a polyhedron. Pedersen found herself studying very hard[8] and looking forward to discussing the new-found aspects of her own models. Pólya and Pedersen also discussed pedagogy and, in fact, Pedersen was Pólya's last co-author (see [PP]).

In 1978 Pedersen was asked to try to get George Pólya and Peter Hilton together[9] in Seattle at the joint annual meeting of the American Mathematical Society and the Mathematical Association of America, to discuss "How to and How *Not* to Teach Mathematics". The suggestion was that Hilton should discuss "How Not to Teach Mathematics" and this would be followed by Pólya giving "some Rules of Thumb for Good Teaching". Pólya agreed to participate on the condition that Pedersen would handle the travel details of getting him to and from Seattle. Hilton also gave only conditional approval for the plan. Hilton's idea was that it would be much more interesting, and effective, if he were to *demonstrate* a thoroughly bad mathematics lecture (instead of simply talking about it). Hilton also suggested that Pedersen should be the moderator for the program.

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[7] After George Pólya's death, Pedersen continued to visit Stella Pólya at least once a week until her death in 1989, just before her 94th birthday.

[8] Figure 9 is an example, in Pólya's own handwriting of a page he once gave Pedersen saying "see if you can figure out what it means". It is connected with what we've been writing about in this article, so we leave the reader to do Pólya's homework assignment for the week! The only hint Pólya gave was to say that H = *H*exahedron (cube).

[9] This was how Hilton and Pedersen met and began a collaboration that has resulted in over 70 papers and four books – to date!

Groups S_n symmetric, $n!$
A_n alternating, $\frac{n!}{2}$

C_n	D_n	I H	O	D I	
cyclic	dihed.	$T = A_4$	$O = S_4$	$I = A_5$	group
n	$2n$	$12 = \frac{4!}{2}$	$24 = 4!$	$60 = \frac{5!}{2}$	order

F face D_i diagonal$_i$ | Df $\frac{\text{face}}{\text{Diag.}}$ | ✱ rect. coord.
V vertex $A_{1/n}$ axis $(C_n, \frac{2\pi}{n})$ of symmetry (rot.)
E edge P_i plane of symmetry$_i$
$\measuredangle$ face angle

	whole	why not?	$A_{1/2}$	
	I		D_2	(next index, opposite order)
T	1	2	3	
	12	6	4	
	$\frac{C_1}{\measuredangle}$	$\frac{C_2}{E, P}$	$\frac{C_3}{F, V}$	

	whole	inscr T	$A_{1/4}, P_4$	$A_{1/3}$
	O	T	D_4	D_3
H|O	1	2	3	4
	24	12	8	6
	$\frac{C_1}{\measuredangle}$	$\frac{C_2}{E}$	$\frac{C_3}{V \mid F}$	$\frac{D_2; C_4}{(P_2)\ F \mid V}$

	whole	why not?			inscr H	$A_{1/5}$
	I				T	D_5
D|I	1	2	3	4	5	6
	60	30	20	15	12	10
	$\frac{C_1}{\measuredangle}$	$\frac{C_2}{E}$	$\frac{C_3}{V \mid F}$	$\frac{D_2}{P_3 A_{1/2}}$	$\frac{C_5}{F \mid V}$	$\frac{D_3}{A_{1/3}}$

Figure 9

All conditions were met and the Seattle presentation duly took place. It was a tremendous success. Hilton's part was hilarious and some said it nearly ruined the rest of the meeting as participants saw many of Hilton's intentional errors *unintentionally* repeated by some of the other speakers. Pólya's contribution was, as you might expect, superb and had the unmistakable mark of a master teacher. A month or so later Pedersen was asked by the National Council of Teachers of Mathematics to arrange that the Hilton-Pólya performance be repeated at their San Diego meeting in the fall of 1978 so that it could be videotaped. After a few more meetings with tea and cakes, and some long distance calls, this was done.

At the San Diego meeting Pedersen invited Hilton to visit SCU in October to give a colloquium talk. He did, and when he saw the models in Pedersen's office they again sparked long discussions, but this time the discussions centered on the differences between the ways geometers and topologists classify surfaces.

In 1982, while Peter Hilton was on sabbatical leave as a visiting professor at the ETH and Pedersen was visiting there for a quarter, they began looking seriously at the paper-folding. Hilton suggested to Pedersen that she should try to devise a really *systematic* way of constructing the polygons from the folded strips (since the 2^n+1-gons seemed to have very special features that didn't generalize). The first result of Hilton's suggestion was the *FAT*-algorithm. This innocent-looking algorithm, in fact, opened the flood gates for both the development of the general folding procedures and the number theory that grew out of the paper-folding.

After 1978 whenever Hilton visited SCU he went with Pedersen to visit the Pólyas and together they continued the tradition of mathematics, tea and cakes. In 1981 Hilton and Pedersen, along with Alexanderson, cooperated with Pólya to bring out the Combined Edition of *Mathematical Discovery* (see [P2])[10]. During many of the tea parties at Pólya's home, Pólya talked about his idea of homologues, and on one occasion told us that he had never written about them and that someday he would like us to write about them – in fact, he extracted a promise from us that we would do so. Thus we are very grateful to Dénes Nagy for giving us such a splendid opportunity to fulfill our promise to our dear friend and teacher George Pólya, and to convey the flavor, and a few of the details, of our friendly relationship with him.

[10] Alexanderson updated the references, Hilton wrote a foreword, and Pedersen provided an expanded (and less esoteric) index.

REFERENCES

[C1] Coxeter, H M. S. (1993) *Regular Polytopes* (Third edition), Dover, New York.

[C2] Cromwell, P. R. (1995) Kepler's work on polyhedra, *The Mathematical Intelligencer* 17, No. 3, 23 - 33.

[HP1] Hilton, P., and Pedersen, J. (1983) Approximating any regular polygon by folding paper; An interplay of geometry, analysis and number theory, *Math. Mag.* 56, 141 - 155.

[HP2] Hilton, P., and Pedersen, J. (1985) Folding regular star polygons and number theory, *Math. Intelligencer* 7, No. 1, 15 - 26.

[HP3] Hilton, P., and Pedersen, J. (1987) Geometry in Practice and Numbers in Theory, *Monographs in Undergraduate Mathematics* 16, 37 pp. (Available from Dept. of Mathematics, Guilford College, Greensboro, North Carolina 27410).

[HP4] Hilton, P., and Pedersen, J. (1993) Geometry: A gateway to understanding, *Coll. Math. Journ.* 24, No. 4, 298-317.

[HP5] Hilton, P., and Pedersen, J. (1994) *Build Your Own Polyhedra* (Second Printing), Addison-Wesley, Menlo Park, CA.

[Rec] Hilton, P., and Pedersen, J. *Symmetry in Practice - Recreational Constructions.*

[N] Newman, J. R. (1956) *The World of Mathematics*, New York: Simon and Schuster, Vol. I, p. 502.

[PP] Pedersen, J., and Pólya, G. (1984) On problems with solutions attainable in more than one way, *College Math.* J. 15, 218-228

[P1] Pólya, G. (1963) Intuitive outline of the solution of a basic combinatorial problem, *Switching theory in space technology* (ed. H. Aiken and W. F. Main), Stanford University Press, 3-7.

[P2] Pólya, G. (1981) *Mathematical Discovery - On Understanding, Learning, and Problem Solving* (Combined Edition), John Wiley and Sons.

[W] Wilde, C. (1978) *The Contraction Mapping Principle*, UMAP Unit 326, Lexington, MA: COMAP, Inc.

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PERFECT PRECISE COLOURINGS OF TRIANGULAR TILINGS, AND HYPERBOLIC PATCHWORK

Dedicated to the memory of Raphael M. Robinson

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Abstract: *We consider the problem of colouring the regular triangular tiling* $\{3,n\}$ *with n colours, in such a way that one tile of each colour occurs at each vertex. Such a tiling will be called* precise, *and the most interesting precise colourings are those that are also perfect. We shall find a method of enumerating a large class of perfect precise colourings, and shall also consider various related problems. Other aspects of the subject are discussed in Yaz (1997) and Rigby (1998), which were written after the present article was first submitted.*

1. INTRODUCTION

The following problem appeared in the American Mathematical Monthly, proposed by Raphael M. Robinson (Robinson 1993).

> *The hyperbolic plane is tiled with equilateral triangles meeting seven at each vertex. Can the tiles be colored with seven colors in such a way that no two tiles of the same color meet even at a vertex? (This problem was suggested to the proposer by David Gale.)*

The proposer mentioned to me in a letter that he had already solved the problem in 1984, but that David Gale was interested in the more general problem: *can we find a colouring of the regular tiling* $\{3,n\}$ *with equilateral triangles (on the sphere for* $n = 3, 4, 5$, *in the Euclidean plane for* $n = 6$, *and in the hyperbolic plane for* $n \geq 7$*) using* n *colours, such that one triangle of each colour occurs at each vertex?*

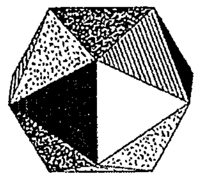

Figure 1

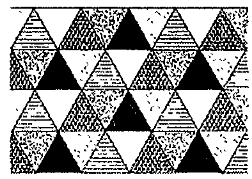

Figure 2

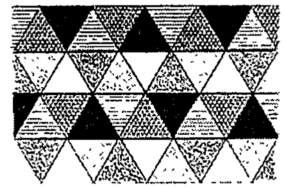

Figure 3

We shall say that such a colouring is *precise*. There is no precise colouring of the tetrahedron $\{3,3\}$. The unique precise colouring of the octahedron $\{3,4\}$ is obvious. The precise colouring of $\{3,5\}$ shown in Figure 1 is essentially unique, but it occurs in left- and right-handed versions. The most obvious precise colouring of $\{3,6\}$ is shown in Figure 2; it is fully perfect, which means that every symmetry of the tiling (every transformation that maps the tiling to itself) permutes the colours instead of jumbling them up. Another precise colouring is shown in Figure 3; here each horizontal strip contains just three colours, and the colours in any horizontal strip can be permuted, thus producing an infinity of highly imperfect precise colourings.

In Section 2 we give a solution to the problem when $n = 7$ which results in a colouring that is not only precise but also chirally perfect (a notion that will be explained later). In Sections 2 and 3 we find a way of viewing this solution that enables us to generalise it to construct solutions for all values of n. In Section 4 we find all perfect precise colourings of a standard type, and a unique notation for them. Other related problems then present themselves, notably the existence of fully perfect precise colourings (Section 5), which occur only when n is even and $n \neq 8$, and the existence of fully perfect (but not precise) colourings when n is odd (Section 7). In Sections 8 and 9 we briefly consider non-standard perfect precise colourings and semiperfect precise colourings. In Section 10 we show how an infinity of imperfect precise colourings can be produced; many of these still have a high degree of symmetry. Finally, in Section 11, we introduce the notion of *equivalent colourings,* which is useful in the investigation of the groups of permutations of colours induced in perfect colourings by the direct symmetry group of the tiling.

The concepts, constructions and proofs in this article were conceived in a very visual manner, and are here presented in the same manner. The reader is encouraged to make photocopies of the blank tilings on the final page (Figure 31), then to get a feel for the constructions to be described in the article by creating various trees and sewing them together, and by creating various partial colourings.

2. PERFECT PRECISE COLOURINGS AND TREES

Figure 4 shows the black tiles in a precise colouring of {3,7} (Rigby 1990, Fig. 16; Rigby 1991, Fig. 7). If we stand on any black tile, it is easy to find two simple rules for proceeding to the adjacent black tiles; these rules are independent of which tile we are on, and which edge we are facing. As a consequence (a) any direct symmetry of the tiling that maps one black tile to another maps the set of black tiles to itself (this is not true of the opposite symmetries, the ones that "turn the tiling over") and (b) starting with any other tile coloured with a second colour, we can use the same rules to colour all the tiles that are to have that second colour, and proceeding in this way we can complete the colouring with seven colours. The colouring is said to be *chirally perfect*, because every direct symmetry of the tiling permutes the colours, but opposite symmetries jumble up the colours. This particular colouring has fascinating group-theoretical properties; see Mackenzie (1995) written in response to Robinson's problem, and Rigby (1991, p.60). A partial colouring of black tiles such as the one we started from in Figure 4 can be called *self-consistent.*

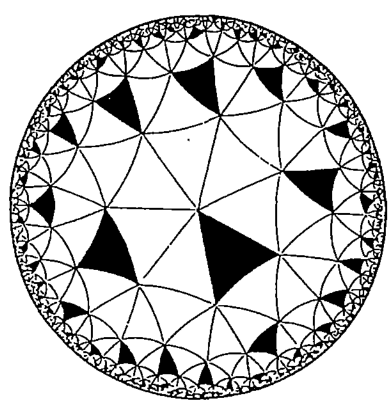

Figure 4

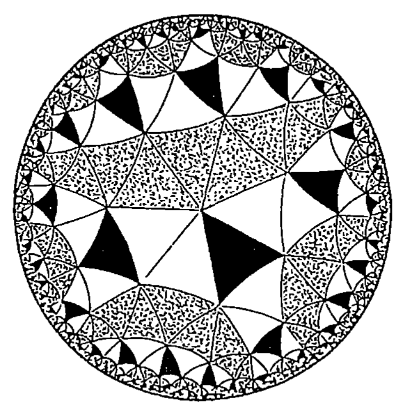

Figure 5

If we start with the mirror image of the partial colouring of black tiles in the previous paragraph, we obtain the mirror image of the chirally perfect colouring: every chirally perfect colouring occurs in left- and right-handed versions.

To solve the problem for other values of n, we need to analyse Figure 4 further, with a view to producing similar self-consistent partial colourings for other tilings. Figure 5 shows the partially coloured tiling of Figure 4 divided up into pieces; alternate pieces have been stippled in order to show the division more clearly. The stippled pieces are strips extending to infinity in both directions. The remaining pieces are all directly congruent; they contain both white and black tiles, and will be called *4-trees* because of their branching shape. The prefix 4 refers to the fact that four tiles of the 4-tree come together at each vertex of the 4-tree. Note also that no vertex of the complete tiling lies inside the 4-tree: every tile of the 4-tree has all three of its vertices on the boundary. Each 4-tree is *vertex-transitive:* there is a direct symmetry of the 4-tree mapping any vertex to any other vertex. The 4-trees are *partially coloured* (some tiles are black), and this partial colouring of each 4-tree is *precise* and *(chirally) consistent*: there is one black tile at each vertex, and every direct isometry of the tree maps the set of black tiles to itself. The stippled strips, with three tiles at each vertex, will be called 3-trees, even though no branching occurs.

Patchwork quilts are often made by first creating portions of the quilt, then sewing the portions together to make the entire quilt; hence the title of the article.

Our initial technique (to be modified later) for dealing with the general $\{3,n\}$ tiling will be to construct k-trees for all values of k ($k \neq 2$), and then to sew together partially coloured k-trees and plain $(n-k)$-trees alternately (for a particular value of k) to cover the entire plane; this will produce the black tiles in a precise chirally perfect colouring of $\{3,n\}$. We shall require these k-trees to be vertex-transitive under direct isometries, and their partial colourings to be precise and chirally consistent (as defined above for 4-trees); also all vertices of each tile in a k-tree must lie on the boundary of the tree. The k-trees must all be isomorphic, as must the $(n-k)$-trees. The next three figures show examples of this technique. Figure 6 shows partially coloured and plain 4-trees sewn together to make a self-consistent partial colouring of $\{3,8\}$. In Figure 7 the partially coloured and the plain 4-trees have opposite orientations; this produces a different partial colouring of $\{3,8\}$. In Figure 8, partially coloured 5-trees and plain 3-trees are sewn together to produce a third partial colouring of $\{3,8\}$. Note that precise consistent partial colourings of 3-trees do not exist.

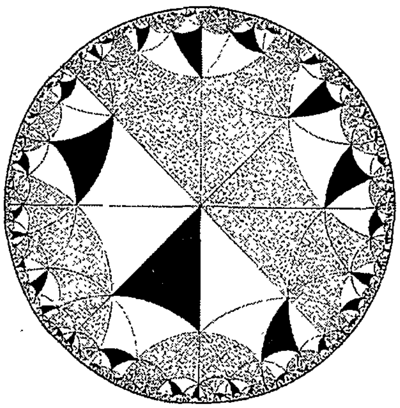

Figure 6

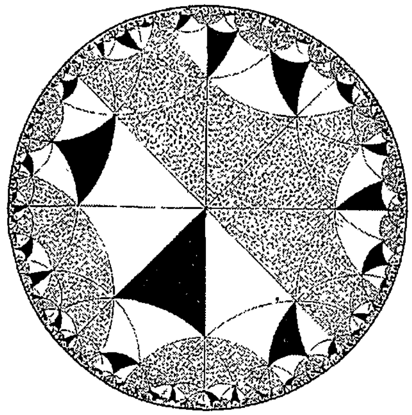

Figure 7

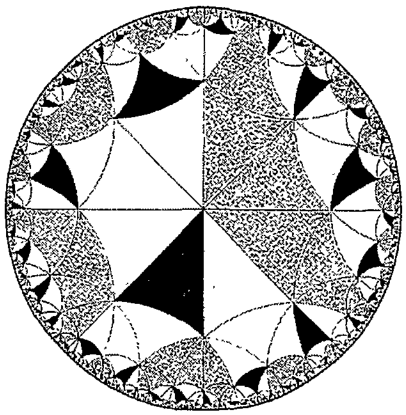

Figure 8

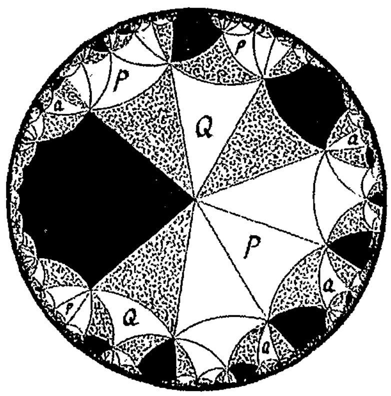

Figure 9

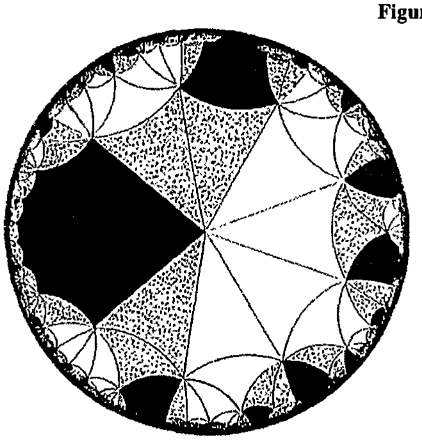

Figure 10

3. THE CONSTRUCTION OF STANDARD TREES

A single tile can be regarded as a 1-tree, and a single edge of the tiling as a 0-tree; there are no 2-trees.

Figure 9 shows a 7-tree; all tiles not belonging to the tree have been coloured black. (In this figure the underlying tiling is {3,9}, so the branches of the tree join up to form loops; but it is easy to construct the same type of tree for $n \geq 10$, so that we then have a genuine tree without loops.) All the tiles of the tree that have an edge along the boundary of the tree have been stippled. The remaining tiles (the white tiles) form 3-trees and 1-trees. Each stippled tile has its base along the boundary, its left side adjacent to a 3-tree (labelled P in the figure), and its right side adjacent to a 1-tree (labelled Q), so we shall denote this 7-tree by the symbol (31). Figure 10 shows a different type of 7-tree; if we break this down in the same way as before, we find that the left side of each stippled tile is adjacent to a 4-tree, and the right side is adjacent to a 0-tree (a single edge), so the symbol for this 7-tree is (40). But we can break down a 3-tree in the same way: all its tiles will be stippled, so the symbol for a 3-tree is (00). Similarly the 4-trees in Figure 10 have the symbol (10). Hence the symbols (31) and (40) for the 7-trees in Figures 9 and 10 can be further refined to (00)1 and (10)0. There is a third type of 7-tree with symbol (01)0, and the mirror images of these three 7-trees have symbols 1(00), 0(01) and 0(10), obtained by reversing the previous symbols.

In Rigby (1996) I asserted that any tree, of type X say, can be broken down in the way just described, using stippled tiles, into trees of types P and Q say, so that every stippled tile has its base on the boundary of X, its left side adjacent to a tree of type P, and its right side adjacent to a tree of type Q. This is incorrect, as I discovered by finding a counterexample whilst drawing various figures to illustrate Section 7 (see Section 8). But the converse *is* true: let P denote any type of p-tree and let Q denote any type of q-tree; then we can put together trees of types P and Q in the manner just described, with the help of stippled tiles, to form a $(p+q+3)$-tree denoted by (PQ). As long as $p+q+3 \leq n-3$, the tree (PQ) will be a genuine tree, with no loops, and the construction of (PQ) will not run into any snags such as the possible overlapping of different branches of (PQ).

We shall now define *standard trees*, using an inductive definition: 0-trees and 1-trees are standard trees, and if P and Q are standard trees, then (PQ) is a standard tree. The symbol for any standard tree is a sequence of 0s and 1s, correctly bracketed as in the previous examples. Any correctly bracketed sequence will represent a type of standard tree, which will always exist if n is large enough, and because of the uniqueness of the breakdown of a standard tree into the form (PQ) different sequences represent different types of tree: to enumerate all types of standard tree we need only enumerate all correctly bracketed sequences. In the interests of clarity and legibility the outermost pair of brackets will frequently be omitted: for instance 0(10) is preferred to (0(10)).

When P and Q are of the same type it is important that each tree used in the construction of (PQ) should be labelled either P or Q, to prevent any tree from being used as both P and Q. The importance of this remark will become clearer in Section 8.

We shall discuss non-standard trees in Section 8; non-standard k-trees exist only when $k \geq 9$. We have already enumerated all the 7-trees. As a second example, the standard 9-trees are (00)(00) which has mirror symmetry, together with ((00)0)0, (0(00))0, (11)1 and their reversals or mirror images.

A reduction formula for s_k, the number of standard k-trees, is

$$s_k = \sum_{r=0}^{k-3} s_r s_{k-3-r} \ (k \geq 3), \ s_0 = s_1 = 1, \ s_2 = 0.$$

When $k = 2m + 1$, the number of standard k-trees with mirror symmetry is s_{m-1}.

How do we obtain partially coloured standard trees? If the partially coloured standard tree X is broken down into $X = (PQ)$, it is easy to see that the stippled tiles are not coloured, and hence either the P-trees or the Q-trees must be partially coloured. Continuing the breaking down, we eventually find that at some stage 1-trees occurring in the breakdown must be coloured. We can denote coloured 1-trees by the symbol $\overline{1}$. Thus the partially coloured 4-trees in Figure 6 and 7 are denoted by $(0\overline{1})$, and the 5-trees in Figure 8 by $(1\overline{1})$. The other possible way of partially colouring the 5-tree (11) is $(\overline{1}1)$; this is just the mirror image of $(1\overline{1})$. In contrast, the 9-tree (11)1 has three essentially different partial colourings, namely $(\overline{1}1)1$, $(1\overline{1})1$ and $(11)\overline{1}$.

When partially coloured p-trees of type P and plain q-trees of type Q are sewn together alternately to cover the entire tiling and to form a self-consistent partial colouring of $\{3,n\}$, where $n = p + q$, we can denote this partial colouring by $P.Q$; then the colourings of Figures 6, 7 and 8 are $(0\overline{1}).(01)$, $(0\overline{1}).(10)$ and $(1\overline{1}).(00)$.

Note the distinction between "sewing together trees of types P and Q alternately" to cover the entire tiling, giving a patchwork or a partial colouring denoted by $P.Q$, and "putting together trees of types P and Q with the aid of stippled tiles" to form a tree denoted by (PQ). Note also that $P.Q = Q.P$, but (PQ) and (QP) are in general distinct.

4. THE ENUMERATION OF ALL STANDARD PERFECT PRECISE COLOURINGS

When $n \leq 6$, the only way of obtaining self-consistent partial colourings of $\{3,n\}$ is to use trees whose branches join up to form loops, as indicated below.

$n = 4$. There is only one self-consistent partial colouring. We can think of the six white tiles as forming a 3-tree (00), so the colouring is $(00).\bar{1}$. Alternatively, we can regard it as $(0\bar{1}).0$ or $(\bar{1}0).0$, where all eight tiles form a 4-tree $(0\bar{1})$ or $(\bar{1}0)$ whose bounding edges are joined up to each other.

$n = 5$. The four black tiles of the icosahedron shown in Figure 1 give a self-consistent partial colouring. The remaining tiles form a 4-tree (01), so the partial colouring is $(01).\bar{1}$; but it can also be regarded as $(\bar{1}0).1$ or $(1\bar{1}).0$.

$n = 6$. The partial colouring given by the black tiles in Figure 2 is $(11).\bar{1}$, $(\bar{1}1).1$ or $(1\bar{1}).1$.

These examples show that the same partial colouring can be arrived at in different ways by sewing trees together; but they also provide two clues as to how we can obtain a *unique* symbol for each *standard* partial colouring (a concept to be defined below), and therefore for each standard perfect precise colouring. First, we must make use of trees with loops; secondly, each self-consistent partial colouring is uniquely determined by the $(n-1)$-tree (with loops) formed by the uncoloured tiles. If this $(n-1)$-tree is standard, we shall say that the partial colouring and the associated precise colouring are standard. Thus there is a one-one correspondence between standard partial colourings of $\{3,n\}$ and standard $(n-1)$-trees, *as long as we can be sure that every possible standard $(n-1)$-tree (with loops) that can be constructed according to the method described above actually exists within the tiling* $\{3,n\}$, i.e. as long as the branches of every standard $(n-1)$-tree join up correctly to form loops and do not overlap in an irregular way. Let us see why no irregular overlapping occurs.

Let P, Q and R denote types of p-, q- and r-trees respectively, and consider the patchwork formed when $(p+q+3)$-trees of type (PQ) and trees of type R are sewn together alternately to form the partial colouring $(PQ).R$ that completely covers the tiling $\{3,n\}$, where $n = p+q+r+3$. This patchwork contains stippled tiles, each stippled tile surrounded by trees of types P, Q and R (R along its base, P on its left, Q on its right), and at each vertex trees of types P, Q and R alternate with stippled tiles. Figure 9 can be used to illustrate the general situation if the black regions are labelled R, but it is not a "genuine" example, because the regions labelled R are not trees: each is a 9-gon with a vertex of the tiling at its centre. (The Q-tiles in Figure 9 provide a partial colouring leading to a perfect colouring of $\{3,9\}$ in ten colours. But that is another story; it leads to a way of obtaining perfect colourings of $\{3,n\}$ in $n+1$ colours except when $n = 4$, 5 or 8.) Thus we see that the patchwork initially denoted by (PQ). R can also be denoted by $R.(PQ)$, $(QR).P$, $P.(QR)$, $(RP).Q$ or $Q.(RP)$. (This is reminiscent of the scalar triple product $(a \times b).c = c.(a \times b) = (b \times c).a = \ldots$ in vector algebra.).

We are concerned here with whether the $(p+q+3)$-tree (PQ) (with loops) exists when $n = p+q+4$, or equivalently whether the patchwork $(PQ).1$ exists when $n = p+q+4$. Now either $p \geq 3$ or $q \geq 3$ unless $n \leq 6$, and we know all about the cases $n \leq 6$. Hence without loss of generality $p \geq 3$, and $(PQ).1$ can be written as $P.(Q1)$ which certainly exists since P and $(Q1)$ are genuine types of tree without loops which can be sewn together alternately to cover the tiling $\{3,n\}$.

The number of standard perfect precise colourings of $\{3,n\}$ is therefore s_{n-1}, in the notation of Section 3. The number of colourings increases rapidly with n; for instance, when $n = 20$ there are 6 standard fully perfect precise colourings (see Section 5 below) and 1548 standard chirally perfect precise colourings occurring in 774 left- and right-handed pairs.

5. FULLY PERFECT PRECISE COLOURINGS

The perfect precise colourings of $\{3,4\}$ and $\{3,6\}$ are *fully perfect* rather than chirally perfect: *every* symmetry (direct or opposite) of the tiling permutes the colours. But, as we shall see, there is no fully perfect precise colouring of $\{3,8\}$. We shall show that a fully perfect precise colouring of $\{3,n\}$ exists if and only if n is even and $n \neq 8$.

A fully perfect precise colouring of $\{3,n\}$ occurs if and only if the underlying self-consistent partial colouring has mirror symmetry. Suppose that in Figure 11 we try to construct a fully perfect precise colouring when n is odd. Each of the n colours occurs once at the central vertex, and the rotation α through an angle $2\pi/n$ about the central vertex permutes the n colours cyclically. Since α induces this same permutation of colours on the outer ring of tiles $A, B, C, \ldots$, exactly one tile in this ring must be black. But reflection in the dotted line in the figure maps black to black; hence we obtain *two* black tiles in the ring $A, B, C, \ldots$ *unless the black tile occurs at* A. This is impossible, and hence no fully perfect precise colouring occurs when n is odd. The same conclusion can be reached, for standard colourings only, by considering the (0,1)-symbol for the associated $(n-1)$-tree: for a fully perfect colouring this tree must have mirror symmetry, and hence its (0,1)-symbol must be symmetric, which cannot occur when $(n-1)$ is even.

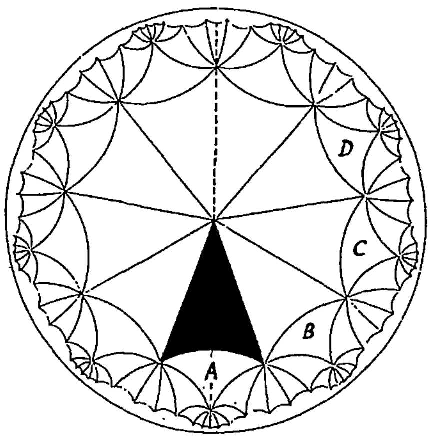

Figure 11

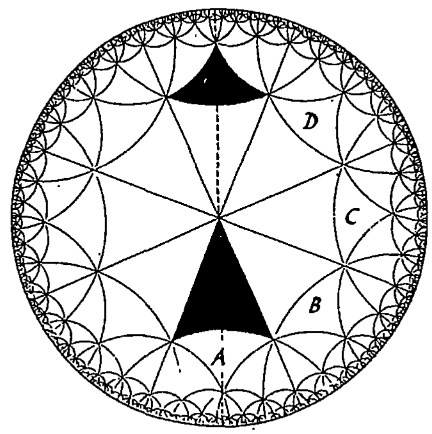

Figure 12

The corresponding situation when n is even is shown in Figure 12. We see by a similar argument that the unique black tile in the outer ring of tiles must occur in the position shown. When $n = 8$, this figure now gives us a rule for proceeding from one black tile to the neighbouring ones, and it can easily be checked that this rule leads to a perfect colouring in 10 colours rather than 8 (Rigby 1991, Fig.18; Rigby 1994, Fig.23). Hence no fully perfect precise colouring exists when $n = 8$.

Let P be any p-tree (not necessarily standard), and let P' be its mirror image. Then (PP') is a $(2p+3)$-tree with mirror symmetry, and the associated precise colouring for $n = 2p + 4$ will be fully perfect. Since p-trees exist whenever $p \geq 0$ and $p \neq 2$, fully perfect precise colourings exist whenever n is even and $n \neq 8$.

6. A SPECIAL TYPE OF COLOURING

The black tiles in Figure 13 determine the same colouring as Figure 6, and the stippled tiles are the tiles of a second colour in this same colouring. The two colours of tile are directly opposite each other at every vertex; they form branched chains of alternate colours. Does such a type of colouring exist for other even values of n? Certainly it does not when $n = 4$ or 6.

One of the patchwork symbols for the partial colouring of black tiles in Figure 13 is $(0\,\bar{1}).(01)$. We can extend our notation and use instead the symbol $(0A).(0B)$, where A denotes the black tiles and B denotes the stippled tiles; the figure is obtained by the sewing together alternately of 4-trees partially coloured with A-tiles, and directly congruent 4-trees partially coloured with B-tiles. Since partially coloured k-trees exist except when $k = 2$, 3 and 6, we can use the same method to obtain precise colourings of $\{3,n\}$ of this special type, when $n = 2k$, by piecing together alternately k-trees partially coloured with A-tiles and with B-tiles, except when $n = 4$, 6 and 12.

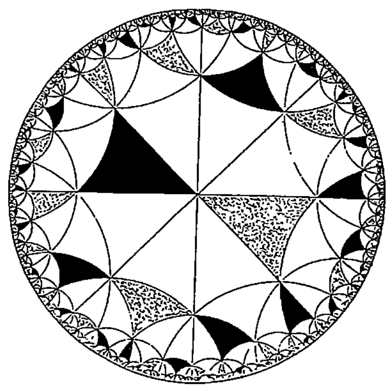

Figure 13

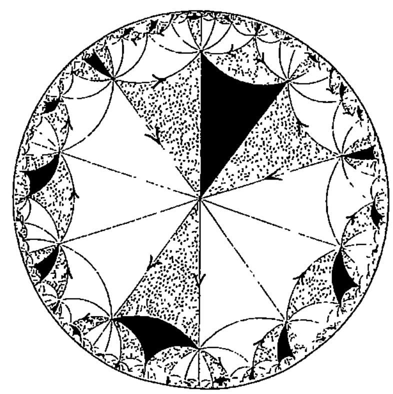

Figure 14

7. FULLY PERFECT COLOURINGS WHEN n IS ODD

The main subject of this article is precise colourings, but it is now natural to ask the question: *are there any fully perfect colourings of* $\{3,n\}$ *(not precise of course) when n is odd*? I had not come across any such before starting this investigation, apart from the obvious colourings of the icosahedron $\{3,5\}$ in ten colours, with opposite faces having the same colour. We can ignore colourings using one colour only, and colourings of polyhedra in which all faces have different colours. In the hyperbolic case, only a finite number of colours are to be used. We shall show that such colourings exist, with twenty-eight colours when $n = 7$, and $2n$ colours for all other odd values of n except $n = 3$.

We need to construct a self-consistent partial colouring with the extra requirement of *complete mirror symmetry*: every symmetry of the tiling that reflects any black tile to itself must reflect the complete set of black tiles to itself.

Let us first look again at fully perfect precise colourings when n is even, taking $n = 10$ as a convenient example. The partial colouring (00)(00). $\overline{1}$ with $n = 10$ (shown by the black tiles in Figure 14) leads to a fully perfect colouring. We shall refer to a tile with its three adjacent tiles as a *node*. Figure 14 can be regarded as black-and-stippled nodes and 3-trees sewn together. The fully perfect partial colouring when $n = 6$ can be regarded in the same manner as nodes and 1-trees sewn together. In general, p-trees and their mirror images can be sewn together with nodes to produce a partial colouring leading to a fully perfect colouring of $\{3, 2p + 4\}$, except when $p = 2$ (see the end of Section 5).

In Figure 14, the nodes are joined together to form *branched chains*. An attempt to produce something similar when $n = 7$ results in Figure 15: we now have branched chains consisting of nodes connected by links (single edges). The chains are separated by strips of white tiles of a type that we shall denote by $s(2,3)$. These strips are not trees. They have vertices of the tiling in their interiors; at each boundary vertex of the strip there are 2 or 3 tiles of the strip, and 2- and 3-vertices occur alternately. The strips are directly transitive on 2-vertices and on 3-vertices: there is a direct symmetry of the strip mapping any 2- or 3-vertex to any other. (Each $s(2,3)$ also has mirror symmetries, but this is not relevant here.)

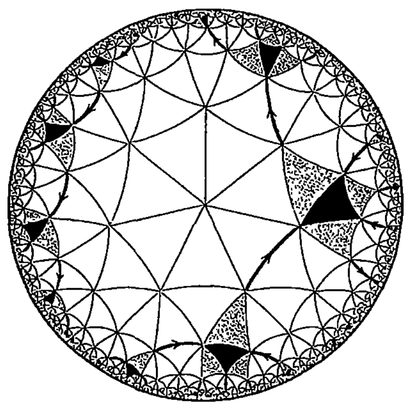

Figure 15

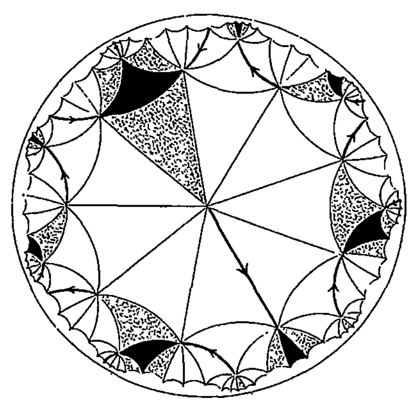

Figure 16

We can indicate an orientation for the boundary of each $s(2,3)$ by means of arrows. If, wherever we stand on the boundary facing in the direction of the arrow, the interior is on our left, the orientation is positive, if "left" is replaced by "right", the orientation is negative; we shall require that the two different disconnected parts of the boundary of an $s(2,3)$ have the same orientation. We can also indicate an orientation for the chains (the same for each chain) as shown in Figure 15. When we sew chains and $s(2,3)s$ together, we require that arrows on chains and adjacent strips must point in the same direction; this ensures that the partial colouring of Figure 15 (and of Figure 16 later) can be completed in a unique manner. The resulting partial colouring is self-consistent and has complete mirror symmetry. Note that adjacent strips have positive and negative orientation alternately. It is easy to check that one twenty-eighth of the tiles in Figure 15 are coloured black, so the complete colouring requires twenty-eight colours.

The corresponding partial colouring when $n = 9$ is shown in Figure 16. The strips are now replaced by what we may call (3,4)-trees, because no vertex of the tiling occurs in their interiors (They are called *alternating trees* in Yaz (1997, Chapter IV).) These strips will be denoted by $t(3,4)$ or its mirror image $t(3,4)'$. At each vertex of $t(3,4)$ there are either 3 or 4 tiles of $t(3,4)$, and $t(3,4)$ is directly transitive on 3-vertices and on 4-vertices; vertices with 3 or 4 tiles occur alternately along the boundary. A way of describing the construction of $t(3,4)$ will be given later. Each $t(3,4)$ in Figure 16 has positive orientation; each mirror-image $t(3,4)'$ has negative orientation. Since $t(3,4)$ does not have mirror symmetry, if we had used instead copies of $t(3,4)'$ with positive orientation and copies of $t(3,4)$ with negative orientation, whilst retaining the existing orientation for the chains, we should have obtained a partial colouring different from Figure 16.

All we need to do now is to construct, in an inductive manner, $t(k,k+1)$s for each k ($k \neq 2$), directly transitive on k-vertices and on $(k+1)$-vertices; $t(1,2)$ is just a pair of adjacent tiles, and $t(2,3)$ does not exist which is why we had to use $s(2,3)$ in Figure 15. These $t(k, k+1)$s are more complicated than the trees in Sections 3 and 4; we shall not attempt to construct all such trees, and only a brief description of the construction will be given.

Basically, we take an existing $t(k, k+1)$ of type P say, and an ordinary q-tree of type Q, and put together trees of types P and Q with the aid of stippled tiles as in Section 3, to produce a $(k+q+3, k+q+4)$-tree of type (PQ), or alternatively of type (QP). We must perform the construction carefully so that (PQ) has $(k+q+3)$-vertices and $(k+q+4)$-vertices alternately; if this is done it can be verified that (PQ) is transitive as required.

Figure 17 illustrates a $t(4,5)$ with $P = 0$ (a 0-tree) and $Q = t(1,2)$. Figure 18 shows a $t(5,6)$ with $P = t(1,2)$ and $Q = 1$.

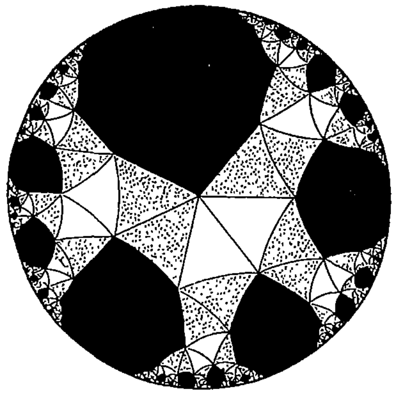

Figure 17

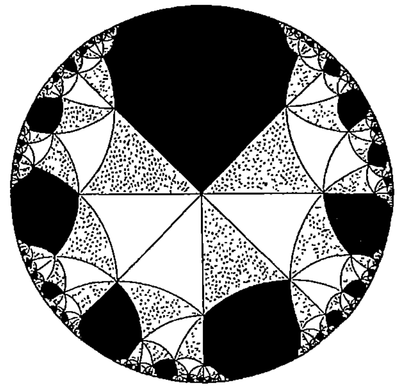

Figure 18

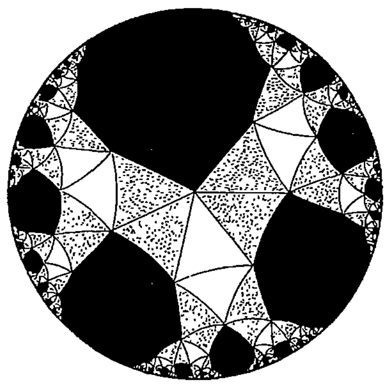

Figure 19

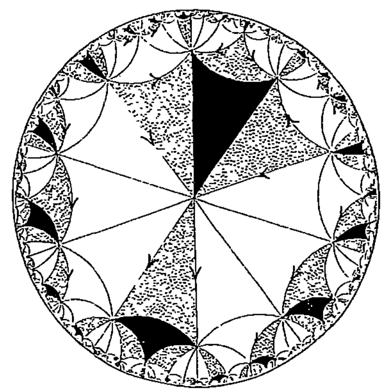

Figure 20

The tree in Figure 17 is made up of white $t(1,2)$s and stippled diamonds; every edge of each $t(1,2)$ is joined to a diamond, but only two edges of each diamond are joined to a $t(1,2)$. In Figure 19 the $t(1,2)$s and diamonds are joined differently to obtain a different $t(4,5)$. We can always join $t(k, k+1)$s and diamonds in two different ways to produce two $t(k+3, k+4)$s. The tree in Figure 18 is made up by joining white $t(1,2)$s and white-and-stippled nodes; if we try to "reverse" half of the nodes (by joining their other three edges to the $t(1,2)$s) as we did with the diamonds in Figure 18, we obtain a $t(4,7)$. This by itself is of no direct use in the present context, but if we join diamonds or nodes correctly to $t(4,7)$ we can get a new type of $t(8,9)$ or $t(9,10)$. The possibilities seem endless.

We can now join $t(k, k+1)$s with positive orientation, and their mirror images with negative orientation, to branched chains as in Figure 16, to create a partial colouring of $\{3, 2k+3\}$ with mirror symmetry, except when $k = 2$. Since a black tile occurs at half the vertices of this partial colouring, it leads to a fully perfect colouring in $2(2k+3)$ colours.

The colouring of the icosahedron $\{3,5\}$ in ten colours can be obtained in this way: the chain then consists of two nodes joined by six links, and the remaining tiles form six $t(1,2)$s.

Finally, when $n = 7$ (compare Figure 15) we can reverse the directions of the arrows on one of the two sides of the boundary of the strip $s(2,3)$, and the strip will still be transitive on 2-and 3-vertices, because there are glide reflections interchanging the two sides of the boundary. We can then sew together such newly oriented $s(1,2)s$ with branched chains to create a new self-consistent partial colouring with mirror symmetry, different from Figure 15. Is there a similar procedure sometimes with $t(k, k+1)s$?

8. NON-STANDARD PERFECT PRECISE COLOURINGS

In Figure 14 arrows have been inserted so that each 3-tree has either positive or negative orientation. But suppose we take 3-trees each of which is assigned one positive edge and one negative edge, and sew them to chains of nodes oriented as before, so that the directions of the arrows match. The result is shown in Figure 20; this is a partial colouring with mirror symmetry which yields a fully perfect precise colouring of $\{3,10\}$, but it is not derived from a standard 9-tree.

Before we analyse exactly how the 9-tree of non-black tiles in Figure 20 differs from a standard tree, let us make the analysis clearer by constructing another non-standard tree. First I constructed, in an *ad hoc* manner, a *pseudo-tree* in which some sections of the boundary have p tiles at each vertex, and some have q tiles at each vertex. The simplest example with $p \neq q$ seems to occur with $p = 5$; it is shown as a pseudo-tree with loops in Figure 21 (my original construction) and without loops in Figure 22. This pseudo-tree is directly transitive on 4-vetices and on 5-vertices, and any pseudo-tree of this type will be called a pseudo-tree of type ps(4,5). Let us now put together copies of $ps(4,5)$ with the aid of stippled tiles to form a tree of type X, in such a way that each stippled tile has its base along the boundary of X, its left side adjacent to a 4-boundary of a ps(4,5), and its right side adjacent to a 5-boundary of a ps(4,5); and every edge of every ps(4,5) must be adjacent to a stippled tile. The result can be seen to be a 12-tree X, satisfying the conditions for a tree given in Section 2.

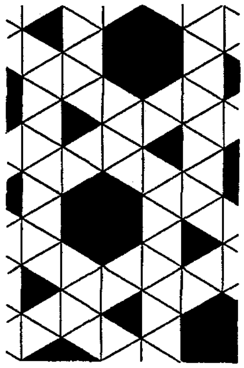

Figure 21

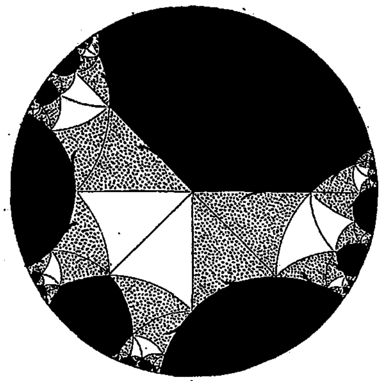

Figure 22

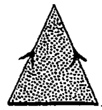

Figure 23

We can describe this process differently. Assign a positive orientation to each 4-boundary of each $ps(4,5)$ and a negative orientation to each 5-boundary, as shown in Figure 22, and assign orientations to the left- and right-hand edges of each stippled tile as shown in Figure 23. Then put together $ps(4,5)$s and stippled tiles so that the orientations match. This is just what we did in Figure 20, where the 3-trees must now be regarded as (3,3) pseudo-trees. In contrast, when we put together trees of types P and Q with stippled tiles in the standard way to produce the tree (PQ), we can regard every section of the boundary of each P as having positive orientation, with negative orientation for the boundary of each Q.

Various questions remain. Do pseudo-trees provide the only method of constructing non-standard trees? How do we construct pseudo-trees? Is there an algorithm for constructing all pseudo-trees? Without doing further research, I can give a sketch of an answer to the second question only.

If we break down the $ps(4,5)$ in the usual way with the aid of stippled tiles, as shown in Figure 22, we see that it is obtained by putting together 0-trees, $t(1,2)$s and stippled tiles in a suitable way (but compare Figure 17 where the same elements are put together differently). I conjecture that we can put together copies of $t(p,q)$, $t(r,s)$ and stippled tiles to produce $ps(p+r+3, q+s+3)$; but then we need a method of constructing all types of $t(p,q)$. See Yaz (1997, Chapter IV).

9. SEMIPERFECT PRECISE COLOURINGS

A colouring in which half the direct symmetries and half the opposite symmetries of the tiling permute the colours, but the remaining symmetries jumble the colours, is called a *semiperfect* colouring. When n is even, we can label the tiles of $\{3,n\}$ positive and negative alternately, like a generalised chessboard. If we can find a partial colouring with black tiles such that *there is a set of rules for proceeding from any positive black tile to the neighbouring black tiles, and the mirror images of those rules lead from any negative black tile to the neighbouring black tiles,* then this partial colouring leads to a semiperfect colouring. There is a semiperfect colouring of $\{3,8\}$ in seven colours; two colours of this colouring are shown in Figure 24, reproduced from Rigby (1990, Figure 18), but let us now look for semiperfect *precise* colourings. *All trees in this section are standard trees.*

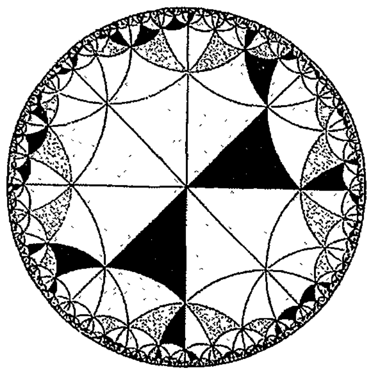

Figure 24

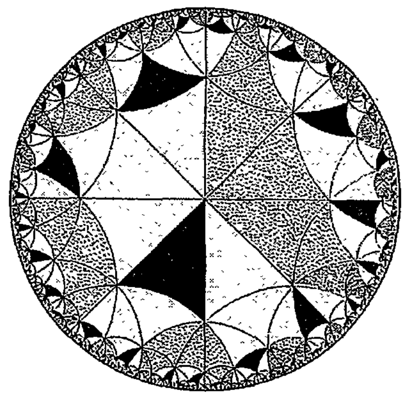

Figure 25

Consider the patchwork $(1\bar{1}).(00)$ (Figure 8), and label the tiles positive and negative alternately, as just described. The two bounding edges of a (00) can be described as positive and negative, since all the tiles adjacent to one boundary are positive, and all those adjacent to the other are negative. But the bounding edges of a $(1\bar{1})$ are either all positive or all negative, so a $(1\bar{1})$ can be called positive or negative. The black tiles of a positive $(1\bar{1})$ are all negative, and vice versa. Now, change each positive $(1\bar{1})$, in the patchwork $(1\bar{1}).(00)$, to $(\bar{1}1)$; the resulting partial colouring satisfies our requirements. This partial colouring is shown in Figure 25; note that it does not have mirror symmetry like the partial colouring of black tiles in Figure 24, but there are opposite symmetries (glide reflections with axis running down the middle of one of the 3-trees) interchanging positive and negative tiles and mapping the set of black tiles to itself.

When n is even, suppose we can find (a) a k-tree $t(k)$ where k is odd, such that there exists a glide reflection mapping $t(k)$ to itself and interchanging positive and negative edges, and (b) a partially coloured $(n-k)$-tree $T(n-k)$ all of whose edges have the same sign. Suppose that $T(n-k)$ has positive edges, and let $T(n-k)'$ be a mirror image of $T(n-k)$ with negative edges. We can cover the whole plane by piecing together directly congruent copies of $t(k)$, $T(n-k)$ and $T(n-k)'$, always joining a positive edge to a negative edge, and negative to positive. This will produce the black tiles of a semiperfect precise colouring.

To show that $t(k)$ and $T(n-k)$ exist with the required properties, we need to consider the symmetries of trees. Consider a particular example. Figure 26 is a redrawing of Figure 10, and shows a 7-tree (10)0. The white tiles form 4-trees, corresponding to the (10) component of the symbol (10)0. In the construction of these 4-trees (as described in Section 3), the unlabelled white tiles correspond to the symbol 1, the dotted edges correspond to the symbol 0, and the tiles labelled s are stippled. The heavily drawn edges in the figure correspond to the second 0 in the symbol (10)0. It should be clear from the method of construction of standard trees that, *in any standard tree, the centre of any tile corresponding to a* 1 *in the* (0,1)*-symbol is a centre of threefold rotational symmetry of the tree, and the mid-point of any edge corresponding to a* 0 *in the* (0,1)*-symbol is a centre of twofold rotational symmetry (or "centre of symmetry" for short) of the tree. These remarks still hold if the tree is partially coloured, indicated in the* (0,1)*-symbol by replacing one of the* 1*s by* $\bar{1}$. We note also that in a symmetrical standard tree (one whose (0,1)-symbol is symmetrical), the perpendicular bisector of every bounding edge is a line of symmetry.

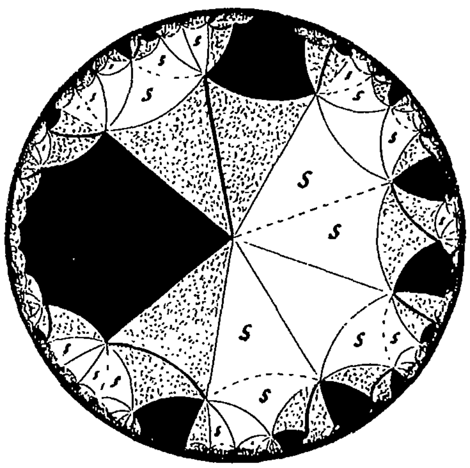

Figure 26

Since a half-turn about the centre of an edge interchanges positive and negative tiles, a partially coloured $T(n-k)$ as described above will have all its black tiles of the same sign only if there are no 0s in its tree-symbol; we can prove by induction that conversely, if there are no 0s in its tree-symbol, all the black tiles of $T(n-k)$ have the same sign, and so do all its edges. A $T(n-k)$ with these properties exists if and only if $n - k$ is of the form $4m + 1$. But $T(1)$ is of no use in the present context because it is symmetrical when partially coloured, and therefore when it is combined with a symmetrical $t(k)$ (as described in the next paragraph) it will yield a fully perfect colouring.

Let $t(k)$, where k is odd, be a symmetrical tree with a 0 in its symbol. Let α denote the reflection in m, the perpendicular bisector of a bounding edge of the tree, one of its lines of symmetry, and let β denote the half-turn about the point A, one of its centres of symmetry. Since A is the midpoint of an internal edge, A does not lie on m; also α preserves signs and β interchanges signs. Hence $\alpha\beta$ is a glide reflection that maps $t(k)$ to itself and interchanges positive and negative edges. A $t(k)$ with these properties exists except when $k = 1, 5, 7$ or 13.

We conclude that semiperfect colourings can be constructed by this method except when $n = 4, 6$ or 10.

10. IMPERFECT PRECISE COLOURINGS

In our construction of chirally perfect precise colourings of $\{3,n\}$, we have used $(n\text{-}1)$-trees to determine the tiles of a single colour; it should be noted that the trees associated with tiles of different colours are quite distinct from each other, although of the same type.

We shall now use trees in a somewhat different way. Each 4-tree in Figure 4 can be precisely coloured in four colours as in Figure 27. "Precise" here means that one tile of each colour occurs at each vertex of the tree. A different but equally logical precise colouring is shown in Figure 28, and "random" precise colourings are also possible. The 3-trees in Figure 4 can be precisely coloured in three other colours. Since the three or four colours in each tree can be permuted, we can obtain in this way an infinity of imperfect precise colourings of the entire tiling. Moreover, when joining the trees together, we can at any time use a tree of type (10) instead of (01), producing further irregularity or imperfection.

I conjecture that it is always possible to create a precise colouring of any k-tree in k colours, except possibly a k-tree with loops embedded in the tiling $\{3, k + 2\}$, and that this colouring can be carried out in a reasonably symmetrical and perhaps a perfect way (so that every direct symmetry of the tree permutes the colours).

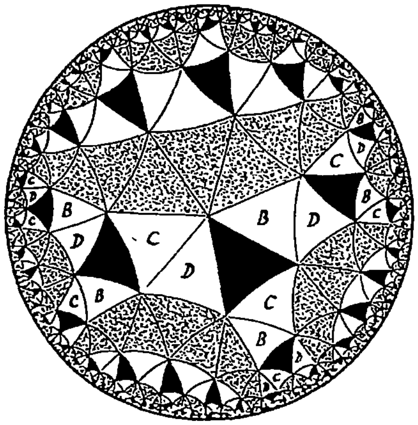

Figure 27

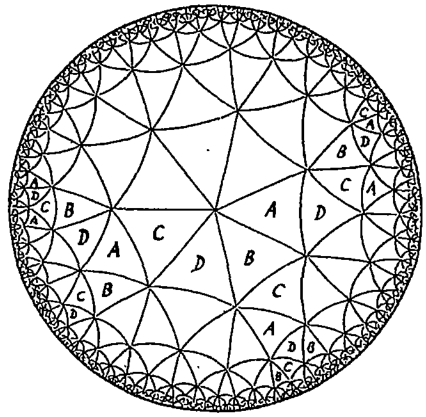

Figure 28

For larger values of n, more possibilities occur. For instance, there are three distinct types of 7-tree, each with its mirror image; when $n = 11$ we can join different types of 7-tree (precisely coloured in seven colours) and 4-trees (precisely coloured in four further colours) to produce an infinity of precise colourings of $\{3,11\}$.

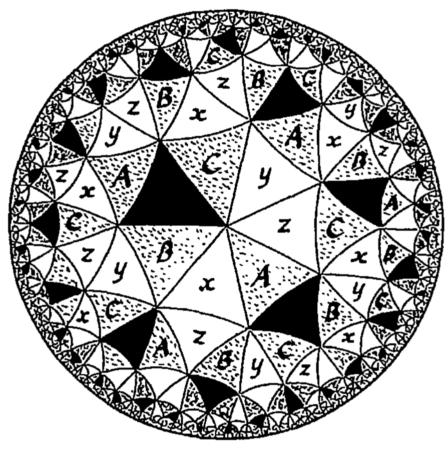

Figure 29

I found the pleasing pattern of Figure 29 when I was just experimenting with methods of making patchwork; flowers with a black centre and three petals alternate with leaves formed by two tiles. (In the notation of Section 8, the non-black tiles form a pseudo-tree with loops: *ps*(6,6).) This can be used as the basis of a subtle precise colouring: the flowers are of two types, according to the cyclic order of the three colours in the petals, and the two types of flower are surrounded in quite different ways by the *x*, *y* and *z* colours of the leaves.

11. EQUIVALENT COLOURINGS

We shall introduce the idea of *equivalent* precise colourings of $\{3,n\}$ by considering just one example – the simplest non-trivial example. In Figure 30, where $n = 9$, the tiles labelled *A* form a self-consistent partial colouring (leading to a chirally perfect precise colouring); the remaining tiles form an 8-tree whose symbol may be written $(CB)0$ rather than (11)0, using the notation introduced in Section 6. The complete patchwork of Figure 30 can be denoted by any of the symbols $(CB)0.A = CB.0A = C.B(0A) = B(0A).C = B.(0A)C$. This notation shows that the *A*-tiles lead to the precise colouring (11)0, the *B*-tiles lead to (01)1, and the *C*–tiles to 1(01). These are three distinct non-isomorphic precise colourings. It is important to emphasize that the patchwork is self-consistent: any direct symmetry of the tiling that maps one *A*-tile to another *A*-tile maps all *A*-tiles to *A*-tiles, all *B*-tiles to *B*-tiles, and all *C*-tiles to *C*-tiles.

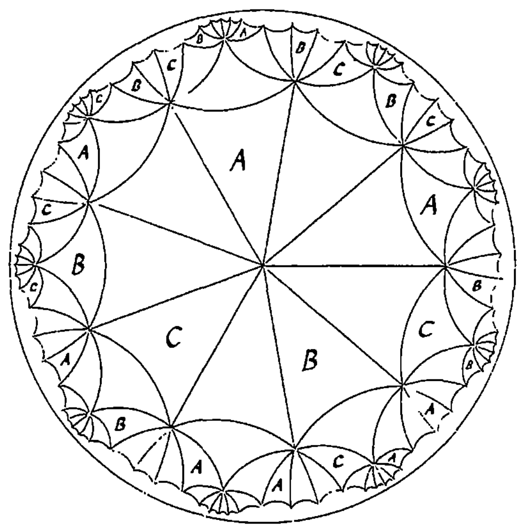

Figure 30

The direct symmetries of the tiling $\{3,n\}$ induce a group of permutations of the colours in any perfect colouring: the *(direct) colour permutation group* of the colouring. We shall show that *any direct symmetry of the tiling induces the same colour permutation in the three colourings described above,* so that the three non-isomorphic colourings have ismorphic colour permutation groups; for this reason we call them *equivalent* colourings. But note that the statement in italics makes sense only if the three tilings are suitably coloured using the same nine colours (as described in the next paragraph), and only if the three colourings are suitably overlaid so that they become three colourings of *the same tiling* (rather than of isomorphic tilings).

Let us then consider in more detail how we can produce the three colourings. Consider three copies of the tiling $\{3,9\}$ overlaid on each other; label the copies A, B and C. Think of Figure 30 as a template that can be placed over the copies of the tiling in any position (as long as edges are placed over edges). (This is of course a hyperbolic template: Figure 30 regarded as a Euclidean figure can only be placed in nine ways over the Euclidean representation of the tiling.) Now perform the following sequence of operations. (1) Place the template over the copies of the tiling. Colour the A-tiles black in Copy A, colour the B-tiles black in Copy B, and colour the C-tiles black in Copy C. (2) Place the template in a new position over the copies of the tiling, in such a way that a non-black tile in Copy A is now marked A. Because of the self-consistency of the template, no black tile in Copy A is now marked A, no black tile in Copy B is marked B, and no black tile in Copy C is marked C. Colour the A-tiles red in Copy A, colour the B-tiles red in Copy B, and colour the C-tiles red in Copy C. (3) Place the template in a new position, in such a way that a non-black and non-red tile in Copy A is now marked A, and proceed as before: use a third colour to colour the A-tiles, B-tiles and C-tiles in Copies A, B and C respectively. Continuing in this way, we eventually arrive at the three perfect precise colourings in nine colours.

Nine positions of the template are used to produce the colourings (we can refer to "the black template", "the red template", etc.), and *any direct symmetry of the tiling simply permutes the nine positions of the template, and hence produces the same permutation of the nine colours in all three colourings;* but this statement needs more explanation and discussion to make it convincing, so we shall use a different approach that leads to the same conclusion and that exhibits an extra property of the colourings.

Denote the angle $2\pi / 9$ by θ. Rotation through 4θ (clockwise) about any vertex of the template maps the A-tile at that vertex to the B-tile. Let P be any vertex of the three copies of the tiling. Rotation through 4θ about P maps the black tile at P in Copy A to the black tile at P in Copy B, and the same is true for any other colour. Hence, *rotation* *through 4θ about P maps the complete ring of colours around P in Copy A to the ring of colours around P in Copy B.* Similarly *rotation through 2θ about P maps the ring of colours around P in Copy B to the ring of colours around P in Copy C.* We emphasize that this is true for every vertex P.

Let α denote the rotation through θ about P; then α induces the same permutation of colours in all three colourings, namely the 9-cycle determined by the clockwise order of the colours around P. Let β denote the rotation through θ about an adjacent vertex Q; then β induces the same permutation of colours in all three colourings. But the direct symmetry group of the tiling is generated by α and β (Yaz, 1997, 2.1); hence any direct symmetry of the tiling induces the same colour permutation in all three colourings, and therefore the direct symmetry group of the tiling induces the same colour permutation group in all three colourings.

The "extra property" exhibited by this approach is that the colours around P in Copy B can be obtained from the colours around P in Copy A by means of a rotation *whose angle is independent of P* (namely 4θ in this instance).

The reader can easily investigate further examples of equivalent colourings. When $n = 5$, the two perfect precise colourings of the icosahedron are equivalent as well as being reflections of each other. When $n = 8$ and 10, some of the colourings are equivalent to their reflections. When $n = 11$, various non-isomorphic colourings occur in equivalent pairs. The next interesting cases, with triples of equivalent non-isomorphic colourings, occur when $n = 12$; this is because the (0,1)-symbols of all 11-trees contain two 1s.

12. FURTHER ASPECTS OF THE SUBJECT

Robinson's quite different method for finding precise colourings when n is even is described in Rigby (1998). A fuller discussion of non-standard trees, an algorithm for obtaining generators for the colour permutation group associated with the colouring determined by a specified tree, and an investigation of perfect colourings of $\{3.n\}$ using $n + 1$ colours, can be found in Yaz (1997). I am currently investigating the connection between perfect precise face colourings and perfect precise *edge colourings.*

I am grateful to Douglas Dunham for providing me with computer drawings of $\{3,10\}$. All the remaining drawing and painting was done by hand.

REFERENCES

Mackenzie, D. (1995) A hyperbolic plane coloring and the simple group of order 168, *American Mathematical Monthly* 102, 706-715.

Rigby, J.F. (1990) Symmetry in geometry: a personal view, *Symmetry: Culture and Science* 1, 63-76.

Rigby, J.F. (1991) Compound tilings and perfect colourings, *Leonardo* 24, 53-62; also reproduced in: Emmer, M., ed., *The Visual Mind,* Cambridge Massachusetts: MIT Press, 1993, xvii + 274 pp., 177-186.

Rigby, J.F. (1994) Fun with tessellations, in: Guy, R.K. and Woodrow, R.E., eds., *The Lighter Side of Mathematics,* Washington: The Mathematical Association of America, 1994, viii + 367 pp., 74-90.

Rigby, J.F. (1996), Perfect precise colourings of triangular tilings, in: *Katachi U Symmetry*, Tokyo: Springer-Verlag. 201-206.

Rigby J.F. (1998) Precise colourings of regular triangular tilings, *The Mathematical Intelligencer,* 20, no. 1, 4-11.

Robinson, R.M. (1993) Problem 10349, *American Mathematical Monthly* 100, 952.

Yaz, N. (1997) *Perfect Precise Colourings of Regular Triangular Tilings.* [M.Phil. Dissertation], Cardiff: University of Wales, Cardiff, 117 pp.

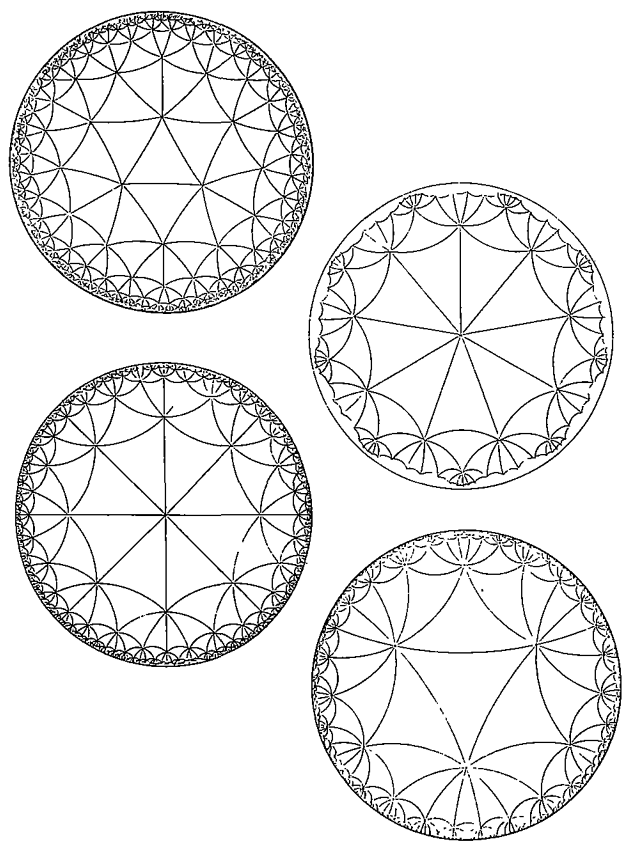

Figure 31

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DIGITAL EVOLUTIONS FOR REGULAR POLYHEDRA

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Visualization of geometric forms has been realized for centuries using two-dimensional representations on paper. This type of representation works well for two-dimensional shapes but is limited for three-dimensional forms. A two-dimensional representation is often only one of the many views necessary for the complete visual understanding of a three-dimensional form. Orthographic projections such as plan and elevation views are not sufficient to visualize complex shapes. Physical models provide a better understanding of three-dimensional forms (Cundy and Rollet, 1961) but they are cumbersome to construct and limited by size; sometimes a form should be seen from its inside for a complete comprehension of its spatial characteristics. Furthermore, the materials used to construct the model often distract from the properties of the "ideal" shape as geometrically defined.

Computer graphics has revolutionized the visualization process and the realization of models. As with physical models, electronic models can be visualized from any viewpoint at any position and distance, but are not limited by size and can be seen from inside as well outside.

The images shown in the each of the illustrations represent top, frontal and perspective views of computer generated models of regular polyhedra as seen from the outside; some of the models are also represented as viewed from the inside.

COMPUTER-AIDED DESIGN AND VISUALIZATION

Beyond its potential for visualization, the computer offers the possibility of generating electronic models using computer-aided design (CAD) software. This type of computer graphics application offers a quite specific data structure, characterized by the use of *instances* (Bertol, 1994). An instance can comprise any of the geometric entities, which can be generated in CAD with associated parameters such as position, scale, and rotational angle. These parameters are related to the other main engine for the development of a computer model, that represented by the *geometric transformations* (Bertol, 1994). In a CAD model each element/part subjected to repetition can be purposefully defined as instance. A model made of instances focuses on the simulated shape as a *whole* comprised of *parts.*

The content of an instance can be redefined with different geometric elements while the parameters associated with it are conserved. The change of content in one instance propagates to all the other instances bearing the same name. The redefinition of instances in a model can generate completely different geometric characteristics even if the spatial and symmetry relations between the elements defined as instances are conserved. A model made of instances can therefore *"evolve"* in completely different forms which have in common the symmetry of the initial configuration: e.g. a row of triangles with a side of 5" at a distance of 9" can be transformed into a row of spheres with a radius of 3". The use of instancing is particularly effective in the case of forms strongly characterized by symmetry, such as the regular polyhedra.

REGULAR POLYHEDRA AND SYMMETRY

The number of possible regular convex (two-dimensional) polygons - triangle, square, pentagon, hexagon, etc. - is infinite. Conversely the number of the analogous entities - regular polyhedra - in three-dimensional space is limited to five: tetrahedron, octahedron, icosahedron, cube and dodecahedron. The regular polyhedra are also called the Platonic solids, after the philosopher Plato (IV century B.C.), who thoroughly discussed the characteristics of these solids in his dialogue *Timaeus* (Plato, 1965). Many artists and mathematicians through the centuries have been fascinated with the regular polyhedra.

In each regular polyhedron all the vertices, edges and faces are equivalent, and the faces are regular polygons (Hilbert and Cohn-Vossen, 1952). The name of the polyhedron expresses the number of faces. The definition of each solid is therefore given by the type and the number of *faces*, *edges*, and *vertices* as well as their position and relations in three-dimensional space.

POLYHEDRON	VERTICES	EDGES	FACES	EDGES PER FACE	EDGES PER VERTEX
TETRAHEDRON	4	6	4	3	3
CUBE	8	12	6	4	3
OCTAHEDRON	6	12	8	3	4
DODECAHEDRON	20	30	12	5	3
ICOSAHEDRON	12	30	20	3	5

In the present discussion the emphasis is on the symmetric structure specific to each polyhedron. The regular polyhedra represent the possible regular symmetric configurations in three-dimensional space (Weyl, 1952); rotational symmetry about the center of the polyhedron rules the position of each vertex, edge and face.

The polyhedron significance goes beyond its geometric shape since it becomes a definition of compositional rules. The models shown in the illustrations follow this approach: different types of shapes replace the geometric elements defining the polyhedron, which becomes a schematic diagram embodying its potential evolutions.

POLYHEDRA AND INSTANCES

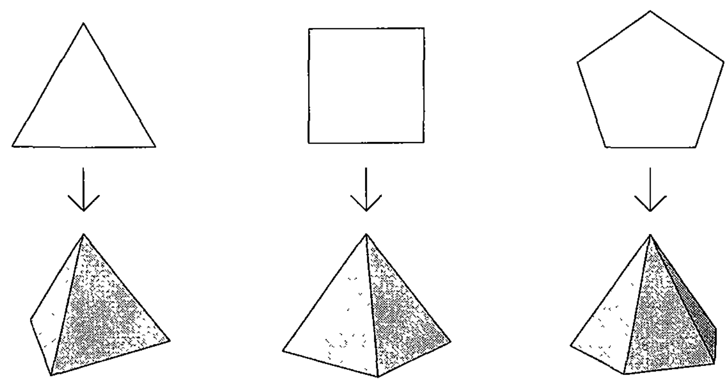
Each of the geometric elements inherent to the architecture of a regular polyhedron - vertices, edges and faces - can be defined as instance, characterized by a *position* and a *rotation* angle relative to the center of the polyhedron. The creation of a CAD database defining a polyhedron exemplifies a situation where the model definition based on the symmetry relations between elements is pregnant of many different possible transformations. The *evolutions* - as defined in previous sections - for each polyhedron depict possible formal transformations involving the change of geometric characteristics while keeping the spatial relations between faces, edges and vertices.

In the evolutions of the models of polyhedra shown in the illustrations, the geometric entities which make the original polyhedra are completely transformed. The evolved model, in some cases, does not resemble a polyhedron any longer. While the contents of the instances are changed, the symmetry relations are kept. The numbers of faces, edges and vertices defining each polyhedron are conserved as well.

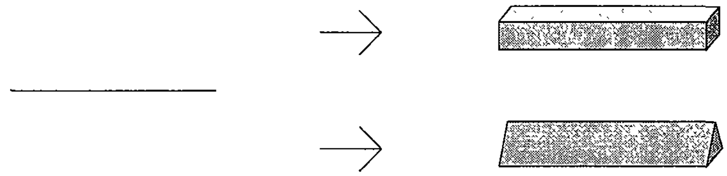
The definition of the elements which make a polyhedron in terms of instances bridges the realm of ideal geometry, made of mere geometric entities, such as points, lines and planes, to the realm of constructable realizations, made of physical materials which are subject to many constraints. For example, the polygons defining the faces of the polyhedra can be assigned a thickness, or similarly, the edges can evolve from a line to a cylinder or prism, changing their geometry characteristics of two-dimensional and uni-dimensional entity to three-dimensional objects assimilable to physical materials.

EVOLUTIONS OF THE REGULAR POLYHEDRA

One of the simplest evolutions is the transformation of each polyhedron into its stellated match (Kepler, 1619). In this transformation the faces defining the polyhedron are replaced by pyramids whose base is identical with the polygon which define each face of the polyhedron.



The "open" polyhedra, already illustrated by Leonardo da Vinci (Pacioli, 1509), can easily be evolved from the basic polyhedra by replacing the edges with prismatic elements. In this type of polyhedra the definition of enclosure, present in the regular polyhedra, no longer exists.



Further evolutions are possible from the basic models of the regular polyhedra by replacing vertices, edges and faces, with a variety of forms resembling natural and man-made elements. A well known example is Haeckel's *Kunstformen der Natur*, where many illustrations of natural forms, such as protozoa, sponges, starfishes and several plants, clearly resemble the regular polyhedra (Haeckel, 1974). In this type of "evolution", vertices, edges and faces completely loose their geometric meaning, assuming complex formal characteristics.

Beyond the stellated and open polyhedra evolutions, the models shown in the illustrations include two additional types of evolutions. In some of the polyhedra evolutions, such as those shown for the tetrahedron, cube and octahedron, the redefined geometric elements are assimilated to physical elements: vertices are replaced by joints and edges by bars. This type of model can bring insights about their physical construction and can be purposefully used to represent and investigate space frame structure.

In the other models derived by the cube and octahedron, vertices are replaced by revolution surfaces, generated by the rotation of a circular arc around the axis defined by the original vertex and the center of the polyhedron. The re-defined faces connecting the revolution surfaces create interesting enclosures: the relationship between inside and outside is completely transformed from that of the original polyhedron.

Other shapes represent ideal, gravityless structures, such as the model evolved from the dodecahedron, where vertices have been replaced by space frame towers. Both of the additional models derived by the icosahedron resemble spaceship architectures: in one model trussed arches replaces the edges while in the other the original vertices evolve in helices.

The digital models shown in the illustrations represent only some of the unlimited number of models which can be derived from the regular polyhedra. The potential presented by electronic models can offer insights in the investigation of symmetric configuration, offering a contemporary interpretation of the regular polyhedra, which have continued fascinating scientist and artist for thousands of years.

REFERENCES

Bertol, D., (1994) *Visualizing With CAD*, New York: Springer-Verlag.

Coxeter, H. S. M., Greitzer, S. L. (1967) *Geometry Revisited*, New York: Random House.

Cundy, H. M., Rollet, A. P. (1961) *Mathematical Models*, Oxford University Press.

Ghyka, M., (1977) *The Geometry of Art and Life*, New York: Dover.

Haeckel, E. (1974) *Art Forms in Nature*, New York: Dover

Hilbert, D., Cohn-Vossen, S. (1952) *Geometry and Immagination*, New York: Chelsea.

Kepler, J., (1619) *Harmonices Mundi*, Lincii Austriae.

Pacioli, L. (1509) *De Divina Proportion*, Venetiis: A. Paganius Paganinus.

Plato, (1965) *Timaeus*, Baltimore: Penguin Books, Baltimore.

Weyl, H. (1952) *Symmetry*, Princeton: Princeton University Press.

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TETRALECTICS - AN APPROACH TO POSTMODERN LOGIC*

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Abstract: *Tetralectics is a new type of logic with a novel way of validation and a demand for a three dimensional geometrical representation, a metatheory for scientific theories, a logic of scientific theory-building. Its intellectual background includes the causal theory of Aristotelian philosophy, Hegelian dialectics and the postmodernist preference for plurality. The rich heritage of them allows tetralectics to become a method for treatment of several different co-represented oppositions.*

The four Aristotelian causes are transformed into four reinterpreted concepts, which are arranged in tetrahedron-form and their relations are analysed applying the symmetry properties of this perfect body. The symmetry elements of the tetrahedron represent oppositions. The four concepts, their arrangement, the oppositions and the assignment of the oppositions make up the formal system of tetralectics.

In the system of tetralectics, treatment of the difficult consequences of the interrelatedness of elements of knowledge reveal that the central concepts have the characteristics of metatheories. These metatheories in the tetralectics of natural sciences are: material, space-time, action and change. The metatheories are comparable to conventional theory families of sciences. The division of the central concepts into sub-concepts facilitates the construction of a specific theory.

The three level description of tetralectics allows the development of a new validation of statements, and crosswalks between the (meta)theories created by symmetry adopted transformations guarantee a flexible nature to tetralectics, in this context the Gödel argumentation has a more friendly face.

TETRALECTICS AS A NEW LOGIC

The aim of the different sorts of logic is to arrive at the general rules of reasoning. To achieve this, a logic has to represent all the relations between the constituents of the given logical universe. The proper representation of a given logical structure can significantly stimulate a deeper understanding of the similarity and dissimilarity of different logics. But what is the suitable representation of these relations? If one tries to represent the main known logical systems geometrically, only some very simple geometry can be applied. Most logics have only one – in some cases two – dimensional representations. This is sufficiently illuminating in the case of most logics, but unable to reflect the rich logical context of many real situations. For example: how can we represent the multiple connections of a scientific concept? Tetralectics suggests a solution: to move over to the third dimension. So *tetralectics is a logic in three dimensions.*

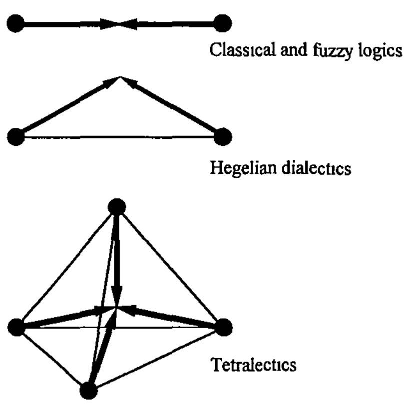

Figure 1: A possible representation of the classical and fuzzy logics, Hegelian dialectics and tetralectics.

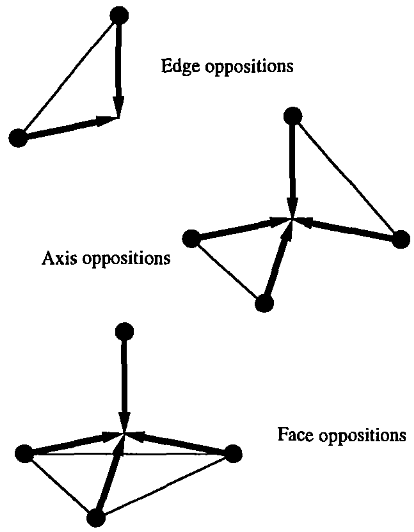

Figure 2: Different classes of oppositions in tetralectics.

Figure 1. shows one dimensional representation for classical and fuzzy logics, a two dimensional for Hegelian dialectics and a three dimensional for tetralectics. The way in which each logical system treats the oppositions and their relations is characteristics. Therefore, we depict the oppositions coexistent in the same logical system as pairs of arrows which meet at their peak. The directions of the arrows show the functioning of the oppositions. Different methods are applied in these different logics at the treatment of the problem of the coexistent opposites, as can be seen in Figure 1. We can call the classical and fuzzy logics one dimensional since both the starting points and the concluding point can be put on one line. In Hegelian dialectics the emergent concept has new qualities as compared to those of its determinant oppositions. The concluding point cannot be represented in terms of the starting points, therefore, it opens a new dimension. Thus the system of the starting and concluding points span a two dimensional space. Tetralectics uses not two, but several oppositions. To hold them together and to express their sophisticated relations, we arrange them on a tetrahedron, thus we need a three dimensional representation.

In fact, there are three different classes of oppositions in tetralectics (which cannot be distinguished in Figure 1.), namely, the edge, axis and face oppositions in the order of their appearance in Figure 2. In this figure the similarities and dissimilarities between the dialectical and the tetralectical treatment of oppositions can be considered. Each edge opposition is equivalent to one in Hegelian dialectics (we also call these oppositions 1 - 1 oppositions), each axis opposition composed of two Hegelian dialectical one (these are the 2 - 2 oppositions), and each face opposition (1 - 3) link three Hegelian dialectical oppositions. Tetralectics possesses 6 edge, 3 axis and 4 face oppositions - as we can see later. There we will use a maybe more transparent notation for these classes of oppositions.

As an important consequence of this choice the strict *unambiguity* of the logical manipulations is lost and a special type of *plurality* appears in tetralectics. For example: a scientific concept embodied in a theory has many - but very different - relations to other scientific concepts. Following the logic of the standard way of scientific theory building, this complex context of concepts must be essentially simplified. Tetralectics suggests a treatment for the contextual plurality of concepts. The objective of tetralectics is to collect the greatest possible number of oppositions of the system studied, to keep them together and to let them work. The results of their work, i.e., the more perfect description of the system in question cannot be derived from one theory, but a class of interrelated theories. Tetralectics suggests a treatment for a plurality of theories. This treatment for plurality includes a new way of validation, which will be described in the chapter concerning the levels in tetralectics. In such a way tetralectics *is a postmodern logic.*

This allows, for example, the structure of a scientific theory to be analysed. Tetralectics can be used to analyse and build scientific theories possessing many different logical structures. In this sense *tetralectics is a logic of theory building*.

It is an obvious statement that tetralectics refers to the mental representations of reality and not the reality itself. Therefore, if we can find some general rules, some regularities in the context of tetralectics, these relations will be valid for scientific theories and not for reality in a direct way. *Tetralectics is a metatheory or a version of general systems theory.*

As in the case of other logics, tetralectics also has a formal and a non-formal aspect. The formal aspect aims to establish the rules or laws of treating and keeping together the diverse oppositions, while the non-formal deals with an actual interpretation of a formal theory, that is to take into account the actual meaning of the opposites. In what follows we tend to separate the general rules and working of tetralectics (the syntax of tetralectics) from its interpreted version (the semantics of tetralectics) applicable to natural sciences. However, this separation seems difficult to us because the development of tetralectics was a consequence of a continuous re-thinking of unsolved problems of natural sciences.

THE BACKGROUND OF TETRALECTICS

We accept the universal validity of the *Aristotelian causal theory* in the world of theories. Of course, it seems useful to re-interpret the four Aristotelian causes in our time. However, only the maintenance of four concepts has significance in the formal side of tetralectics, even the original Aristotelian concepts (or any other reasonably chosen four concepts depending on the subject matter of the analysis) could be used in it. On the other hand, our reinterpretation presented below can be used for a tetralectics applied to natural sciences and so leads to the set of tetralectical concepts for theory building in natural sciences.

Aristotelian concepts	*Concepts in a tetralectics of natural sciences*	
Matter	**Matter**	(M)
Form	**Space-time**	(S)
Efficiency	**Action**	(A)
Aim	**Change**	(C)

Of course, many other versions of reinterpretation can be constructed. Our choice is motivated by our detailed analysis of these scientific concepts which will not be presented here. Applying the above abbreviations, the version of tetralectics based on this scientific interpretation will be called the tetralectics of MASC (or simply MASC).

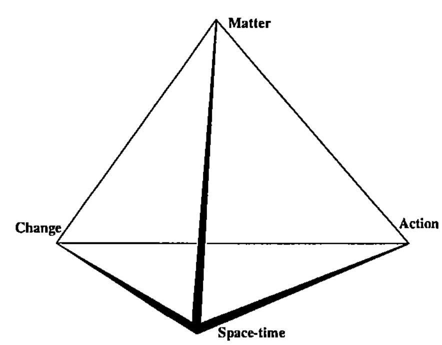

Figure 3: The framework of MASC

We accept the universal validity of the Hegelian view on the contradictory nature of beings. However, in *Hegelian dialectics*, the prominent role is played by *a pair* of oppositions, whereas in tetralectics we prefer the consideration of *many pairs* of oppositions studying both the formal and scientific conceptual contexts. In a tetralectics, of course, the different oppositions are not equally treated. The different oppositions may be represented by different symmetry elements of tetrahedron, since their behaviour in tetralectics may also differ. A symmetry operation transforms some of the oppositions to others while leaving some others unchanged. All of the four Aristotelian causes can be regarded as the coexistence of a number of certain kinds of oppositions. *Tetralectics is a method for the treatment of several different co-represented oppositions.*

We accept the universal validity of the *plurality preference of postmodernism* in all the areas of human activity including science. Tetralectics gives more than one valid description of an investigated object, but these form a unity in tetralectics. Tetralectics does not lose true statements in general but it tries to construct different contexts for them.

The demand for plurality is a common feature in the Aristotelian and in the postmodern view of the world. However, unlike tetralectics, neither Aristotelian, nor postmodern philosophies construct an exact treatment for plurality. In tetralectics the plurality is put into a geometrical context. These aspects can be shown in a possible way drawing an adequate tetrahedron, as can be seen in Figure 3.

THE FORMAL SYSTEM OF TETRALECTICS

The formal system of tetralectics consists of four *central concepts*, a definite number of *oppositions*, a three dimensional *tetrahedron* with its symmetries, a concrete *arrangement of the central concepts* on the vertices of the tetrahedron and the *assignment of the pairs of opposites* to the symmetry operations of the tetrahedron.

In order to evolve the system of tetralectics in a more plausible way, here we will use a scientific representation of the tetralectical system instead of the more formal description. In such a way the four Aristotelian causes are reinterpreted and applied as the very basic concepts of this implementation of tetralectics. As we mentioned earlier, the Aristotelian concepts of matter, form, efficiency and aim are transformed into the concepts of matter, space-time, action and change. These concepts are arranged in a tetrahedron-shaped form and their relations are analysed applying the rich symmetry properties of this perfect body. The edges and symmetry axes of a tetrahedron represent the basic oppositions (e.g., finite-infinite, discrete-continuous, open-closed, local-global, static-dynamic, etc.) that we use in our scientific analyses. An arrangement of the oppositions is shown in Figures 4 to 6. The concepts and their relations represented in Figures 4 - 6 completely define a tetralectics of natural sciences, i. e., the tetralectics of MASC.

Since symmetry plays such a crucial role in tetralectics, here we present a brief review of the connections between the oppositions and the symmetry of the tetrahedron. At first we use the edge oppositions to give an impression of the use of symmetry, then we give a second, maybe more elaborated approach.

The symmetry operations of the tetrahedron are described in the first table. Each reflection (σ) leaves one edge opposition unchanged and another reversed. If we take the plane containing the vertices M and S and the point halving the A-C edge, reflection through this plane leaves the M-S edge (the discrete-continuous opposition) unchanged, A-C (global-local) reversed, and mixes the other oppositions (e.g., $A\text{-}S \rightarrow C\text{-}S$, $M\text{-}C \rightarrow M\text{-}A$, etc.; cf. Fig. 4.). Figure 6. helps us to find a C_2 axis of the tetrahedron, since the MS-AC (static-dynamic) axis opposition that is shown there lies on one of these C_2 axes. The C_2 operation reverses the two edge oppositions which the C_2 axis goes through, namely, the M-S (discrete-continuous) and A-C (global-local) and mixes all the other the edge oppositions (e.g., $A\text{-}M \rightarrow C\text{-}S$, $M\text{-}C \rightarrow A\text{-}S$, etc.). The S_4 axes are coincident with the C_2 axes, which is not surprising since S_4 applied twice yields C_2 ($S_4^2 = C_2$). The S_4 operation mixes all the edge oppositions (e.g., $A\text{-}M \rightarrow S\text{-}A$, $M\text{-}C \rightarrow A\text{-}M$, etc.). The equipositional-hierarchical (M) face opposition shown in Fig. 5. lies on one of the C_3 axes. The C_3 operation mixes all the edge oppositions (e.g., $A\text{-}M \rightarrow S\text{-}M$, $S\text{-}C \rightarrow C\text{-}A$, etc.). Finally, E, the identity operator leaves every opposition unchanged.

We can assign each opposition to one symmetry element (rotation axis or reflection plane) which is invariant for the corresponding symmetry operation (e.g., rotation, reflection). For this, we need the characters of the symmetry operations' representations on the bases of different oppositions. The character shows the result of the symmetry operation as the sum of the invariant (1) and antisymmetrical (-1) subjects. (An opposition is antisymmetrical to an operation if the latter reverses the current opposition.)

Table 1

Symmetry operation	Characters on the basis of Edge (1-1) oppositions (6)	Face (1-3) oppositions (4)	Axis (2-2) oppositions (3)	Description of the symmetry operation
E	6	4	3	Identity
σ	0 = **1** + (-1)*	2	1	Reflection through a mirror plane
C_2	-2*	0	**1**	Rotation through 360/2 degrees
C_3	0	**1**	0	Rotation through 360/3 degrees
S_4	0	0	-1*	Rotation through 360/4 degrees followed by a reflection through a plane perpendicular to the axis of rotation

* a negative number indicates that the operation reversed the opposition(s) [change in polarity]

Since three (σ) mirror planes contain the same (1-3) and two σ's contain the same (2-2) oppositions, to get an unambiguous assignment we should assign the mirror planes to the class with which they have a one-to-one correspondence (which is the class of (1-1) oppositions). The assignment of the remaining classes of oppositions is straightforward. The second table shows the symmetry operations, symmetry elements of the tetrahedron, and the assignment of the oppositions to the symmetry elements:

Table 2

Symmetry operations	No. of symmetry operations	Symmetry element	No. of symmetry elements	No. of assigned oppositions	Class of the assigned oppositions
E	1	E	0	0	
σ	6	σ	6	6	Edge (1-1)
$3C_2$	3	C_2	3	3	Axis (2-2)
$4C_3$, $4C_3^2$	8	C_3	4	4	Face (1-3)
$3S_4$, $3S_4^3$	6	S_4	3*	0	
SUM	24		13+3	13	

* coincident with the C_2 axes.

The above mentioned property of the mirror planes, which may make finding the proper assignment more difficult, draws our attention to the necessity of axis and face oppositions. To make this clear, let us consider the example of the face (1-3) oppositions. Let us choose one vertex of the tetrahedron. There are three mirror planes which contain it. They also contain the three edges [(1-1) oppositions] connecting the chosen vertex to the other three. The equivalence of these three connections along the edges (which is also supported by the C_3 symmetry element) shows that this vertex has a special relationship to the triangle formed by the others, which the edge oppositions cannot express. Similar conclusion can be drawn in the case of the (2-2) oppositions as well.

Besides the properties described above, σ, C_2 and S_4 have a special feature, which may add some peculiarity to the oppositions assigned to them, namely, that they change the polarity of some of the oppositions not invariant to them (and S_4 even swaps some oppositions), adding new elements to the relations between the oppositions. At this stage of the development of MASC presented here these relations have no crucial role, therefore we will disregard some of them. All the relevant assignments in MASC are shown in Table 2.

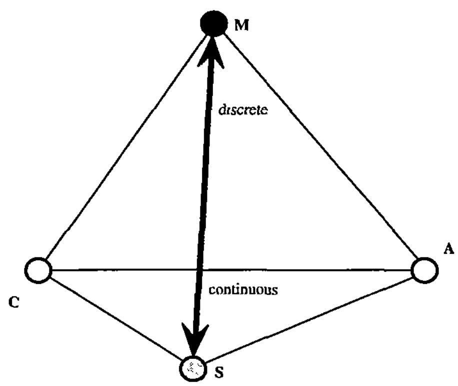

Figure 4

Edge (1 - 1) oppositions of MASC

discrete - continuous *(M - S)*
global - local *(A - C)*
stochastic - deterministic *(C - S)*
homogeneous - inhomogeneous *(S - A)*
causal - teleological *(A - M)*
ordered - disordered *(C - M)*

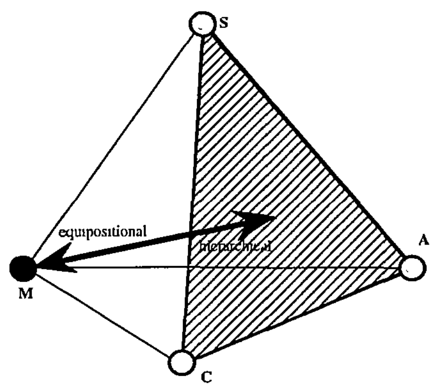

Figure 5

Face (1 - 3) oppositions of MASC

hierarchical - equipositional *(M)*
quantity - quality *(S)*
possibility - reality *(A)*
finite - infinite *(C)*

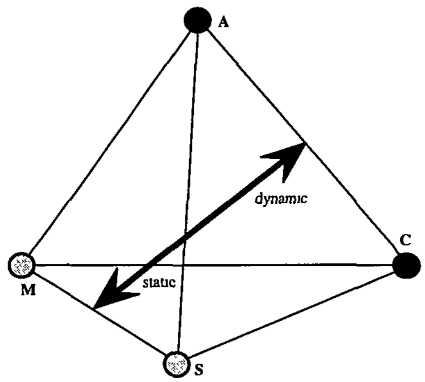

Figure 6

Axis (2 - 2) oppositions of MASC

static - dynamic *(MS - AC)*
closed - open *(MC - AS)*
individual - collective *(SC - AM)*

THE SYSTEM OF TETRALECTICS OF THE NATURAL SCIENCES

Here we present some *general* aspects of the tetralectics of MASC and those special versions of metatheories which can be constructed on the basis of this tetralectics.

As follows from the preceding ideas, the tetralectics of MASC consists of four central concepts (*M*, *A*, *S* and *C*), 13 pairs of oppositions (6 edge, 3 axis, and 4 face oppositions; specified in Figures 4. - 6.), a tetrahedron with its symmetry elements given in Table 2., and the arrangement and assignment of these constituents presented in Figures 4. - 6.

A symmetry element, which is not assigned to any specific opposition, the identity (*E*), represents the whole selected and assigned system of oppositions of the given version of tetralectics. Those pairs of oppositions were chosen into the class of edge or (1 - 1) oppositions whose constituents determine each other without any other relations, as in the cases of discrete - continuous or homogeneous - inhomogeneous. The conceptual structure of the axis or (2 - 2) oppositions is more complicated, because they are basically connected with two pairs of edge oppositions. For example, the meaning of static - dynamic pair of concepts is influenced by both the discrete - continuous and the global - local oppositions. These relations call attention to the understanding of the concept of static as a result of the coexistence of the discrete - continuous and global - local oppositions. This is an unusual aspect of this concept, so its appearance in tetralectics makes our conceptual analysis more sensitive and complex. Of course, the situation is the same in the cases of the other axis oppositions, too. As was mentioned earlier, the face (1 - 3) oppositions unify three pairs of edge oppositions. This relation emphasises the very rich conceptual content of the face oppositions. For example, the meaning of the quantity - quality oppositions is influenced by the ordered - disordered, the global - local and the causal - teleological oppositions. The details of the relations of the face - edge oppositions will not be analysed here, it is, perhaps, noteworthy that these aspects of quantity - quality concepts would probably be important, for example, in evolution theories. All these are very important relations between the three classes of oppositions, but it would be important to recognise that, at the same time, the reverse relations are working too, i. e., the meaning of an edge opposition is influenced by those axis and face oppositions in which the edge opposition appears.

So MASC is a very complex system of concepts, relations and operations. In our view MASC is able to represent all the elements of scientific discourse: the concepts, the statements, the theories, the disciplines, the word views and so on. Some of them preserve their original form in MASC, but most of them have to be reinterpreted applying the view of MASC. So we can use MASC to analyse the conventional scientific practice and products, and following the suggestions of this tetralectics we can construct a new type of knowledge about nature.

How is MASC able to reflect the scientific descriptions of our world? There is no room to present all of details here, but a few typical aspects will be mentioned below.

In the framework of MASC we can consider the *interconnectedness of elements of knowledge* produced by the natural sciences. According to the received view the right approach to this problem is to construct more and more universal scientific theories and, finally, science can produce one unified theory which describes all aspects of nature or of a piece of nature at least. However, following this practice science has to face many difficult conceptual and methodological problems. These problems appear at different levels of science. If we take the level of theories, the Gödel theorem causes difficulties; within the specific scientific disciplines some important dualities (particle - wave, locality - globality, etc.) seem to be problematic; between the different disciplines, the unsatisfactory treatment of the reductionism - holism problem manifests itself; some views and the ideas of natural philosophy are without enough clearness, and so on. MASC tries to treat these difficulties together applying a peculiar methodology to the analysis of natural sciences. This peculiarity appears in both of the - more or less - special elements of the methodology (conceptual analysis, study of analogies, construction of a method for building new disciplines, etc.), and the functioning of MASC in a special way.

First of all, the system of the relations of concepts applied in MASC will be outlined here. The relations between the classes of oppositions have already been treated. Now the relation of oppositions and central concepts is the topic. According to MASC all four central concepts have to apply to describe nature, however, most scientific theories do not follow this idea, so they can treat only one or other aspect of it. MASC suggests a tetralectical combination of the different aspects formulated in different theories. The total description is the matter of four different, but interrelated theories, which are built up around the four central concepts. The characteristics of these theories depend on the features of the given central concept. The feature of the central concepts can be characterised by the systems of oppositions of MASC. A central concept is associated with a vertex of a tetrahedron. Three edges meet in a vertex and three axes indicate its position, moreover, a vertex is a basis of a face in tetrahedron. Because of the symmetry operations associated with these geometrical elements representing oppositions, the four central concepts can be characterised by the relevant different set of oppositions. Here we apply a naive topology: in determining the properties of a central concept at a vertex of a tetrahedron, the nearer element from the pair of the opposite will be the dominant contributor. The face oppositions situated at certain central concepts have an eminent role: the main property of the given central concept is determined by this pair of oppositions. In such way we can characterise the concepts of matter, space-time, action and change. The conceptual structure of these central concepts of MASC seems to be so highly complex that perhaps it would be better to call it a system of concepts or some kind of a theory. However, real scientific theories have a rather close relation to experiences, so seeing that the constituents of this system of concepts or theory in this sense do not have a very concrete content, they are not interpreted in their details, the name *metatheory* would be more expressive instead of the other ones suggested earlier. We will adopt this idea. So we can speak of four types of metatheories in the tetralectics of MASC: material, space-time, action and change metatheories. In Table 3. the set of the properties associated with the central concepts or metatheories in MASC is presented.

According to tetralectics, a metatheory in MASC is not able, even at this high level of complexity, to describe nature, but the system of the four metatheories can do that. Of course, there is a series of symmetry operations to transform them into each other. In such a way the elements of the different metatheories (concepts, statements, laws, problems, theorems, etc.) can be assigned to each other. A great number of analogies can be found, but the different conceptual context makes these theories different ones. The tetralectics of MASC, in fact, is a meta-meta theory, a metatheory for the four metatheories in MASC. It is interesting that its epistemological nature is rather similar to a normal theory than a metatheory. There seems to exist, in this case, some kind of analogy between the dialectical operation of double negation and the doubly applied meta-level view.

Table 3

The collection of the properties connected to central concepts/metatheories

Central concept/Metatheory	*Properties*
M/Material	static, closed, individual discrete, stochastic, disordered hierarchical - equipositional
S/Space-time	static, open, collective continuous, homogeneous, causal quantity - quality
A/Action	dynamic, open, individual global, deterministic, inhomogeneous possibility - reality
C/Change	dynamic, closed, collective local, teleological, ordered finite - infinite

Different families of theories or metatheories can be found in the standard scientific description of nature, too. For example, if we consider the physical theories, four groups can be identified. The four metatheories in tetralectics can be compared to these four groups of theories of physics:

Metatheories in tetralectics	*Theory families in physics*
Material	**Corpuscular**
Space-time	**Field**
Action	**Variation principles**
Change	**Conservation laws**

A comparison of the relevant theories and concepts reveals a deeper connection between MASC and theoretical physics. There are some - more or less - obvious similarities between the metatheories and the physical theory families. However, a very important difference is that each of the physical theory families claim absolute universality in the description of nature, so the different families can be considered as alternatives which exclude each others, whereas metatheories in tetralectics cooperate with each other, and only the whole of MASC claims to its absolute universality. Like in MASC, in the case of conventional physical theories one can also identify some important transformation rules, principles or theorems (i.e., Noether theorem) between the groups of theories, but in the conventional physical theories these transformation rules and theorems do not constitute a clear system, in their appearance, working, form and interrelatedness the contingency has a dominant role.

There are some other possibilities to find connections between the MASC and the conventional scientific practice. Sometimes it is interesting to know how a specific scientific concept situated in the conceptual field of tetralectics; which concepts are similar in different scientific theories, and what features are common in them. To study these questions sometimes we have to leave the vertices of the tetrahedron. and we have to move along the direction determined by the very nature of the concept. Not only this results in a more realistic representation of a conventional scientific concept, but it gives a special "fuzzy" features to MASC. Consequently, MASC is a proper tool to find similarities or analogies between the concepts, laws, principles or problems of different disciplines of natural sciences.

Here we should mention, that a theory in the standard scientific practice and a theory in tetralectics of MASC is not the same thing. They differ from each other in many aspects. To make these differences clearer we have to say something about the position of theories in MASC. As was mentioned earlier, tetralectics of MASC itself is a meta-metatheory which includes four metatheories. But within these metatheories we can, of course, identify or construct theories. For this purpose we have to find some more concrete concepts, which are closer to the level of experience, so applying them we can state some relevant basic statements, or axioms for theories. To achieve this, the reduction of complexity of the central concepts would perhaps be a natural aspiration. So we divide all the four concepts into a pair of sub-concepts. These sub-concepts play the same role in the tetralectics as the central concepts do. Because they represent different sides or aspects of the certain central concepts of the tetralectics, so they have a reduced complexity. In MASC we chose the following sub-concepts dividing central concepts:

Central concepts	*Their sub-concepts*		
Matter	**substrate**	and	**structure**
Space-time	**space**	and	**time**
Action	**action**	and	**interaction**
Change	**transformation**	and	**equilibrium**

A theory in MASC gives some relations between the sub-concepts presented here. There are several (but finite) possibilities to find relations between these sub-concepts. We prefer such theories in MASC, whose axioms consist of borrowed constituents from the all the different central concepts. For example an interesting theory can be constructed defining the action - time, space - transformation and the interaction - equilibrium relations. Such theory can be represented by a graph connecting the sub-concepts on the tetrahedron. As a consequence of the general features of MASC, it is clear that we can move between the different theories of MASC by applying the symmetry relations, and these transformations can be represented by the relations of theory-graphs. The domains of validity of these tetralectical theories depend on the chosen sub-concepts, so many very special aspects of the nature can be described with tetralectical theories. The borderlines between the different tetralectical "disciplines" situated in other positions related to the conventional ones. However the details of any kind of concrete theory in MASC is not the topic of this paper.

So the tetralectical theories can be very different from the conventional theories because of their very different conceptual structure and their mutual, well-defined relationships to each other, but the practical use of them is the same as that of the standard scientific theories. They have to fulfil the same role in the scientific discourse.

In some cases it is handier to consider the projections of MASC instead of its 3D form. Many interesting consequences became clear if we study only a proper projection of the tetrahedron. Of course, the MASC is equivalent with a definite system of its projections. We can project the tetrahedron into the plane of its two axes along its third axis. These three projections are the most important projections and they are very useful in the scientific analyses. An illustration can be find in Figure 7.

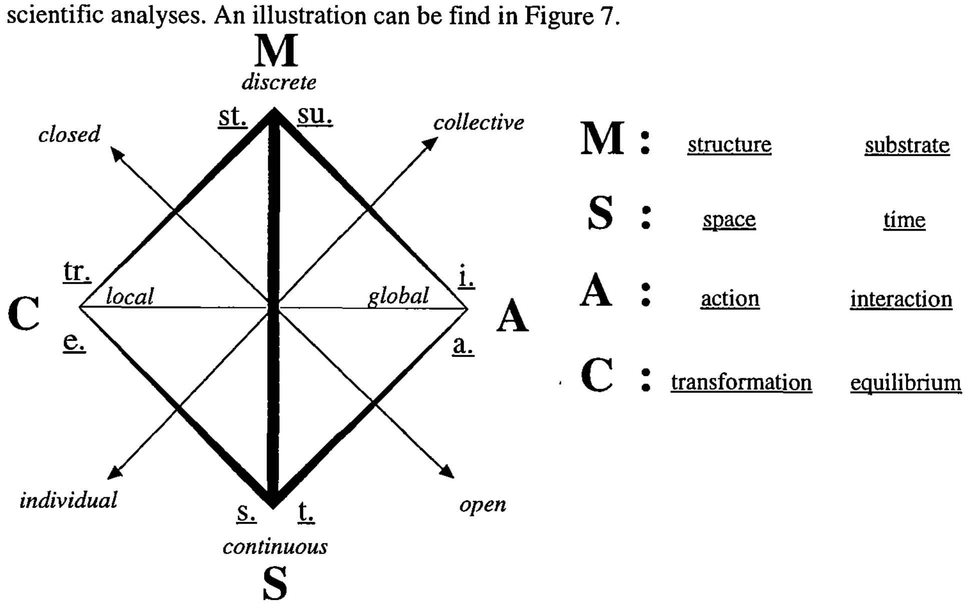

Figure 7: The projection of MASC along the static - dynamic opposition.

ON LEVELS OF TETRALECTICS

From the construction rules it is evident that in tetralectics we do not want to describe reality by only one general theory, but we have a *three-level description.*

The lowest level is the level of standard scientific theories. Above this level is the level of metatheories. The highest level in tetralectics is the whole system of tetralectics, which organises the four metatheories into one system.

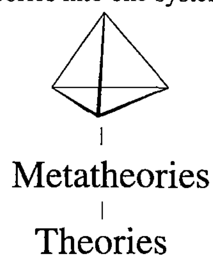

Figure 8: The three levels of tetralectics.

The *validation of statements* is possible at both of these levels but the rules and the results may be different. At the lowest level (the standard scientific theory level) we accept the conventional rules of logic, so we can use all the methods and results of scientific theories. At the middle level, this is not so simple. First we should introduce our non-conventional logical definitions for this level. A statement is *true in a metatheory* if it was true at the scientific theory level, otherwise we call it *not-true*. We do not distinguish between a false and an undecidable statement at this level. So there will be not-true statements which are false or which are independent from the axioms of that theory (it is impossible to prove that they are true or false from this set of axioms). Finally we arrive at the top level.

A statement is *true in tetralectics*, if it was true in any of the metatheories (at the middle level). A statement is *false in tetralectics*, if it was true in none of the four metatheories. At this level the excluded middle principle operates.

There is still the question, why use tetralectics? What does it avail? We can get help from the system relations of tetralectics.

On the level of metatheories we can *transform a set of axioms* of a metatheory into another one by applying the symmetry relations of the tetrahedron. The problems are not necessarily transformed, but they can be transformed with the same method as the axioms. The statements can be stated in any metatheory, and can be validated as well, the question is only the result. They can be handled much more easily within the framework of tetralectics, because as was mentioned already, we do not distinguish between the false and the undecidable case.

The transformation of the problems between theories is a question of symmetry relations. We prove a theorem in one metatheory and we want to know whether it is true or false in other metatheories. How can we validate that the statement is true or false in another metatheory? Let's take the following two statements as examples:

(1) $1 + 1 = 2$

(2) Bolzano-Weierstrass theorem: A continuous function on a closed interval has all the function values between the endpoint function values.

Both statements are true in tetralectics. If the axioms contain the integer numbers, then in the first case the transformed axioms of the other metatheory will also contain these axioms without any change, so statement (1) will be true in any metatheory. This is because the prerequisites of the statement are independent from the transformed axioms. However, if we transform the axioms with a symmetry operation reversing the opposition of continuous - discrete, then the situation is quite different in the case of statement (2). The prerequisites of statement (2) contain the continuity so it will be true only in the metatheory of the continuous side of opposition, because this metatheory will contain an axiom defining "continuous". In the discrete side it will not be true, because this other metatheory will contain "not-continuous".

If our transformed axioms do not guarantee the prerequisites of the statement, for example "continuity", then the theorem will be false. But if the transformed axioms are independent of the prerequisites of the statement, then the statement will be true in the other metatheory as well. So we should transform the axioms by the symmetry relations and deduce the statement from the axioms. If we get the same result, then the changed axiom is not relevant for the statement. If we get the opposite result, then the changed axiom is important in the statement. With this method we can achieve a deeper understanding of the essence of the statements.

Such transformations have a crucial role in the famous Gödel theorem.

Gödel's incompleteness theorem states, that if we have a set of axioms which is meaningful enough, and the system does not contain contradiction, then we can always construct a statement which is true, but we cannot prove the statement, and cannot prove the opposite of the statement, so this statement is undecidable in this system.

The Gödel's incompleteness theorem will be valid at the level of theories. But with tetralectics, there is a chance to transform a so called Gödelian statement into another theory, where it is not necessarily Gödelian even at this level. The proof of the Gödelian theorem is based on the numbering of every axiom and statement. Strikingly, the transformation may change this numbering. This change implies that the formerly Gödelian statement can become decidable, and another statement can become Gödelian. At the meta-meta-level of tetralectics, the Gödelian problems have a quite similar treatment to those of at the level of theories.

What is more, the metatheory level may contain apparent contradictions, because it is not a strict theory, but a framework for theories, which consists of concepts, principles and features. For example, the metatheory of geometry (which is not identical with the theory of geometry) contains the axiom of parallels and the negation of the axiom of parallels at the same time. So this means that a metatheory in tetralectics may contain contradictions. In this sense the metatheory concept, and so tetralectics, can be complete without breaking Gödel's incompleteness theorem.

As should be clear from the foregoing, we can live with the Gödel's incompleteness theorem in a new, friendlier way.

SUMMARY

We presented here some ideas to introduce a new type of logic named tetralectics. How are the features of this logic determined? In any kind of logic there is a *formal* framework which is, more or less, independent from the contents of its statements. In tetralectics we have four concepts, thirteen pairs of oppositions as the very constituents of this logic, and a collection of the possible operations on these constituents manifested by the symmetry operations of a three dimensional tetrahedron. For the construction of the whole system of tetralectics we have to arrange the four concepts on the vertices and arrange the oppositions on the symmetry elements of tetrahedron. Fixing all of these, a special version of tetralectics is specified and defined. Applying some elementary combinatorics, the exact number of arrangements on the tetrahedron can be calculated getting by this way the possible number of different kinds of tetralectics. In the framework of the given tetralectics there can be identified three levels of the

description. The first level, the level of theories is related directly to the world of experiences. In the next level of tetralectics we have four metatheories dealing with the theories, and finally on the third, meta-metatheoretical level we can treat the relations of metatheories. As follows from the rules of construction in a concrete version of tetralectics we can construct a *definite number* of theories, while we have *four* metatheories and *one* meta-metatheory.

If we want to use this logic to analyse some kinds of phenomena, we have to select four *characteristic concepts* as central concepts and 13 *characteristic* pairs of *oppositions* which probably can characterise the given phenomena. In such a way we can construct and study the tetralectics of different beings. Our presentation given above depicts some details of a *tetralectics of natural sciences*, which we called tetralectics of MASC. It has also three levels: the level of scientific theories, the level of Material, Action, Space-time and Change metatheories, and the meta-meta level of MASC. We can use this tetralectics as an analysing tool for the conventional scientific theories and, on the other hand, we can construct special kind of tetralectical theories. At this point we can determine again all the tetralectical theories that are possible, and we can use the four metatheories and the whole MASC to study them. Of course, a different view on the essential aspects of natural sciences represented by the different central concepts and oppositions could produce different tetralectics.

For analysing other phenomena, we can define their special tetralectics. Any kind of complex phenomenon seems to be a good subject for tetralectical thinking. We have some ideas on the tetralectics of religious systems, social systems, psychological phenomena and the systems of language. Constructing and applying these tetralectics a special kind of complexity and exactness can be simultaneously enforced on the description of these fields - at least in this postmodern logic.

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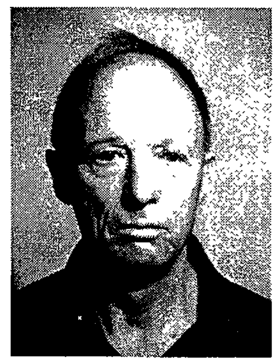
SYMMETRY: ART & SCIENCE

REFLECTIONS ON ROTATIONS

John G. Harries



Exponent of notated visual art, (b. England, 1928)
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Fields of interest: Graphic, computer and video art, movement analysis (also T'ai Chi Chuan).
Publications with relevance to symmetry: Language of Shape and Movement, Tel Aviv: Tel Aviv University (1983); Symmetry and Notation: Regularity and Symmetry in Notated Visual Art, in Hargittai, I., ed., Symmetry: Unifying Human Understanding, New York: Pergamon, (1986) pp. 303-314; Symmetry in the Movements of T'ai Chi Chuan, Computers and Mathematics with Applications, 17 (1989), 4-6, pp.827-835.
Exhibitions with relevance to symmetry: (1969) New Tendencies, Zagreb, Yugoslavia (participation); (1971) Art and Science, Tel Aviv Museum (participation); (1972) Festival de Due Mondi, Spoleto, Italy (in conjunction with performances of Noa Eshkol's Chamber Dance Group); (1973) Graphic Art and Movement Notation, Museum Haaretz Science and Technology Pavilion, Ramat Aviv, Israel; (1992) 'Variations on a Pomegranate' - video installation and paper cuts, Kalisher Five Gallery, Tel Aviv (in conjunction with 'Movement Notes' exhibition); (1993) 'Emergences', ZOA House, Tel Aviv (in conjunction with performances of Noa Eshkol's Chamber Dance Group).

Abstract: *A sequence of 16 pictures employing dynamically conceived shape is used to demonstrate how symmetry operations can constitute one of the connective forces in a work of visual art that is systemic and periodic. The aim of the study was to generate a set of pictures comprehensible by virtue of the appeal to transformations that have been present in art for thousands of years. The sequence was composed in EW movement notation, the principles of which are briefly sketched. The composition of the sequence is explained, using illustrations together with their symbolic expression in movement notation, and also typographical representations of aspects of its overall structure. The visual realization of the structure comprises the full sequence of 16 pictures.*

INTRODUCTION

The study presented here is a part of work begun over 30 years ago, all of it based upon the use of Eshkol—Wachman (EW) movement notation as a tool of composition in visual art, including kinematic forms such as videotapes. Since EW movement notation entails the perception of movement as shape, it is the natural vehicle for this work, in which shapes are perceived as the paths of movements. Indeed, the notation was the catalyst in the conception of this model for the generation of shape. It provides a firm basis for the organisation of the material, because the conception of visual forms as movement traces allows their precise definition in the notation. EW notation is ideal as the support for a serial or periodic approach to composition which rests upon quantities and order. By its use, it is possible to control and pursue complex and extensive variations of shape, within a reasonably concise symbolic scheme. Furthermore, anyone who is literate in the notation is able to follow in depth the formal operations it describes.

The aim of the present study was, however, to generate a sequence of pictures that would be as far as possible comprehensible independently of any knowledge on the part of the viewer, of the ideas underlying movement notation. This leads to the question: upon what could such comprehension be based, if the work of composition is in fact based upon those ideas? The solution is here assumed to lie in the intuitive and probably innate perception of transformations such as symmetries, which have been present in all art for thousands of years (Weyl,1952; Avital, 1996; Avital, under review). These verbally nameable operations would be reflected in any thoroughgoing notation used in the domain of visual art, including them as an integral part of a single compositional act. Where EW is the notation that serves this end, the quantification of these operations will also certainly be possible (Harries,1986).

In this study, operations on the basic set and on its successive transformations are all intended to be intuitively identifiable and verbally nameable, and also quantifiably expressible in EW notation. Movement notation provides for the quantified approach essential to any well defined composition. At the same time, the presence of quantitative differences can be detected even without knowledge of the precise values involved. For the sake of simplicity, the principles through which this study is organised were confined to two types of transformation of the motif series.

The employment of EW movement notation in visual art, is itself inevitably based upon some intuitive transformations: recursion, rotation, scaling, and positive/negative sense of movement; *and it necessarily involves taking into account other operations such as* selection of trace type, colour modification, layering.

In this black and white sequence, the basic motifs are obtained by splitting up a single motif taken from an earlier sequence in colour ("Emergences", 13 paper cuts, first exhibited in Tel Aviv, 1992). The composition of that sequence of pictures was based entirely upon a serial and periodic approach. In the present case, a similar serial and periodic approach manifests an overall systemic structure, in which the ordered reduction of a complex pattern to a basic set of motifs is followed by a cumulative redevelopment of that set through successive transformations and recombinations. These metamorphoses will be described in the text, both verbally and in terms of movement notation.

The use of EW movement notation in visual art has been described extensively elsewhere (Harries, 1969, 1975, 1983); for the purposes of the present article, a brief outline of this application is provided in the next section.

EW movement notation

A subset of Eshkol-Wachman (EW) movement notation serves abstract composition of visual images in several ways: as a record, for communication, for systematizing, and above all for formulating ideas in symbols that make it possible to grasp structure through a scheme of manageable size. Any bounded area can be regarded as the path of the movement of a line, called in descriptive geometry a 'generatrix'. The line sweeps out a trace, the shape of which is determined by the way the generating line moves. This idea is useful so long as it is possible to define exactly how the line does move. This is achieved when the movement is expressed in the symbols of EW.

The primary use of EW notation is in the context of human movement (Eshkol and Wachman, 1958). Movements are treated as the paths produced by the limbs, which are regarded, for the purpose of analysis, as chains of articulated axes.

Using EW, it is possible to define the movement of any line, and thus to describe shapes in terms of movements of articulated generating lines, in relation to a system of reference encompassing two or three dimensions of space, plus time. The instructions of the notation are general in that they apply to any medium, but specific in the way they work for each - for instance: pencil and paper, computer graphics as in the present study, or (when the moving lines are the axes of limbs) human performers.

Seen in this way, a still picture is not only a motionless object but also one stage in a formative process, and the potential point of departure for subsequently emerging form. EW notation preserves the continuity of the static and dynamic, and of the two- and three-dimensional. In three-dimensional space, if a single generating line moves about one of its ends, which remains at a fixed position, the line may sweep out a curved surface or a plane. (In the case of a solid limb, rotation about its own axis is also significant.) In two-dimensional space, a circular shape results. A more complex shape is obtained if a second generator is articulated with the moving end of the first, and simultaneously moves about their common 'joint'. Modifications of the circular path also result from any changes of length of the generator. Chains of any number of such generators may be formed, moving about the points of linkage.

In every movement of articulated generating links, these are characterised in terms of EW as 'heavy' or 'light', i.e., carrying or carried by a neighbouring link. (A generator may simultaneously carry one neighbour and be carried by another.) When a generator moves, it carries with it all other links that are further away from the origin, thereby changing their positions; the origin of the heaviest link is analogous to the base of support in the case of a living organism.

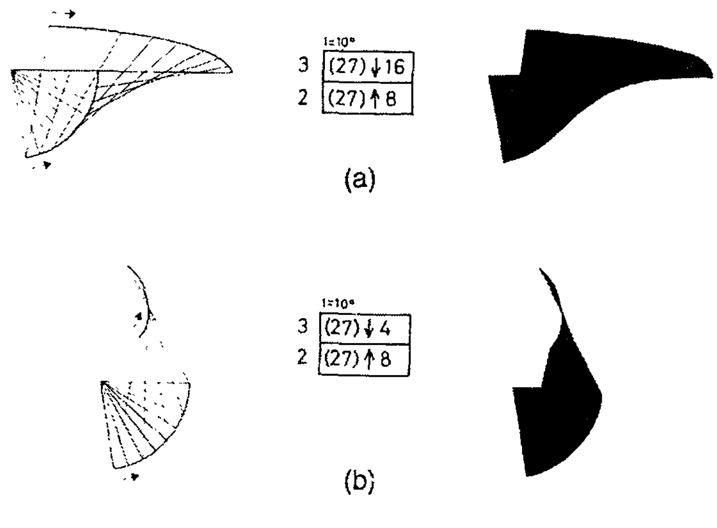


Figure 1

When independent movements of the light links occur at the same time as they are carried by a heavy link, the change of position of each link is the result of the simultaneous movement of the carried link together with the movements of the heavier links. The movement of each generator is written as though in relation to an immobile carrying link; but in fact the path of this movement will be modified because its heavy neighbour moves as well; see Figure 1. The figure shows simultaneous movements of generators, both as successions of positions and as the shapes that they sweep out. In (a) the carried link moves at twice the rate of the carrying link; in (b), it moves at half the rate of the carrying link. The varied synchronisation of the movements of two or more articulated generators is the source of the apparently endless wealth of shape that can be obtained and composed using this system of representation.

The notation of each of the shapes in the figure indicates, to the left of the frame, the relative lengths of the generating links. Above the frame, the value of the unit of movement is given (one unit = 10 degrees). The plane in which the movements take place is given in parentheses; following these, arrows indicate the sense of the movement (clockwise or counterclockwise), and numbers specify the amount (in units) of the movement of each generator. If the length of a generator were to change, this too would be indicated in the appropriate horizontal space. Specifications of colour can be added in an additional space, parallel with the movement score itself.

Work designed to be displayed in time is conveniently provided for in EW, where the measured flow of time is represented by the columns of the basic grid, upon which the synchronised patterning of movements of the generating links is written and easily perceived. The information conveyed in these scores is implemented as abstract moving computer graphic images, or as videotapes. The quantitative nature of EW makes it ideal for computer input. The software I have developed provides for the entry of data in an EW score on screen; when this score is completed, the visual process it represents is displayed in movement on the screen, and can be recorded on videotape (Harries, 1981,1983).

All of the work of visual composition is written in the same notational system and does not require new parameters or new modes of symbolisation for different projects. The generality of the notation is more than sufficient and it can equally well encompass the domain of three-dimensional structure. Furthermore, shapes and processes are defined with as much accuracy as can be matched in the chosen medium; this allows both for a maximum of control and for great subtlety of variation.

The picture sequence: 'Reflections on Rotations'

This picture sequence is intended to be *in principle* comprehensible without special knowledge of the means used in its composition. The use of movement notation in generating such a picture sequence involves processes that can be learned, and any EW-literate person could understand the structure by studying its score. It is, however, doubtful whether even such a person would be able to follow the structure in its entirety *without* studying the score, if there were no visually explicit integrating principle. Therefore, while this study is firmly anchored in every detail to definitions and procedures of variation formulated in EW notation - which guarantees full control of the compositional procedures - at the same time, all compositional choices were made in cognizance of the nature of the transformations that result from those procedures, and from the way in which they are bonded together by their ordering in relation to one another. To understand the meaning of the structure is to understand the place of a motif (in any one of its transformed states) within a picture; the place of a picture within a group of pictures; and the place of a group within the whole. (This is the essentially hierarchical principle which Avital [under review] has pointed out as being at the foundation of all viable art.) If the bonding principle is one that is universally understood in visual terms without the help of verbal or other symbolic explanation, it is reasonable to hope that both an observer who is not EW-literate and an observer who is EW-literate but had no access to the score, would be able *through the visually apprehended structural procedures* that emerge in the composition, to perceive the meaning of the whole in visual terms. It may be that even such a simple composition as the present is not immediately obvious; instant comprehension is, however, not the aim.

The concern was, then, to present a serially formed picture sequence, designed to be directly – i.e., visually – understood as a single systemic structure, by virtue of its being bound together by familiar and intuitively comprehensible transformations. The transformations chosen were threefold rotational symmetry, and reflective symmetry. These transformations were to be unambiguously specified in EW Movement Notation, no less explicitly than were the details of the motifs and series out of which the work was formed. The whole sequence is shown in Figure 7.

In the following exposition, no special previous knowledge is assumed on the part of the reader. It includes illustrations in which the same idea is conveyed symbolically in movement notation, and also visually by the shapes generated in accordance with the notation. Some simple typographical schemes are also used to indicate aspects of general structure without going into detail.

G: { : 1=1°

70 (90)	↓40	↓30	↓20
0 (90)	+8	-6	+4
25 (90)		↑10	↓20
70 (0)	↑40		

}

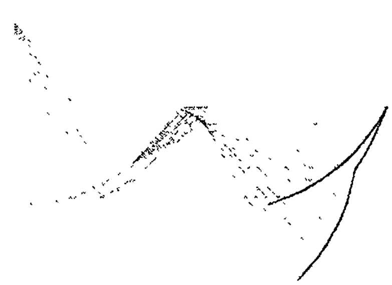
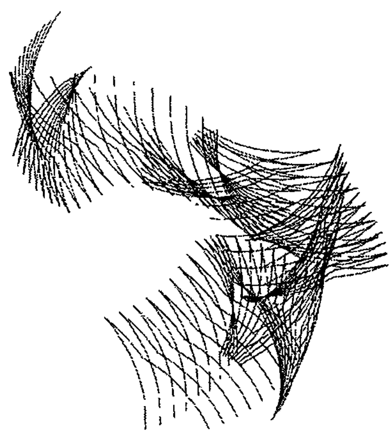


Figure 2

Movements of a linkage of four articulated axes are specified (Figure 2), and the trajectory of the distal end defines a curve, *G*.

The curve *G* is used as the visual generating link in a movement sequence; i.e., its movements are assumed to generate shapes. The curve *G* is substituted for the 'lightest' (upper) link in the three-movement sequence which gave rise to the curve itself. (Its axis for this purpose is the line joining the two ends of the curve.) This is illustrated in Figure 3; the states of this new curved link are shown as they appear at the beginning and end of each movement.

Figure 3



The other three links are not visual, but contribute to the form of the shape generated, by carrying the (independently moving) curved generating link. This sequence is repeated a further three times, from the states reached. Figure 4 shows the notation of the movements, and position lines selected from the path they sweep out.

	G [0]	↓40	↓30	↓20	G [270		G [180		G [90	
E	0 [0]	+8	-6	+4	6 [0	·/·	12 [0	·/·	18 [0	·/·
E	25 [90]		↑10	↓20	25 [80		25 [70		25 [60	
E	70 (-)	↑40			70		70		70	
	a				b		c		d	

Figure 4

The four elements *a, b, c, d,* are generated as the four consecutive parts of this extended sequence. They are recombined (with a single shift of the sequence) to form four motifs, *A, B, C, D,* as shown in Figure 5.

a		b		c		d
d		a		b		c
Motif **A**		Motif **B**		Motif **C**		Motif **D**

Figure 5

These form a group of four pictures: | **A : B : C : D** | which constitutes the central group in the whole series. Each motif contains one element in common with each of its neighbours, emphasizing sequential continuity between the four. Each motif is made more readily identifiable by being displayed as a different type of trace: fully swept trace, trajectory, and selected position lines from the swept path (in two different weights of line).

In the following schematic representation of the structure of the entire series, the groups are separated by vertical bars, and the pictures within the groups are separated by colons:

| ABBCBCCD | ABBC : BCCD | AB : BC : CD | **A : B : C : D** | AB : BC : CD | ABBC : BCCD | ABBCBCCD |

The groups to the left and right of the central group contain increasing accumulations of the motifs, so that their complexity increases with their distance from the centre. In the groups to the left, the pictures at successive levels (groups) are connected by relations of reflective symmetry (σ). In terms of EW movement notation, this means that the movements of the links that form each motif start from positions bilaterally symmetrical to those in the appearance of the same motif in the group next nearest to the central group; and the links move in the opposite sense to the movements that produced the same motif in the group next nearest to the central group. The groups on the right are connected by relations of threefold rotational symmetry (C_3), expressible in EW notation as reiterations of the motif at a series of positions separated by intervals of 120°. In all appearances of the motifs in their various transformations, the *initial* positions of the 'heaviest' links are permutations of the positions (.0), (.3) and (.6) where 1 = 40°.

A diagram of the structure

Since it is intended that the structure be perceived in the picture sequence itself, to describe it in words and diagrams may appear to be merely paradoxical and superfluous. But in fact their use need no more interfere with a proper appreciation of the composition than consulting a map detracts from the direct viewing of the actual terrain. A map represents only certain aspects of the actuality, but this very selectivity may facilitate an overall understanding.

In the typographical scheme given above, only the additive aspect was represented. The next 'map' is a schematic representation of the interrelation of motifs, transforms, combinations, and levels of hierarchic order. In Figure 6, the hierarchic order is shown integrated with the composed sequence.

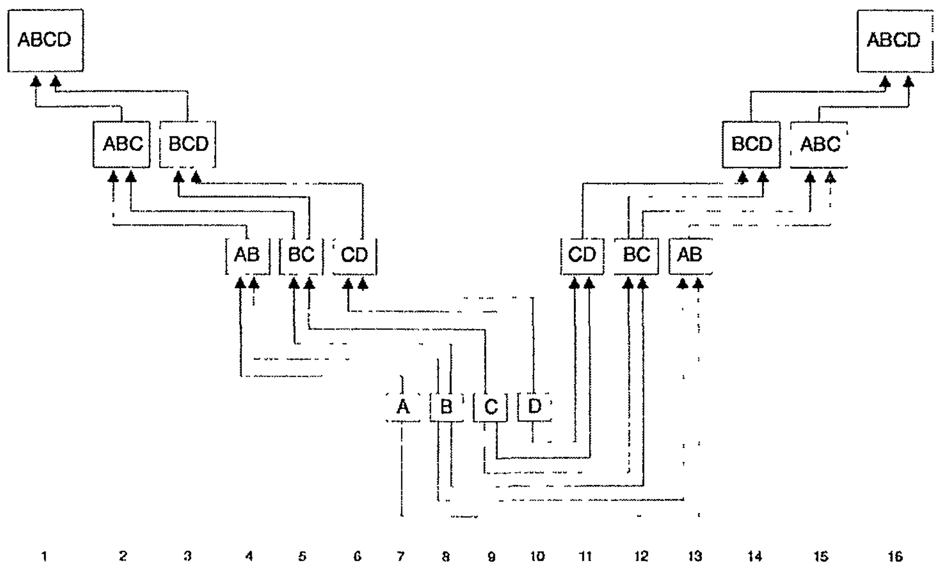


Figure 6

In this scheme, *A, B, C* and *D* again represent the motifs. The boxes indicate pictures, each labelled with a number vertically beneath it. All pictures shown on the same level on the page belong to the same level of order. Arrows indicate the direction of ascent to a new level of order, and transformation of the motif (by reflective symmetry or by rotation). All states of the motif from lower levels appear together with each of its transformations. If the pictures are viewed in order from 1 to 16 (or indeed the contrary - from 16 to 1), the changes consist firstly of subtractions of transformed motifs from level to level, and continuity (similarity) is revealed through the gradual isolation of the original, undeveloped motifs. This process comes to completion in the central group (7 to 10). From that stage onward, the changes consist of *additions* of transformed manifestations of the motifs, while continuity is further maintained through the persistence of the transforms that have already appeared at lower levels.

A 'reading' of the outer groups of pictures depends more upon visual differentiation than is the case in the more central groups, which rather invoke the retention of the images in visual memory when viewing individual pictures as interrelated members of a group.

The full picture sequence is shown in Figure 7 (1-16).

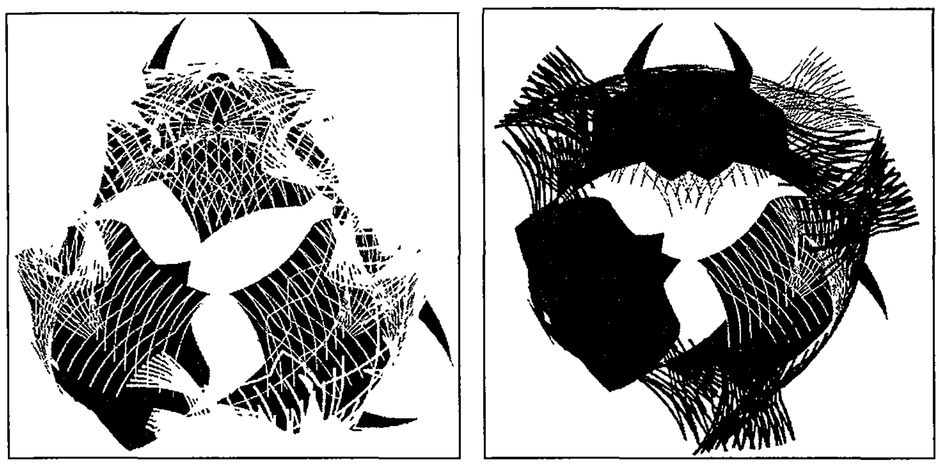


Figure 7 (1-2)

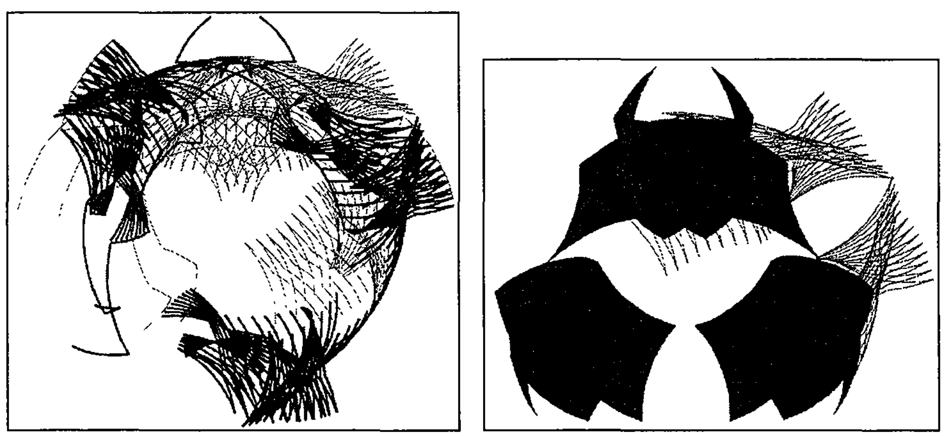


Figure 7 (3-4)

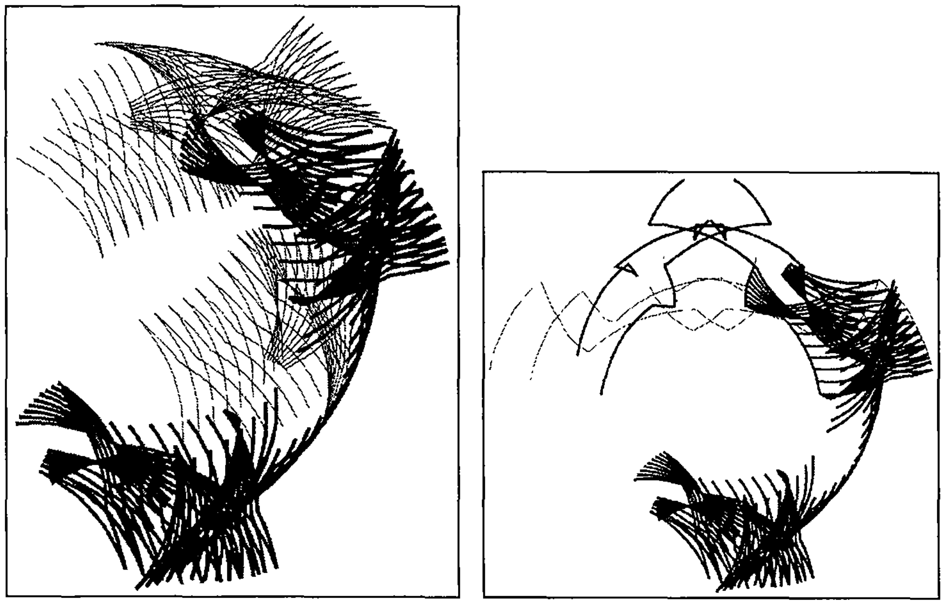


Figure 7 (5-6)

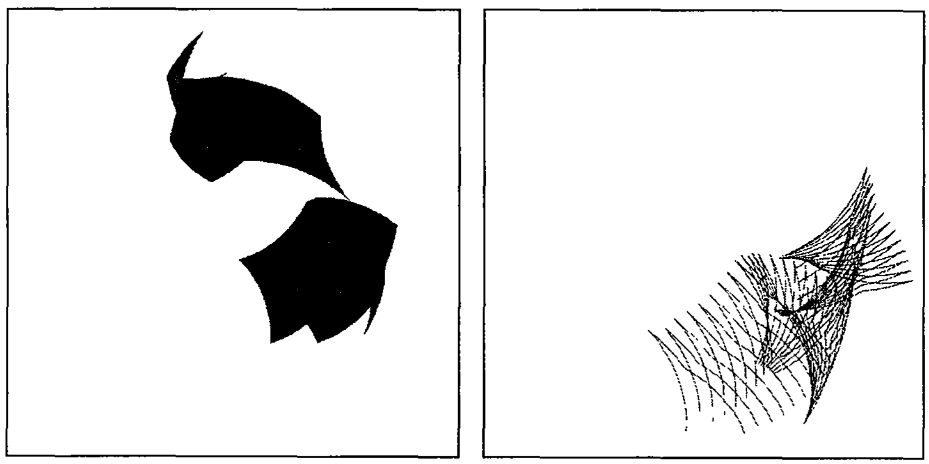


Figure 7 (7-8)

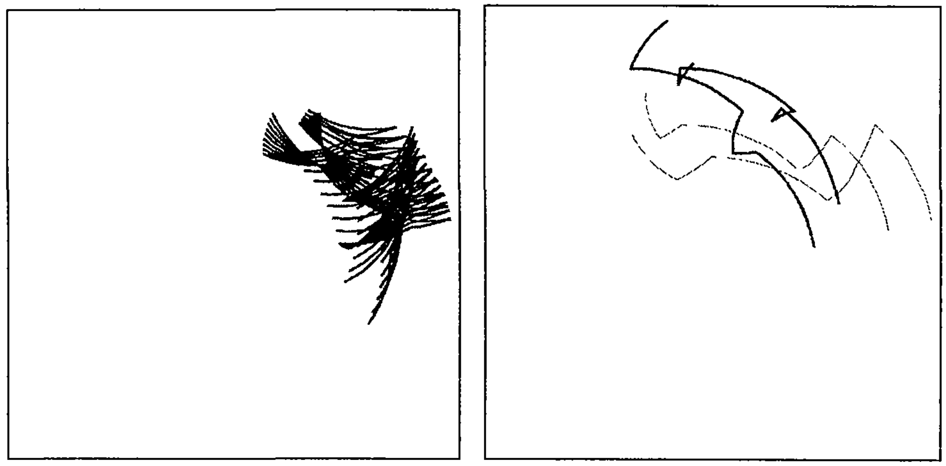


Figure 7 (9-10)

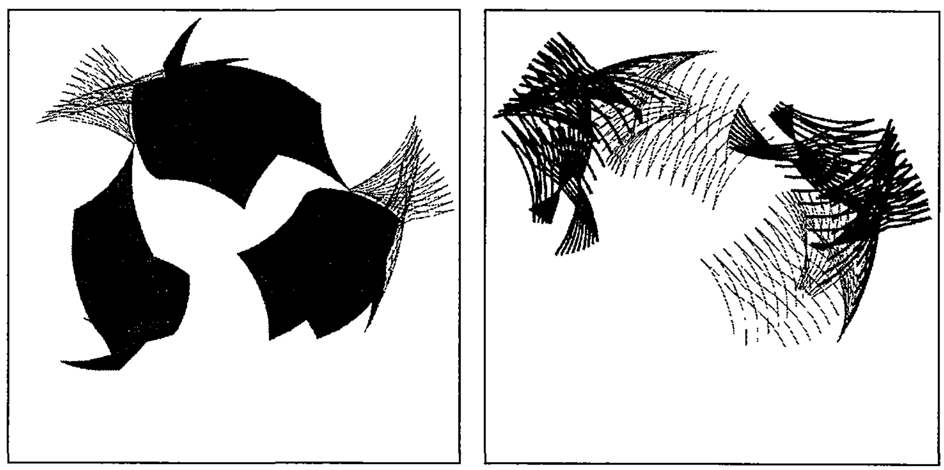


Figure 7 (11-12)

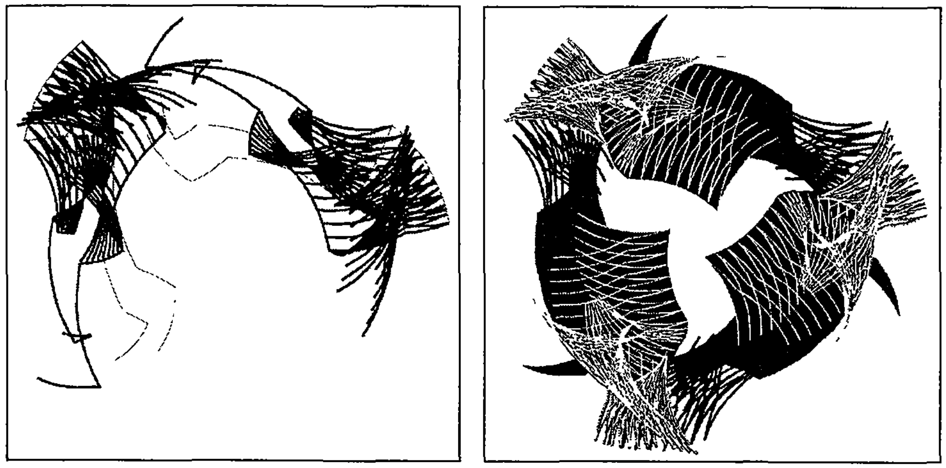


Figure 7 (13-14)

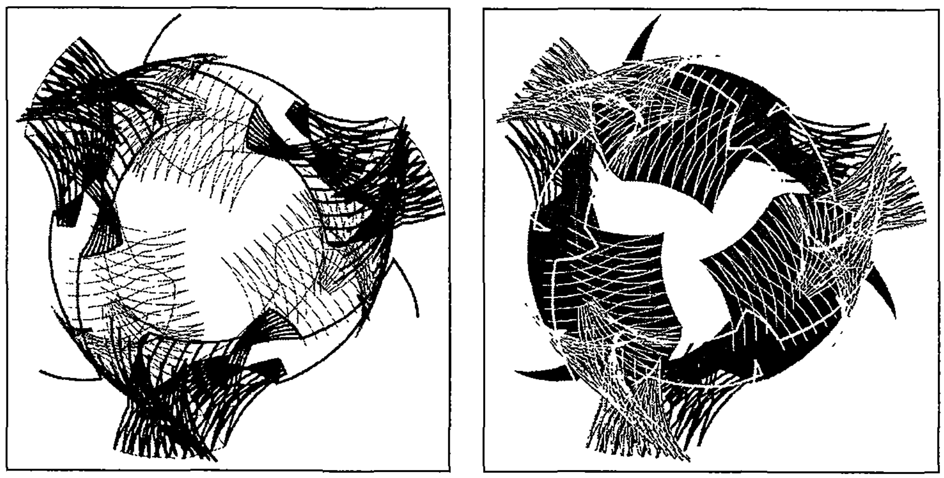


Figure 7 (15-16)

Notating the transformations

A less schematic representation than that given in Figure 6, would include the definition of the orientation and type of transformation involved at each level of the sequence for each motif. This is tantamount to their symbolic and quantitative expression in terms of the notation actually employed in composing the sequence – i.e., EW movement notation.

The motifs *A B C D* have been defined as simultaneous appearances of pairs of the elements *a b c d*. For example, motif *B* is the combination of elements *a* and *b*, the first and second parts of the source sequence shown and notated in Figure 4. The motif appears (alone) in picture no. 7 in a specific orientation, as notated in Figure 8.

	G [0]	↓40	↓30	↓20	G [270	·/·
E	0 [0]	+8	-6	+4	6 [0	
E	25 [90]		↑10	↓20	25 [80	
E	70 (0)	↑40			70 (40)	
	a				b	

Figure 8

The transformation of *B* in picture no. 11, can be seen as the recreation of the elements from scratch, in a new orientation, as notated in Figure 9.

	G [0]	↓40	↓30	↓20	G [270	·/·
E	0 [0]	+8	-6	+4	6 [0	
E	25 [90]		↑10	↓20	25 [80	
E	70 (120)	↑40			70 (160)	
	a				b	

Figure 9

This is equivalent to the rotation of the whole motif *B* through 120 degrees about its origin. An equivalent of further rotation of the motif by the same amount as in picture no. 14 would be notated as shown in Figure 10.

	G [0]	↓40	↓30	↓20	G [270	
E	0 [0]	+8	-6	+4	6 [0	·/·
E	25 [90]		↑10	↓20	25 [80	
E	70 (240)	↑40			70 (280)	

a b

Figure 10

These three (original and two variants) would reveal threefold rotational symmetry.

If the medium had been one in which the display changes in actual time - as in a videotape - this mode of notation would be the only way of expressing the movements to be displayed. However, that is not the present case. The original shapes of the motifs have already been fully defined, and these definitions have served as the basis for the 'macros' *A B C D*. In this structured set of immobile pictures interrelated through symmetry operations, it will therefore be more appropriate to notate them in a way that reflects the symmetry type, while also (since we are dealing with real specific visual forms) indicating *what* forms are involved, and *how* they are orientated. This requirement is fulfilled by treating the paths (shapes) of the motifs as invariant forms, and defining the movements of the total shape, which produce the transformation. For example, in the case of motif *B* the three states of the motif can be adequately indicated as shown in Figure 11.

$(n) = (\underline{n})$

$1 = 40°$

B $\boxed{(0)}$ $\boxed{\uparrow(3)}$ $\boxed{\uparrow(6)}$

Figure 11

The positional series in Figure 12 expresses the whole sequence of rotational symmetries, by indicating the positions of the appearances of the motifs; and also which of the modified motifs 'survive' in each picture. (Note that in the end *all* of them survive - in the sense that they all reappear in each group.)

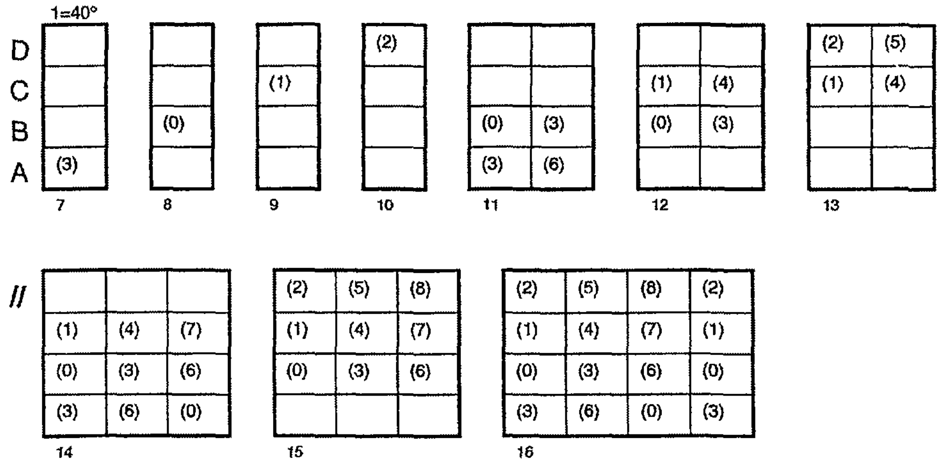


Figure 12

This constitutes a new - third - level of operation: movements of the invariant forms - entire motifs that have already been defined as 'frozen' or 'fossilized' *traceforms* - paths of movement generated by the movements of a curve, the shape of which was itself previously defined as a path of movement.

Similarly, in Figure 13, the transformation of motifs by reflective symmetry can be expressed as the rotation through 180 degrees ('flipping'), of the invariant forms (motifs) about specified axes lying in the picture plane.

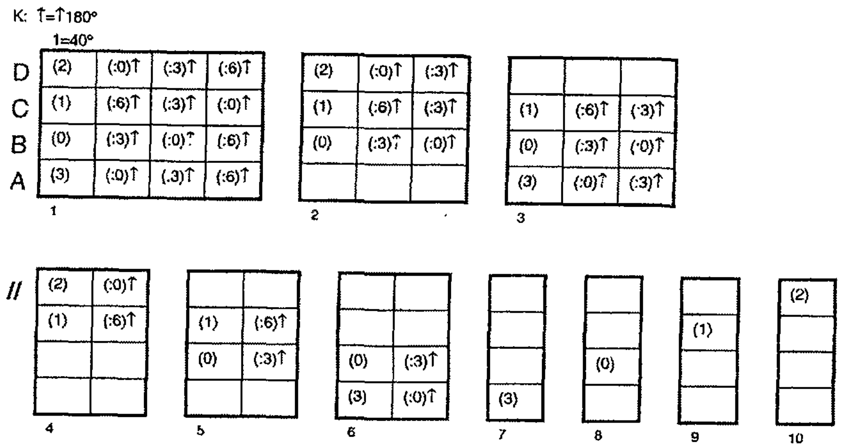


Figure 13

Here again there is explicit indication of which transformations survive in each picture.

We have now explained how it is possible to give in EW notation a fully detailed account of a structured sequence of 16 pictures that includes the definition of generating links, the generation of elements and motifs, and the full deployment of successively transformed combinations of those motifs, including a representation of the detailed operation of the two symmetry types employed in this sequence.

The last picture (16) manifests the total rotational symmetry towards which the second half of the sequence tends throughout its intermediate stages. This *tendency* is the mode of articulation that connects the parts of the sequence that include pictures 7-16. In the reflective transformations, there is complete symmetry between pairs of motifs at every stage. The successive elimination of these symmetries down to the isolated asymmetrical motifs in pictures 7-10, is the mode of articulation - or rather, of separation - of the parts of the sequence that include pictures 1 to 10. Furthermore, the overall connecting principle which gives the sequence its unified structure is that of reduction followed by accumulation. (In its accumulative aspect, the sequence has symmetry about the central group.)

Besides movement notation, we have for the purpose of this explanatory article also used typographical representations of the overall structure in explaining the successive grouping and recombination of the motifs, and in a 'map' of the evolution of the motifs and their recombination in transformations, resulting in increasing complexity as the distance from the middle group of pictures increases. These schemes are comprehensible with a minimum of verbal explanation. While they are not detailed notations, they make sense because their frame of reference is the unambiguous detailed specification fully formulated in terms of concepts of EW notation. Different selections from these schematic representations would yield descriptions of varying generality - some so simple that they could be given in words and commonly accepted symbols such as σ and C_3, as was done in the penultimate paragraph of the section entitled 'The picture sequence...'

Ultimately, only the pictures themselves can give a proper visual realization of the structure, for no description or symbolic representation can fully convey the landscapes of visual experience.

REFERENCES

Avital, T. (1996) Symmetry: The Connectivity Principle in Art, *Symmetry: Culture and Science*.

Avital, T (under review) Footprints Literacy: The Origin of Art and Prelude to Science, *Symmetry: Culture and Science*.

Eshkol, N. and Wachman, A. (1958) *Movement Notation*, London: Weidenfeld and Nicolson.

Harries, J.G. (1969) *Shapes of Movement*, Tel Aviv: The Movement Notation Society.

Harries, J.G. (1975) A Proposed Notation for Visual Fine Art, *Leonardo* 8, 295-300

Harries, J.G. (1981) Personal Computers and Notated Visual Art, *Leonardo* 14, 299-301

Harries, J.G. (1983) *Language of Shape and Movement*, Tel Aviv: Tel Aviv University.

Harries, J.G. (1986) Symmetry and Notation: Regularity and Symmetry in Notated Visual Art, in Hargittai, I., ed., *Symmetry: Unifying Human Understanding*, New York: Pergamon.

Weyl, H. (1952) *Symmetry*, Princeton: Princeton University Press.

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BISHOPS OF OPPOSITE COLOURS: THE IDEA OF SYMMETRY AND THE SYMMETRY OF IDEAS

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INTRODUCTION

Although the last resort of the scoundrel academic is to fall back on the truism that we all reserve the right to be speculative, there still remain some intellectual projects that cannot rid themselves of the whiff of impertinence, and I have to admit that this is one of them. I believe that the general idea was that an arts specialist should offer some personal reflections on the notion of symmetry, and there is sufficient provenance for a line of argument to be developed, perhaps centred largely on the decorative arts and the aesthetics of pattern recognition, with genuflections in the direction of *art nouveau* tiling, traditional wall paper, the spiritual geometry of islamic art and architecture, together with more recent icons coming out of chaos theory, including the ubiquitous Mandelbrot fractals.

But the more I contemplated this agenda, the less it seemed to capture what I actually wanted to say. In part this was because the approach seemed trapped in the second of the classical double definition of symmetry, which juxtaposes a mathematical version (that an expression is symmetrical if it remains merely reflected in the face of operations that might otherwise reconfigure it) with an aesthetic version that adjudicates symmetry to be a matter of pleasing internal balance and proportion. The first kind of symmetry derives its power from its use in explanation within fields where the data is capable of mathematical manipulation such as Biology or Linguistics, with the added practical advantage that design processes in fields exhibiting symmetry, such as Engineering and

Architecture, can and do make use of repetitive iteration (CAD being the paradigm example). The second, as usually formulated, has until recently been little more than a footnote in aesthetic history, although revived somewhat under the influence of the kind of cognitive psychology that has concerned itself with pattern recognition. To put matters this way is already to have formulated the central question as one of what kind of order is to be found in art, moving on to a consideration of the balance it draws between order and disorder, symmetry and symmetry-breaking. And it is increasingly recognised (as for example by Ian Stewart and Martin Golubitsky 1992) that nature itself, so often the object of contemplation in art, exhibits deeply symmetrical inner structures not least living things through DNA, although the actual processes by which living things come into being seem to prefer right-hand spirals. The expectation, then, is that the aesthetics (symmetries and partial symmetries constructed in the eye of the beholder and perceived pleasurably by the audience) bears some kind of (symmetrical?) relationship with the real order that can be considered as an attribute of the data.

Sydney Parkinson. *Portrait of a New Zealand man*, pen and wash, (c. 1769). British Museum Add. MS, 23920.54a.

The Head of a New Zealander, engraving after Parkinson, in Hawkesworth, *Voyages* (1773), pl. 13.

But this line of argument, although part of what I want to say, also does not seem quite good enough. As reflected in the title, I want to take my speculations beyond the idea of symmetry and its application across the discipline, and in the directions determined by a fuzzy[1] and ill-formed project that seeks to see ideas themselves as stable repetitive structures carrying their own tendency towards soft symmetry. Jan Tent's paper (1993) makes its own claims to "soft symmetry", so perhaps the kind of symmetry I have in mind should be seen as liquid rather than soft, with the equivalences on view much more fluid but nonetheless ordered according to their own principles of commensurateness.

The structure of this paper begins with some trivial examples and moves up to more complex ones, finally taking a sustained look at negotiated cultural meanings as they inform the dramatic pageants that make up the English medieval mystery cycle, particularly the impact on its complex dramatic form of the doctrine of equivalence known as typological theology.

MIRROR MIRROR ON THE WALL

Mirrors give the erroneous impression that they pull their trick of reversal only on a right/left basis but not on a top/bottom one, something to do with the vertical axis of symmetry of the human body. But when I lie down horizontally in front of my mirror, the guy looking back at me leans on the wrong elbow. On a similar theme, when giving this paper at the *Symmetry and Structure among the Disciplines* Conference I wore a *sulu* made up of the Welsh flag, which features a dragon rampant.

Since the dye goes right through the flag, one does not need a mirror to reverse it; it could simply be worn back to front. Does it matter what side the dragon is coming from? It certainly would in the theatre where the symbolism of sinister/dextra adds a dimension (like colour in symmetrical tiling) that reduces perceived symmetry by introducing a novel category of non-equivalence. There is only one "right way" for the Welsh flag.

In passing, it is more usual for dragons in the decorative arts to be symmetrically two-headed. Figures 2 and 3 offer two pleasing examples, the first displaying bilateral symmetry, but the second only rotational symmetry.

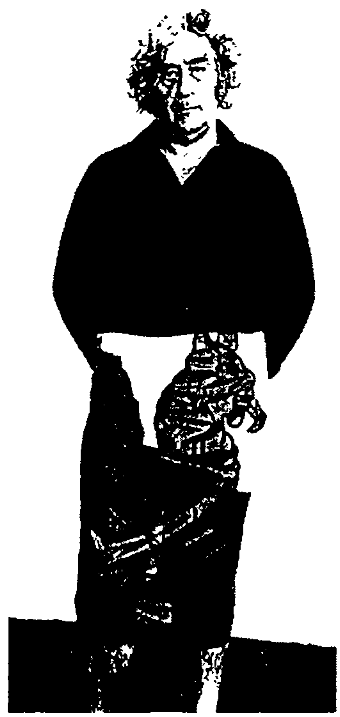
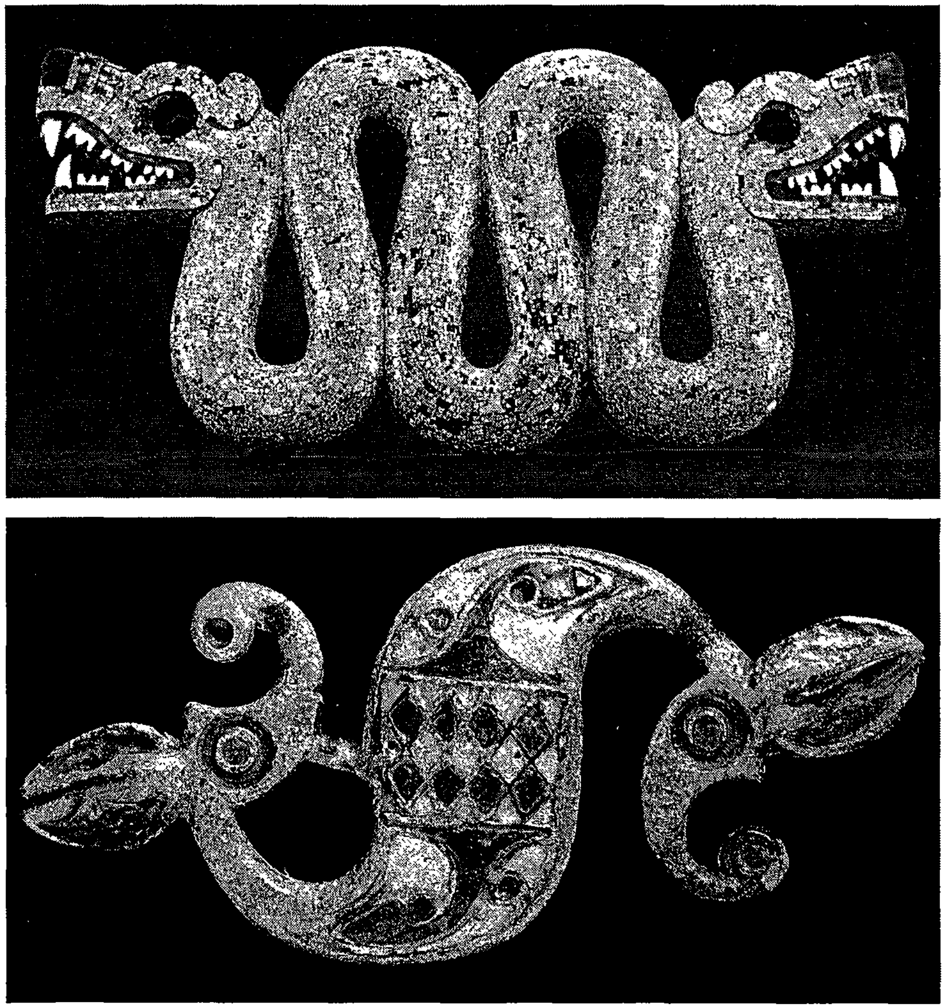


Figure 1: Flag-as-sulu



Figures 2 and 3: Symmetrical dragon ornaments

CROWNS, HEARTS, HANDS

The claddyr ring is a ring of traditional design associated with County Galway on the west coast of Ireland. No matter that I first acquired this particular one to confuse deliberately the semiotic signals during my sojourn as Professor of Education at the University of Ulster (claddyr rings are supposed to be worn only by Catholics); our present interest is in its design. Rings as objects in their most basic shapes exhibit not only a symmetry but a symbolism of shape, suggesting a wholeness and completeness in their sparse roundedness (topological doughnuts, all) but also carrying in the way they are worn on fingers a symbolism owing much to what we might delicately label physiological analogies ("Clarissa's ring" in Shakespeare's *The Merchant of Venice* is clearly handled with such reverberations in mind). But the claddyr ring breaks its symmetry by carrying further emblems on one segment of its circle, in some sense not unlike the bicycle wheel, which allows its rotational symmetry to be broken by a valve. What the claddyrness of the claddyr ring depicts can be variously interpreted, linking together as it does a pair of hands cradling a heart, topped by a crown. Some say it represents the Sacred Heart of Jesus, others that it celebrates the commanding authority of the heart's affections, but there are two other features that command attention. Firstly, that it is reversible and can be worn in two ways; with either the crown or the unprotected heart exposed to the top of the finger; and in Ireland this bifurcation in orientation is deployed culturally to indicate commitment (marriage, engagement or "going steady") or emotional availability. Secondly, and arising out of this possibility, that it is much more aesthetically interesting than a plain ring, precisely because of its symmetry-breaking, whilst retaining bilateral symmetry around a single axis (hearts, hands and crowns echoing the symmetry in their own way).

Figure 4: The "Claddyr" ring

THE RORSCHACH BLOT

First year university students studying the quasi-science of Psychology can scarcely avoid coming across the most famous of the projective psychodiagnostic tests, the so-called Rorschach Inkblot (Figure 5).

As Bootzin, R., Acocella, J., and Alloy, L. (1993) point out, the validity of the test is based on the assumption that unconscious motivations can be drawn out by asking subjects to "interpret" ambiguous stimuli into which they will be disposed to read meaning, first through free association but increasingly via a kind of psychological interrogation. The Inkblot cards vary considerably in detail, but they have one feature in common, displaying bilateral symmetry, almost as though they had been produced by folding a piece of paper to mirror-image an inkblot. It is this bilateral symmetry, of course, that anthropomorphises the image and trades on the (according to Jonathan Miller probably wired-in[2]) capacity of the human brain from infancy to recognise faces and their myriad of expressions and subliminal messages. For similar reasons attributed meanings may also be taken in the direction of flowers or insects as one's mood or psyche might determine. This theme (believing is seeing) resulted in the strangely enduring iconography of Dürer's misobserved rhinoceros. His "plated armour" image was copied assiduously for nearly a century, including the anatomically mistaken dorsal horn[3].

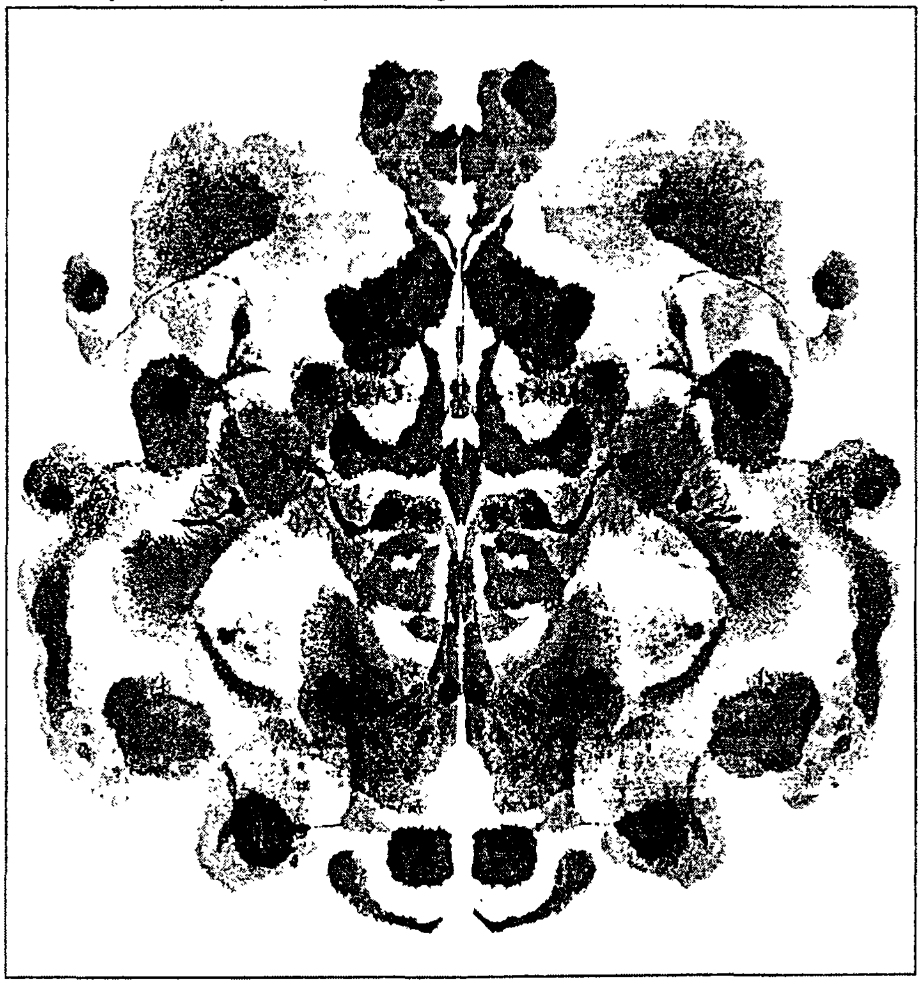


Figure 5: Rorschach inkblot

PIED BEAUTY

Perhaps beauty has little to do with symmetry after all. The Jesuit poet Gerard Manley Hopkins is usually perceived as torn between his ascetic calling to the "God beyond change" who (paradoxically) "fathers" everything forth, and his overwhelming aesthetic engagement with the "dangerous" earthly beauty of an ever-shifting and changing creation itself. His poem *Pied Beauty*[4] glorifies God for His "dappled things", valuing the transient, the ephemeral and the fleeting ("landscape, plotted and pierced, fold, fallow and plough / And all trades, their gear and tackle and trim / All things counter, original, spare, strange / Whatever is fickle, freckled (who knows how)...."). There is perhaps an echo here of the sort of thing James Gleick talks about in *Chaos* (1987) when he describes how the reconstituted geometers, persuaded to look afresh at things like clouds, eroded landscapes and turbulence, have been digging up new words to describe whole families of shapes, words like "jagged", "tangled", "fractured", "twisted" and "splintered". If Gerard Manley Hopkins were alive today he might even have become a topologist.

Figure 6: Monet's *Grain Stacks at Giverny*

The spirit of *Pied Beauty* can be discerned in *fin-de-siecle* French Impressionist painting, particularly the work of Monet in his legendary concern for the ambience of light, subtle colouring and atmosphere that might be snatched from fleeting moments of time. Yet Monet became pre-occupied with "series paintings" (of the facade and Tour d'Albane of Rouen Cathedral, the grain stacks at Giverny etc.). For the grain stack paintings, he often stood in exactly the same position rendering the stacks under different lighting conditions and different seasons of the year ("I have been grinding away, bent on a series of different effects, but this time of the year the sun goes down so quickly I can't keep up with it") Collectively they show an unnerving balance between periodicity and what Hopkins would have called their unique idiosyncratic "inscape"[5].

MORAL SYMMETRY

Ideas of appropriateness and balance often outcrop in areas of life where to assert a mathematical model of commensurateness would appear contrived. But take the *lex taliosis*, the famous Hebrew doctrine of appropriateness in punishment; the eye for the eye and the tooth for the tooth is symmetrical in its patterning, with retaliation aping offence, although the notion of fittingness has at times in our cultural history been subjected to grim excess, as when the adulterers among the damned in San Gimigniano Cathedral's *Last Judgement* have sadomasochistic retributions enacted on their offending parts.

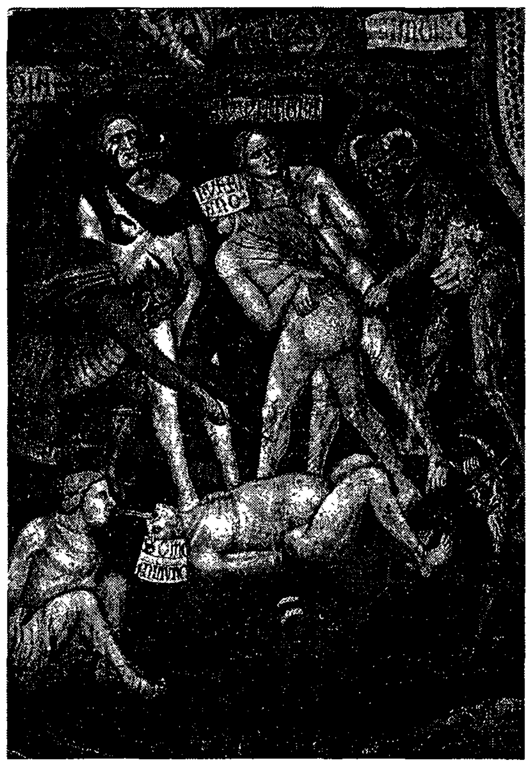


Figure 7: *Last Judgement* in San Gimigniano cathedral

The Greeks, of course, saw matters differently, imagining a moral universe with some asymmetrical unfairness, precisely because the gods were held to be capricious. The symmetry was broken when the victim's number came up, as when a roulette wheel loses its rotational symmetry at the point a ball drops into a slot.

FROM MIMESIS TO "THIS WOODEN O"

Any discussion of symmetry in the arts has to find a way of handling the classical doctrine of *mimesis*. According to Plato, all art is mimetic, either striving for true likeness (the "eikon") or representing from a viewpoint (the "phantasma"). But as Monroe Beardsley (1966) points out, this doctrine in its strongest version implies deep symmetries central to what it is to understand anything, thus linking the idea of symmetry with the symmetry of ideas. ("Not only are objects imitated by pictures of things, but the essence of things is imitated by names, reality by thought, eternity by time.") These are ideas we will take to English medieval religious drama, but meanwhile will jump ahead of ourselves and consider Shakespeare's Globe and Swan theatres.

At the heart of the architecture of the Globe Theatre lay both a conceit and an ambiguity. The conceit punned on the physical shape of the theatre's structure, a "wooden O'" that held in suspension two extreme versions, standing both for the world and nothing, the pun itself echoing the hyperbole by which a stage could name itself after a planet or alternatively understate itself as zero, nothing, zilch, the figure naught. And of course in some symbolic sense both were true and the tension between them was productive, a kind of negative symmetry of opposites. Paradox is not dissimilar, although it is important to remind ourselves that its structure depends on some *doxa* being considered *para*.

The ambiguity relates to the Elizabethan theatre's capacity to accommodate shifts in *mise-en-scene* between outside and inside locations. Historically, this flexibility has been explained by people like S. L. Bethell (1948) as having something to do with stage conventions, in particular that scene-setting speeches are allowed to colonise neutral space; but equal attention might well be paid to the physical ambiguities of theatre architecture. The facade of the tiring house lends itself to being treated as the outside of a building such as Macbeth's castle with its attributed "friezes and jutties".

Banquo

This guest of summer,
The temple-haunting martlet, does approve
By his loved mansionary that heaven's breath
Smells wooingly here; no jutty, frieze,
Buttress, nor coign of vantage, but this bird
Hath made his pendant bed and procreant cradle...[6]

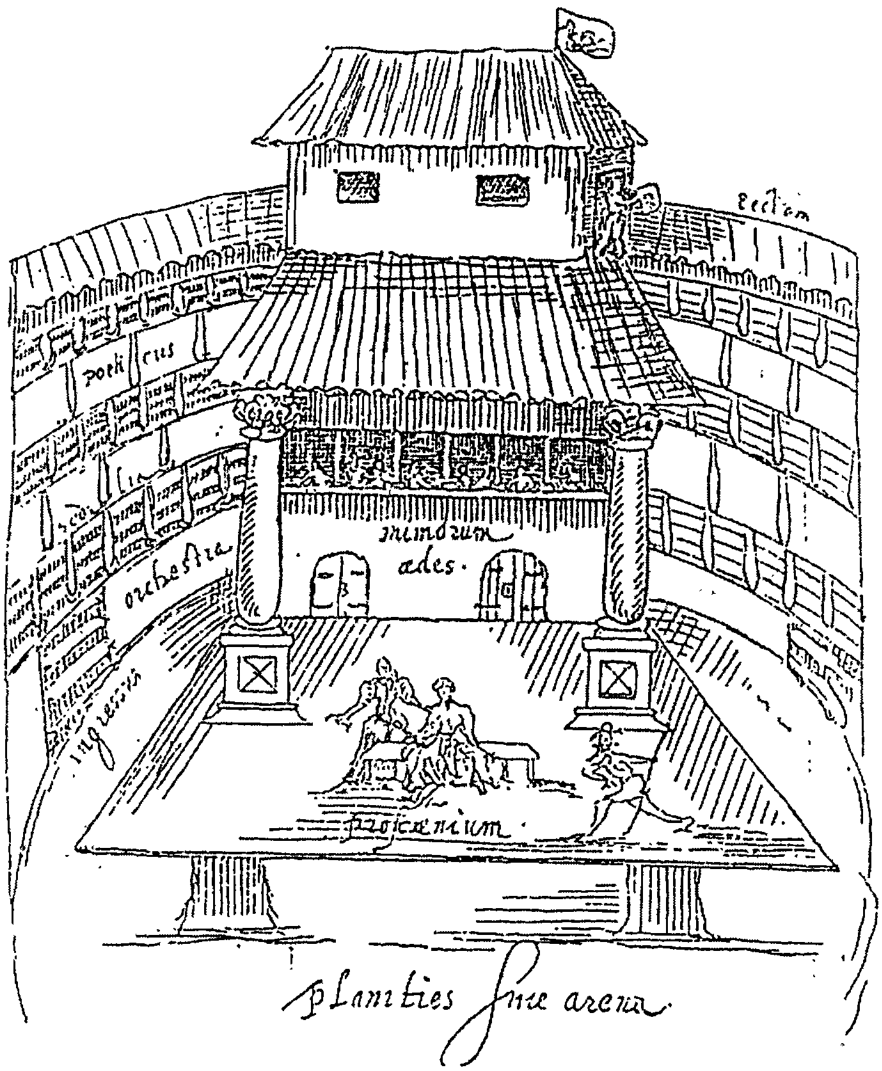


Figure 8: Johannes de Witt's *Sketch of The Swan Theatre*

When seen in relation to the cube-shaped extension that came out from the tiring house above the stage itself (pillars supporting a square roof), sufficient visual ambiguity existed to support the alternative interpretation that the facade represented an inside wall. The fact that the ceiling of the extension was called "the heavens" and painted with stars rendered it also capable of inside/outside dichotomous interpretation, and the plays make consistent use of the implicit optical illusion.

A formal distillation of the same set of ideas can be found in those two-dimensional cube-suggesting symmetrical tiling patterns that retain an optical reversibility as images interpreted in three dimensions, depending on which face or edge is held to be nearest to the eye[7]. The addition of tone or colour can be made substantially to reduce the ambiguity since our brains are more used to interpreting objects lighted from above.

SYLLOGISTIC LOGIC AND METAPHOR

In the mathematical expression $x = y$, it is unproblematical to assert equality at either side of the equals sign since its ordinary meaning is not to compare but equate. One consequence, perhaps the cause, of the symmetrical form of the expression is that it can be written as $y = x$ without affecting its values. Yet ordinary language statements that appear superficially to have an identical configuration carry a particular problem associated with the verb "to be" in that there is often a difference that the traditional formal logic identifies as pertaining to the implied distribution of the terms. Thus "apples are fruit" can be seen as a shorthand version of the longer sentence, "all apples are some fruit", a feature sufficient to make the sentence asymmetric although perhaps retaining a soft bilateral symmetry setting it aside from sentences using transitive verbs, as in "Sitiveni castigated the sleaze-bag".

Syllogistic logic is a pattern of inference (as opposed to truth) by which it is possible to assert some conclusion deduced from two premises. Let us stick with the most famous example:

All men are mortal
Socrates is a man
Therefore Socrates is mortal

Leaving aside that inference is in some sense itself a symmetrical system, consisting entirely of transformations preserving "truth value" equivalence, it is interesting to note that the two premises that lead to the conclusion are technically interchangeable, whilst at the same time possessing a natural order, the major preceding the minor. One of the rules of syllogistic inference is that there can be no increase in the distribution of a term between the premises and the conclusion[8].

When we move on to the world of the arts, the caveats we noticed in relation to propositional logic are present in spades, since simile and metaphor are not only seeking to assert equivalence and commensurateness in the teeth of differences but are often far-fetched, extremely in the deployment of the metaphysical conceit ("Far-fetched, but worth the carriage," observed Dr Johnson dryly). A typical example might be John Donne's *The Flea* which seeks to seduce a mistress with the thought that since the insect has bitten both of them it has pre-sanctified their union in the mixing of blood, as well as being emblematic of the three-in-one Holy Trinity. Nice one, John.

Nevertheless, metaphorical language is capable of carrying sustained parallels of great emotional and intellectual appeal and it would be parsimonious to exclude this area from any discussion of soft symmetry. When parallels of this kind are sustained in narrative the resulting framework is often what we call allegory[9], which is a cultural trick whereby a story is allowed to carry a non-literal message in relation to which it stands as a sign. As ever, the reciprocity between a sign and what it signifies is at one level a symmetrical one, despite the polymorphous perversity of signifiers.

More than perhaps we realise, our interpretations of experience are ultimately dependent on making comparisons, asserting sameness, and using preferred and proven configurations as the basis on which to assimilate anything new. It is small wonder that there is a natural history of metaphors that take them from their first beginnings as arresting novel comparisons (e.g., why not argue a dramaturgy of everyday life?) to their perception as well-worn literal truth (e.g., role theory as put forward in standard structuralist/functionalist sociology textbooks).

Stanley Fish (1980) offers a brilliant exposition of a sermon by Lancelot Andrewes, dating from 16th April 1620. Fish sees the sermon as an example of what Roland Barthes calls "replete literature". Andrewes' sermon deals with the episode when Mary Magdalen encounters the Risen Christ outside the empty tomb and "supposes Him to be the gardener". But Mary is cleared of error, since her mistake concealed a deeper truth (Jesus is the second Adam, can weed the tares of sin from the human heart etc.). Fish realises he is dealing with "structural homiletics", a world of mutual affirmation in which the available meanings are seen as a vast mesh of fitnesses, agreements and correspondences "within a storehouse of equivalent and interchangeable meanings". The narrative of the sermon, as of the Gospel itself, posits a world in which it is not possible to make a mistake as every surface feature carries its load-bearing share of the deeper truths that it reflects in its internal symmetries.

Later in this paper I will be taking a look at extended metaphors, symbolic associations and allegories as they proliferate in English medieval religious drama. But by way of foreshadowing, it may be useful to note three ways in which the dramatic genres leading up to Shakespeare and the English Renaissance echoed the basic configurations of bilateral, rotational and periodical symmetry. These parallels can be seen respectively in the conventions of the sub-plot (mirror-imaging the main plot)[10], the reflected antitheses and double allusions of irony[11] (itself always dependent on comparisons around a twist) and thematic repetition at the level of plot[12]. It was not for nothing that A. P. Rossiter (1950) described the Wakefield Master, responsible famously for the "mock nativity" in the Towneley *Secunda Pastorum*, as having "a peculiar twist to his vision".

THE GAME OF CHESS

It has long been realised that certain repetitive tiling patterns lose some of their geometric symmetries by the addition of colour, a consequence that underlies much of the complex artistic punning of M. C. Escher[13]. A deceptively simple example is the ordinary chess board, which allows alternate squares to be coloured black or white. Since either a black or a white square will appear at the bottom right hand corner, and the rules of chess require it to be a white, the chess board in relation to either of the players evidences rotational symmetry only at 180 degrees, with 90 and 270 precluded by the colouring. Interestingly, the inclusion of colour means that the chess board has bilateral symmetry only around two axis, the black and white long diagonals.

When we add the pieces in the opening position ("the array") there is again a tantalising near symmetry, interestingly broken in three ways, by the colouring of the pieces (raising again M. C. Escher's question of the extent to which colour matters), by the disruption of the pattern on either side of the line between the *d* and *e* files by the non-equivalence of the King and Queen, and in actual play by the alternation of moves so that any tendency towards symmetry in opening theory is a matter of Black echoing to the extent that is possible White's previous move.

Chess players are, of course, deeply aware of the impact of these imperfect symmetries and many have settled preferences either for balanced or unbalanced positions. Several defences are sufficiently reflective of the opponent's opening moves to be labelled "symmetrical", for example the symmetrical defence in the English opening and the Austrian defence in the Queen's Gambit Declined[14]. It is in the endgame, however, that a formal symmetry, perhaps organised around the axis of the long diagonal, enters into solutions, with the practical concomitant that one section of analysis will hold for its symmetrical alternative. An interesting pure example was recently offered by John

Nunn (1994)[15]. The following diagrammed position involves a neat reciprocal zugzwang through which White can force a win by Bishop to g2.

The only way for Black to avoid loss of the rook is by moving it either to c8 or a6, but because of the symmetry of the position either analysis will hold for the other. If c8 then the rook on f3 moves to f7 with discovered check. The Black king is forced to b8, allowing the White rook to b7 forcing the king back to a8. The point becomes apparent when White gently moves his king to h2 creating another reciprocal zugzwang. Black to move must lose his rook to some discovered check. Although it is rare for chess to achieve this level of symmetrical geometric precision, the aesthetic qualities of the game are substantially linked to its subtle balancing of true, partial or broken symmetries and it is these qualities that make the game so appealing. Like many games, it also has the oppositional symmetry of zero sum since in any chess game there is one point to be distributed between the players. A draw gives both half a point.

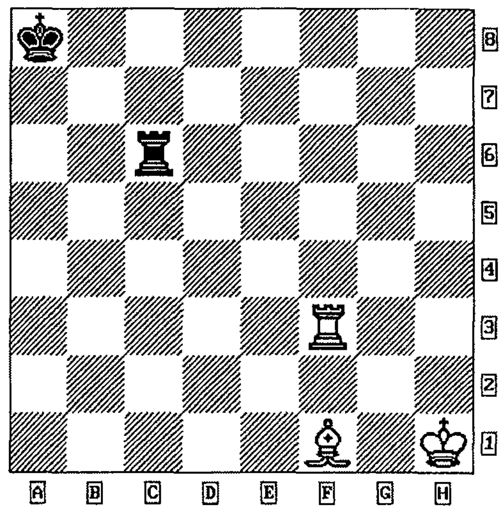


Figure 9: Reciprocal zugzwang with symmetry

IMPLICIT BIOGRAPHY

The possibility of projective structuring is not confined to the Rorschach blot, as it is increasingly seen as legitimate in literary criticism to plot an unconscious soft symmetry between texts and lives. One fine example of this biographical criticism can be found in Ellen Moers (1985) which develops the argument that Mary Shelley's *Frankenstein* is in effect a mythological reworking of her tragic experiences of pregnancy and parturition[16]. Mary Shelley's mother, Mary Wollstonecraft, died giving birth to her and she herself lost several babies. Frankenstein's creature (Frankenstein was the inventor and not, as commonly supposed, the monster) was put together with bits of dead bodies gathered from graves and charnel houses in a way clearly drawing on these experiences. My ex-colleague Carolyn Steedman has been able to explicate the links with reference to entries in Mary's diary[17]. Clearly this kind of parallelism must remain imprecise at any level of detail, but the principles of thematic commensurateness are utterly clear[18]. As ever in the Arts, we are dealing with reverberation and resonance rather than formal geometric equivalence but there is a commonality of patterning nevertheless.

STRUCTURAL ANTHROPOLOGY

Already admitted into our consideration is the notion of the underlying patterns, and it is one of the tenets of structuralism that when exhumed in particular contexts they can display a fearful symmetry. Barbara Hardy has observed that narrative is a primary act of mind, one of the absolutely basic ways of bringing order to the intractable complexities of experience. One might expect narrative to reflect this unavoidable and opaque human diversity. But one would be wrong! Hardy (1975) examines the narratives of fiction not as unique aesthetic monads but as heightened versions of common discourse, organised around stable thematic patterns (although clearly exhibiting surface cultural variation). In the novel, the epic and the picaresque narrative carry the largest burden in sustaining the template of the genre. And certainly in relation to that range of narratives that we call myths, their historical subjection to the interpretive approaches of a "structural anthropology" has led to the strong assertion that it is possible to exhume a deep persistent underlying patterning. This approach has been epitomised in the book by Claude-Levi Strauss (1963) that names the game, *Structural Anthropology*. Readers preferring the more tough-minded definitions of symmetry will warm to Levi-Strauss's analogy with mathematical expressions retaining symmetry under transformations ("variants of one myth or several myths which appear different from each other can be reduced to many stages of the same group of transformations as can their corresponding rituals of different people"). An interesting litmus question is whether this insight does or does not appear to be counter-intuitive.

The robustness of the patterning leads Levi-Strauss to assert that the evidence shows "relations of symmetry" between the rituals and myths of neighbouring people. This example perhaps opens up a further dichotomy; elegance in explanation so often strikes one as pleasing precisely because of its sparseness. As we should see later, eloquence in the decorative arts paradoxically often depends more on complex elaboration and variation, in short on symmetry-breaking as much as on symmetry.

CLASSICAL AND ROMANTIC AESTHETICS

One set of ideas from which we might seek some of the answers is classical aesthetics, pursuing some of the issues that Dénes Nagy raised in his introductory lecture "Why Symmetry?" which are reflected in this volume. As Monroe Beardsley (1966) analysed, Plato's answer to the question what kinds of things are both complex and beautiful saw them as exhibiting ideal properties. Thus a temple was imbued with literal mathematical qualities that guaranteed its aura of stillness and self-completedness. What was being asserted was the underlying truth that *metron* ("measure") and *symmetron* ("proportion") between them constitute beauty. Deformity is at heart lack of proportion.

But as ever these notions need to be modified with at least half a nod in the direction of the mathematical models of perception wired-in to the human brain. The Parthenon's famous optical illusion, with its pillars bulging slightly so as merely to look vertical must occasion at least a mild diversion from the platonic argument. A modern version of this digression would be the insistence that any decent aesthetics theory would have proper regard for the mechanics of perception, most sharply framed in neurophysiological terms. Talking as we are of symmetry, it is interesting to record the rectangular geometry associated with the arrangement of rods and cones in the human visual system, and to enquire in what precise ways it might determine perception of shape.

But we need not, of course, allow ourselves to get trapped in a classical aesthetics. In my own country, there was a time when neoclassic canons of taste, at least as far as gardens and landscapes are concerned, shifted abruptly under the influence of that collective cult of individualism and feeling that tends to be collected together under the rubrics of romanticism. As Anthony Ashley the third Earl of Shaftesbury put it, the "horrid graces of the wilderness" were now to be preferred to "the mockery of princely gardens", gardens which sought to tame the rude promiscuity of nature by imposing geometric order on it[19]. The new taste embraced the sublime as well as the picturesque and led to paintings like those by Salvator Rosa and Claude of Lorraine, showing humanity overwhelmed by brooding alpine cliffs, and having titles like *Landscape with Banditti*. At the time taste was seen as detaching itself from mathematics and basing itself on mood. There is evidence that this was a lens through which explorations of the South Pacific were viewed and interpreted, as, for example, in Figure 10.

One of the really interesting things about chaos theory, as we shall see later, is that the gnarled oaks are now once again seen, alongside cloud formations and ferny undergrowth, as possessing their own soft symmetry.

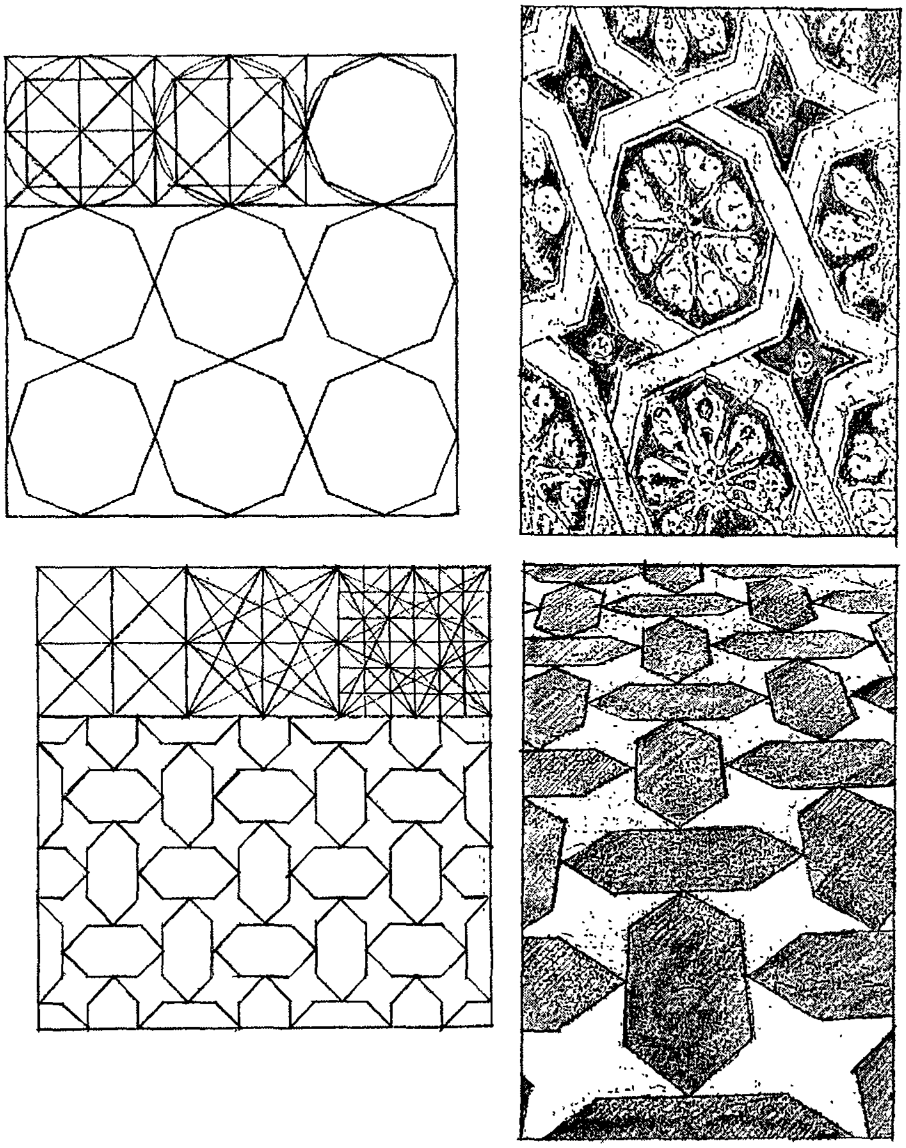
If we are to take on a single lesson into the rest of this paper, it is that in matters of taste and aesthetics there is no such thing as a naive viewer. Any assertion as to why symmetrical patterning may be perceived as pleasurable must cope not only with alternative delimitations of what is symmetrical but must also allow itself to be subsumed under cultural history[20].

Figure 10: Augustus Earle, *Waterfall in Australia*

ISLAMIC ART

My former colleague Richard Yeomans (1993) recently analysed the abstract geometric art of Islam as an abstract expression of its spiritual values, since as Titus Burckhardt has observed, there is a strong rejection of devotional images in Islam. Particularly on the Sufi side, Islam holds that the deepest truths cannot be put into words but must be "manifest" in a uniquely abstract visual language, which subsumes under a spiritual geometry everything from calligraphy to its uncluttered mosques[21].

The resulting decorative art and architecture is unrelievedly abstract, based on a method of mensuration and composition that predates the numerical decimal system in the 8th century, probably at first making use of ropes with equidistant knots. As El-said and Parman (1976) note, its ordering principle is that of *mizan* (balance or order) and depends aesthetically on the systematic arrangement of repeat units in the overall design.



Figures 11: The geometry of Islamic tiles

Figures 11 show the design principles lying behind the two examples. The sheer complexity of the architectural inventiveness never ceases to amaze. The squaring of the circle, allowing a circular dome to cap a square plan, was achieved by Byzantine architects in a way that spawned the complex honeycombed crystalline forms known as squinches and the stalactite forms known as pendentives.

ERNST CASSIRER AND SUSAN LANGER: FROM "PHILOSOPHICAL ANTHROPOLOGY" TO "SYMBOLIC TRANSFORMATION"

The claim made in the introduction for the broad thrust of this paper was that it is attempting to move outwards from a discussion of the idea of symmetry towards some parallel notion of the repetitive patterning through which cultures seek to organise experience, what I called by the hopefully evocative but doubtlessly slippery term ``the symmetry of ideas". It is necessary to lay down some basis on which this project might be attempted, and I shall do so by a large scale borrowing of a set of insights first proposed by Ernst Cassirer in his Philosopie der Symbolischen Formen. Cassirer has been influential in showing us how we might see the arts alongside other preferred explanations as aspects of human culture and his work has been built upon in particular by Susan Langer and more recently by the cognitive psychologist Howard Gardner in Art, Mind and Brain (1982).

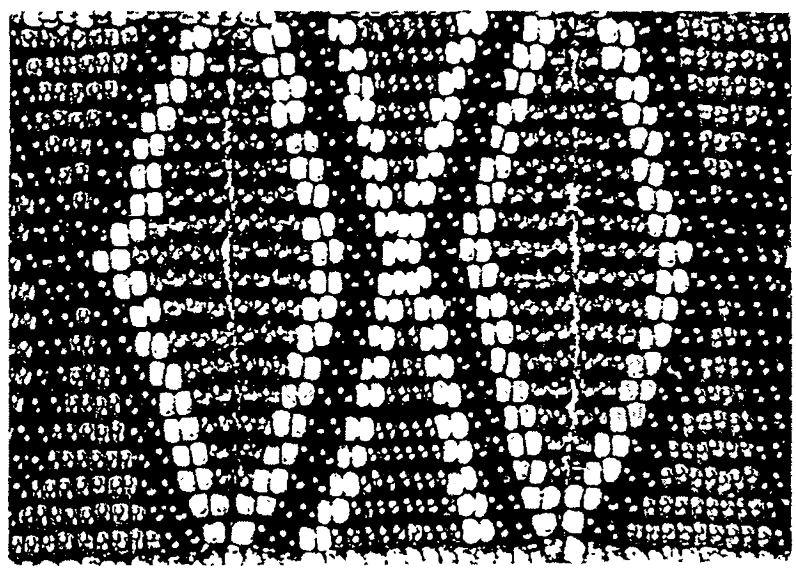


Figure 12: Patterning with beads

The insight at the heart of Cassirer's work is his argument that human forms of intuition (space, time etc.) and categories of understanding (substance, causality etc.) are typically *imposed* on raw data to give it form rather than in any strict sense derive from it. Once again this is an intellectual backwater, or perhaps frontwater, sensitively summarised by Monroe Beardsley (*op. cit.*) who depicts Cassirer surveying evidence from psychology, anthropology, and linguistics before coming to the far-reaching conclusion that the great symbol systems or symbolic forms of world cultures (their mythologies, their art forms, their scientific theories and explanations and their religions) are not modelled on reality but actually seek to model it. If so, the purpose of all the creativity of spirit that we see in these human endeavours is less a nomothetic quest for underlying regularity than an idiographic imposition of interpretive schema of our own invention. The "symmetry of ideas" that I hope to indicate is the product of the coming together in cultures of the twin forces of human inventiveness, often rooted in metaphor, and our preference for elaborating the patterns we already assert rather than switching to new ones. We just thread the beads in different but depressingly predictable ways.

Susan Langer (1942) in *Philosophy in a New Key* hints at the role played by symmetries (essential meanings unaltered by symbolic transformations) in the world of the arts. If tonal structures exhibit a close similarity to forms of feelings, music can be conceptualised as a tonal analogue of the emotional life, presenting as it were a kind of reflection of it with its essential configurations intact.

Towards the end of this paper I shall be taking a sustained look at English medieval religious drama to test these insights in a reasonably well described setting.

HAGIOLOGY AND ICONOGRAPHY

There is a difficulty in arriving at an appropriate visual representation of a saint, with perhaps a tendency to identify by some association with a symbolic object, like St Catherine and her wheel. Representations need to be symbolically commensurate, not with lives as lived, but with the often apocryphal and yet to be demythologised version of them currently being peddled to succour the faithful. In the case of a "composite saint" like Mary Magdelen, there is a kind of bilateral asymmetry around the twin poles of her life-as-processed, taking in her traditional identification as the woman "taken in adultery" in *St Matthew's Gospel* and her subsequent apocryphal life as a wilderness saint. She is also graphically identified by her assertive but touching act of homage to Jesus before his death, smothering his feet with expensive ointment from an alabaster box and drying them with her (often luxurious and auburn) hair. Her association with

Christ's feet extends insistently into the iconography of the crucifixion, and one quite delightful anonymous miniature from the Siena Pinacoteca depicts her quite unmistakably as the patron saint of foot fetishists, crawling under the table to embrace Jesus' feet at an otherwise sombre gathering in the house of Simon the Pharisee.

On the opposite polarity, Donatello's quite striking wooden *Magdelena*, centrepiece of the 1987 *Donatello et son Sui* exhibition at Fort Belvedere, Florence, revealed an anorexic Mary Magdelen with calf-length hair like some wilderness animal. She appears tense and angular, ravaged with grief, her suffering seemingly void of any comfort or faith, confronting us with what to all intents and purposes is a gaunt stare[22].

AMBROGIO LORENZETTI'S ANNUNCIATION AND ITS ARCHITECTURAL SETTING

Figure 13: The *Annunciation* by Ambrogio Lorenzetti

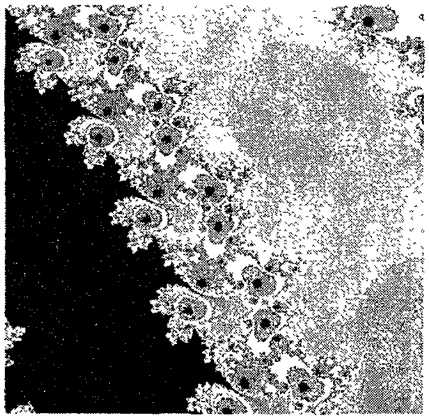
Ambrogio Lorenzetti's reputation rests most securely on the cycle of frescoes of the *Allegories and Effects of Good and Bad Government* painted towards the end of the first half of the 14th century in the Sala dei Novae of the Palazzo Pubblico in Siena, but my own favourite piece is his *Annunciation* in the Pinacoteca. The event, celebrating the moment when the Virgin Mary is told she is with child by the visiting Angel, is variously depicted in art, but at best with a subtlety of reciprocation, most usually expressed as a delicate balance between deference and modesty. Here, however, there is an unusual serenity in the acceptance and an evident easy complicity between the strikingly similar figures in spite of their segregated conventional gestures. But the unusual mutuality of the encounter is achieved architecturally by placing both figures under identical trefoil arches symbolising the eternal Holy Trinity (even as the Son is about to be born). The symmetry of the arches is continued in the expansive ornamental pavement, which has a strong repetitive tiling pattern. Its perspective lines vanish towards a single point, although arrested by the abstract gold background that gives the picture its jewel-like gloss. Interestingly, this application of the rules of Renaissance perspective would not have been experienced as symmetry-breaking; it was simply understood that this is how symmetrical patterns appear when not viewed from above. What the formal tiles contribute to the painting, along with the church-like architecture, is an overwhelming sense of rightness and order, enhanced by the slight all-too-human hint in Mary's expression that perhaps she knows already and is quietly confident[23].

THE AESTHETICS OF CHAOS

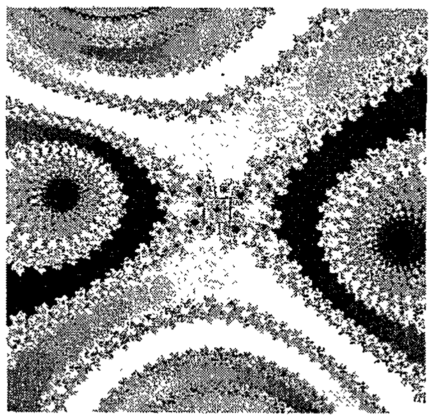
From quintessential order it is appropriate that we next turn to chaos. It is perhaps best to start with an example.

James Gleik (1987) offers a nice account in *Chaos* of the first application of Cantor sets to the distribution to transmission noises. In all systems, "message" is infinitely preferable to "noise", but the standard way of thinking about noise, although recognising its tendency to come in clusters, was that the effects were pretty much random with the worst examples mythologically attributed to somebody somewhere being careless with a screwdriver. But Benoit Mandelbrot brought intellectual order to the noises by seeing them as a Cantor set over time (a Cantor set is a representation produced by removing the middle third of a block, then the middle third of each section etc., thus producing a pattern like the famous rough but robust fractal model of the English coastline, identical on every scale)[24]. In passing, self-similarity across scales can be proposed as a novel form of "Russian doll" soft symmetry with an aesthetic *frisson* of its own which from time to time has informed certain kinds of architecture, although the opposite effect is more common in which a large scale symmetry is subtly undermined at the level of detail (e.g., twelve consecutive identical architectural niches occupied by twelve distinctively individualised apostles).

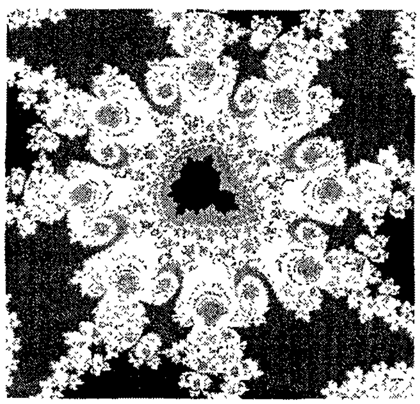
Fractals occupy a world poised surprisingly easily between the realms of the arts and the sciences. Peitgen and Richter's *The Beauty of Fractals* is clearly premised on Mandelbrot's claim that fractals can at one level be read as novel anti-minimalist art, albeit the (accidental?) by-product of dynamic systems, merely offering pictorial visualisation of the encoded mathematical information. But through an act of radical political appropriation they have been taken over by the cyberpunk sub-culture and turned into cyberpunk mandalas by New Age hallucinogenic acidheads ("Just like the inside of my head, man"). The video makers Strange Attractions recently produced a four-minute video, *Where No Penguins Fly* based on a colour-cycled Mandelbrot set. Unsurprisingly, it is now available as a screen saver.



COLOR PLATE 1



COLOR PLATE 2



COLOR PLATE 3

Figure 14: The Mandelbrot set at various scales

One of the shifts in taste associated with fractal aesthetics, is its implicit endorsement of an architecture rich in every scale. Gleick (1987) infers from Mandelbrot's schema an antithesis to the architecture of the *Bauhaus* as the epitome of Euclidean sensibility, "spare, sparse, orderly, linear, reductionist, geometric". By contrast the Mandelbrot set (Figure 14) is the most complex of all objects with its "thorns, spirals, filaments, hanging molecules on God's personal vine". Amazingly, this deep recursive swirling geometry of almost infinite complexity comes, like Creation itself, out of what seems in context so little; the full set can be transmitted using only a few dozen characters of code.

COMPUTER-GENERATED IMAGES: FROM THE UTAH TEAPOT TO VIRTUAL REALITY

One of the axioms of computer-generated artwork within the quasi photo-realistic mode is that it is based on attempts to replicate mathematically optical phenomena arising from predictable relationships between lights, surfaces and reflections. The results have a fashionable plastic, shiny, almost surrealist look, as one would expect from observing algorithms seeking to arrange coloured pixels. As Jane Prophet (1994) suggests, the early pictorial efforts (like the Volkswagen Beetle or the ubiquitous Utah teapot) did not merely establish developmental norms but acquired cult status. Since a Utah teapot is for all practical purposes a data-base, it can be manipulated to appear with a variety of reflective surfaces, from any angle, and under any light conditions, rather like a Monet series painting. The computerised data-base was the more simple to construct because of the partial symmetries characterising the original object.

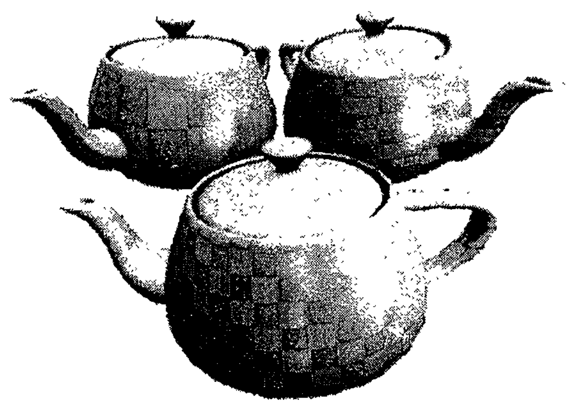


Figure 15: Image of the Utah teapot

It is in VR (virtual reality systems) that emerging computer graphics applications are seeking an experiential symmetry between genuine and simulated happenings, although "virtual reality" is currently a bit misleading, not much more than a manufacturer's hype. But beyond flight simulators for pilot training and the opportunity for CAD architects to walk around inside simulated buildings as if they were real, lie complex problems to do with intellectual property rights (sampling is not coterminous with plagiarism), morality (cybersex will not remain confined to *Lawnmower Man*, and the technology is neutral before the question of whether your partner might be a distant lover or a piece of software that has passed the Turing test[25] and ontology (a concern for truth requires that we can tell the difference)[26]. Clearly, these issues place mimesis at its problematic outer edge.

Figure 16: *Lulu* by Pekka Tolomen, 1992. A computer-generated sex object. The sexual fanatics of hudreds of men (transcribed from pornographic literature and interviews) have been programmed into the system of this program

TYPOLOGICAL THEOLOGY AND FIGURAL REALITY IN THE MEDIEVAL MYSTERY PLAY

We now take these ideas to the sustained example of English medieval religious drama. In spite of its relative unfamiliarity, it exemplifies richly the capacity of developed art forms deploying meta-narratives (in this case rooted in Christian theology) to evolve towards a metaphorical structure of elaborated resonances and equivalences, exactly in the spirit of Stanley Fish's earlier comment on the replete literature of the sermon. A more accessible example might have been to explore the thematic echoes and doctrines of "correspondence" lying behind what E.M.W. Tillyard described as an "Elizabethan world picture"[27] ("Untune that string, and see what discord follows"), but the medieval mystery play has the advantage of offering stronger links between the drama and the iconography. What is at stake, in Cassirer's terms, is the human imaginative capacity to *invent* order and structure. Commensurateness is in the eye of the beholder.

Figure 17: Animated Coco Cola bears

There are four extant texts of English mystery plays, dating from the 15th century, although a continuous tradition began earlier. Cycles have survived from Chester, York, Towneley (Wakefield) and from an unidentified N-town, probably an itinerant cycle travelling at of Bury St Edmunds. Like Gothic cathedrals, the mystery cycles evidence layers of accretions, alterations, excisions and revisions. Mystery plays were street plays, cyclic verse dramas each containing up to forty-six individual plays, and performed under the sponsorship of the medieval craft guilds which were at that time still under the hegemony of the medieval church. In performance they were distinctly "stagey", employing props, machines, traps, tricks, dummy bodies, live animals and exploding furnaces. Craft pride allowed an amusing allocation of the Chester *Deluge* to the "drawers of water from the River Dee" and (more audaciously) gave the *Crucifixion* at York to the nail-makers and ironmongers, turning the sacred nails into commendable craft objects ("large and long"). Although the Feast of Corpus Christi, with which the plays were associated, lies outside the liturgical calendar as such, each in its own way reflects the liturgical year with many individual pageants arranged around the events of Christmas and Easter.

Each cycle of plays tells the story of man from Creation to the Last Judgement, a schema that has been described as "making *War and Peace* thin and *Paradise Lost* parochial". From the point of view of the present argument, it is important to stress that each cycle was written under the arch of scholasticism, in the spirit of St Augustine's *Theatrum Mundi* which quite literally treats history as if it were a play written by God. The critic V. A. Kolve (1986) brought to the plays Northrop Frye's abiding agenda, the quest for the so-called "generic features" that might make the genre stable. Kolve noted that the cycles appear to have been written to a patristic framework, heavily dependent on the teaching of the church fathers[28]. Indeed, Rosemary Woolf (1986) has argued that the patristic tradition of the church was decisive in choosing episodes, and that it is possible to assert an irreducible core including *Adam and Eve, Noah, Abraham and Isaac, The Shepherd Plays, Herod, The Birth of Jesus, The Temptation in the Wilderness, Christ before Pilate, The Crucifixion, The Harrowing of Hell, and The Last Judgement.*

The Crucifixion is sometimes treated with a kind of grim irony, sometimes with an audacious wit. The Towneley Jesus is "done on the rood, tugged, lugged, all-too-torn with traiters", but the cross is also a horse with a wooden saddle, and Jesus depicted daringly as a "jouster" in a tournament. The torturers deliberately mis-drill the holes in the cross piece, requiring Jesus to be stretched with ropes. Emblematically His widespread arms embrace all humanity, but the torturers fail to decode accurately the iconographic symbolism ("Short-armed is He").

Underpinning the use of *Old Testament* material is an elaborate set of ideas known as "typological theology", which asserts a (soft) symmetrical doctrine of equivalence linking *types* with their *antitypes* and developing thematic reverberation at every level (rather like Mary Magdalen being cleared of error in supposing Jesus to be the gardener).

The effect is partly achieved through a "figurative" non-periodic view of time. Although rooted in medieval life, the plays have in common a juxtaposition of narrative time (the sequence of the plot), contemporary time (medieval England) and figurative time (the non-time of the Divine Plan). It is in this last framework that the *fabula* can best be understood, although exemplified at the other levels. Erich Auerbach's defining work *Mimesis* (1957), important to the argument I am trying to develop here, has this to say about figurative time:

> An occurrence on earth not only signifies itself, but others that it predicts or confirms. The link is oneness in the Divine Plan.

Both the theology and the texts are elaborated to resonate with a particular kind of thematic repetition. As well as the deployment of type and antitype, as in the *Abraham and Isaac play*[29], another underpinning theological concept is the "periodic" doctrine of *recapitulatio*, a schema which demands that Christ re-iterates Adam, restoring by repetition. *Recapitulatio* avoids the worst crudities of a "barter" theory of the redemption by emphasising the continuity of God's providence. Something of its flavour is caught in the iconography of the period, so often a bridge between the theology and the drama. The *Holkham Bible Picture Book* uses emblematically the "pelican in piety", the bird pecking its breast to succour its young with little fountains of blood. The strict allusion is to *Psalm* 101 ("I am a pelican in the wilderness") but the fabula references Jesus, justifying the usual non-literal "placement" of the bird's nest on the holy rood tree, itself often sprouting foliage. (The dead tree of the second Adam brings life, while the live tree of the first Adam brought only death). But in the "Fruit Forbidden" of the *Holkham Bible Picture Book* the pelican tops Eden's tree. Only to the uninitiated would this be an inappropriate placing, since the trees have a mystical unity. (Indeed some of the legends of the holy rood assert physical continuity, the rood growing from the pip of Adam's apple). Such complex reverberations appear at time almost capable of exponential growth. The *Arundel Psalter* has a "Tree of Life with Nicodemus" that playfully turns him into spiritual fruit, referring back to the "you must be born again" episode. Similarly a phallic "Rod of Jesse", perhaps the ultimate icon of a disinterested autoeroticism, displays its arboreal genealogy of Jesus and manages to get in on the symbolic act.

A number of important issues arise from the use in medieval drama of the doctrine of *recapitulatio* and the invitation of typological and allegorical interpretation, all of which were already part of the "mind set" of the contemporary believer. In the first place we can dispose of the old accusation of "naive anachronism", which must rank as one of the least discerning misjudgements of all time. More interesting is the extent to which the structure lends itself to the most Sophoclean of ironies in which the audience knows the end from the beginning. Neat examples abound, as when Herod in *The Slaughter of the Innocents* boasts "He will die on a spear" even as the audience references Longinus as its bearer. A more tricksy issue in speculative poetic hands is what might be termed the limits of typological decorum. A willingness to stretch the metaphor to embrace Rahab the hospitable harlot as a type of Christ (her red hair like Christ's blood in Marlowe's *Dr. Faustus* "streaming in the firmament") must be somewhere near the outer edge, but what about the "mock nativity" of the stolen sheep in Mak's wife's cradle in the Wakefield *Secunda Pastorum*?

Among the symmetries and correspondences on view is a rather daring version of moral symmetry in the N-town cycle, which systematically develops the distinction between "pious" and "malignant" fraudulence. The conflict between Jesus and Satan dominates the cycle, but the characterisation is conceived of in a rather daring way, reflecting an earlier classical aesthetic. Cornford (1914) distinguishes a traditional opposition between the *eiron* (from which we get the word irony) and the *alazon* (who is brashly overstated, a braggart in a Herod-the-Great kind of way). In the N-town Cycle Jesus is the *eiron*, but his understatement is given a twist by the imposition on the plays of a speculative theological framework, the so-called "deception of Satan" theory of redemption. Satan has acquired mankind by malignant fraud and must be out-duped in return. Patristic sources for this theme are suggestive, for example Gregory of Nyssa's metaphor of the fish hook in *The Great Catechetical Oration*. Satan is tricked by the humanity of Christ, but it is the divine Logos playing the line:

> the deity was hidden under the veil of our nature, that, as is done by a greedy fish, the hook of the deity might be gulped down along with the bait of the flesh.

The equivalent in chess is the sealed move, known only to the player who has made it. In the plays, the theme is worked through consistently in *The Betrothal of Mary, The Trial of Joseph and Mary* ("By my father's soul here is great gyle!"), *The Temptation in the Wilderness* (after which Satan declares, "His answers were marvellous. I knew not his intention/whether God or man be he I can tell in no degree"), *The Descent of the Anima Christi into Hell* and somewhat equivocally in an incomplete *Last Judgement* play. But the fanciful theology surrounding pious and malignant fraudulence is most fully developed in an exquisite *Christ and the Doctors* play in which the boy Christ ("twelve years old" through his mother but "everlasting" through his Father) turns the temple episode into a theological set-piece. He explains the *Logos* as a *Logos pediagos*, a divine teacher restoring knowledge in kind. A nice exchange occurs with one of the doctors:

Doctor: What need was it for her to be wed
To a man of so great age
Lest that they might both go to bed
And keep the law of marriage?
Jesus: To blynd the devil of his knowledge
And my birth from him to hide
That holy wedlock was a great stoppage
The devil in doubt to do abide.

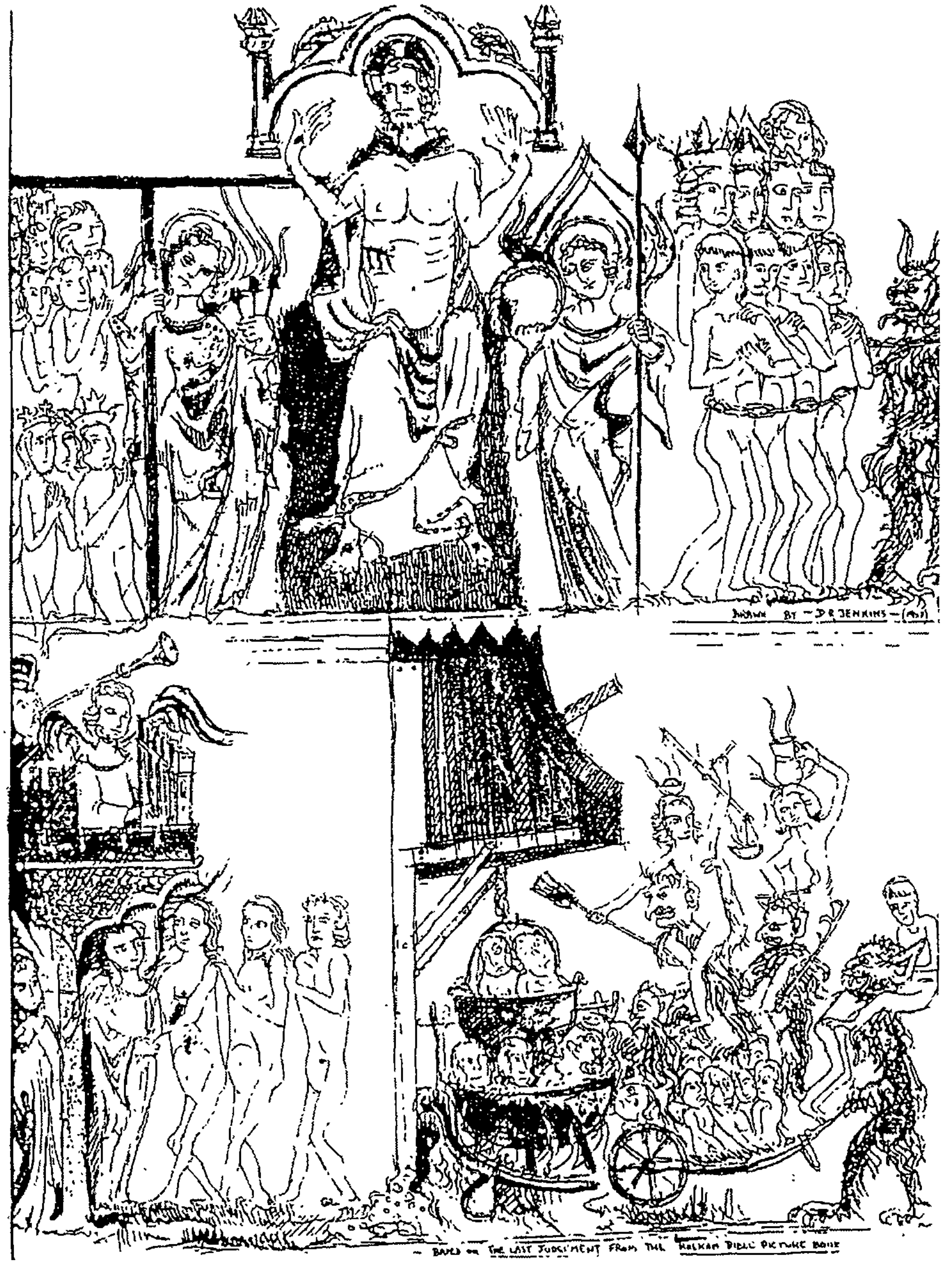


Figure 18: The *Last Judgement* from the *Holkham Bible Picture Book*

Although, as Beardley (*op. cit.*) points out, there was not a strong interest in the Middle Ages in constructing a distinctive theory of aesthetics, the arts (particularly drama and iconography) were seen, beyond initial uncertainties, as sensuous aids to worship. In both the theology and the staging of the N-town Cycle, the incarnation is itself treated as a device producing an illusion, and the two forms of commentary intersect.

One promising strand linking classical aesthetics to the medieval era resides in continuities that can be traced in an implicit theory of interpretation. The work of Homer, as poet and seer, was increasingly held to be open to deep, but rather fanciful, allegorical interpretations and increasingly perceived as carrying profound truths hidden in symbol and allegory. Consider the peculiar problem posed to Christianity in having a double set of texts. At its roughest, the dilemma of the hi-jacked Hebrew Scriptures (turning them into the Christian *Old Testament*) was solved by giving to them the treatment given to Homer, evolving styles of interpretation that transcend their literal or received meanings. Understood in retrospect, in a Christian perspective, the secret truths of the *Old Testament* reveal themselves through typological and allegorical forms of exegesis. As in solving a good crossword clue, the truth once revealed as fitting a symmetrical pattern of complex mutually supportive reverberations, could not have been otherwise. Jonah's experience in the whale's belly in the mode "prefigures" the Harrowing of Hell, an association offering the iconic leviathan hell-mouth to the medieval stage. Such figurative interpretations were legitimised by St. Augustine in terms of their capacity of facilitate spiritual growth. The ultimate inspiration for this theory of interpretation was John Cassian's four levels of meaning to be found in the scriptures – historical, typological, tropological and analogical[30].

A medieval mystery cycle is thematically complex in its reverberations, involving in another context what Douglas Hofstadter (1979) called a "nesting in recursive structures and processes", and involving not a few "tangled hierarchies and strange loops"[31]. Is God a geometer?" ask Ian Stewart and Martin Golubitsky (*op. cit.,* sub-title). In the medieval drama, just to pick the *Noah* plays as offering one of a thousand reverberating images, he is also the metaphorical as well as the literal boat builder, preoccupied with measurements to be sure, but also one whose arc is also an arca – shrine, ship, salvation, church and altar, all at one and the same time. There are no regular solids in poetic geometry.

REMARKS

1. There is increasing recognition recently of the beauty and usefulness of blurred ideas. See, for example, Bart Kosco's *Fuzzy Thinking* (1993).

2. Suggested by Jonathan Miller in a series of (as yet unpublished) lectures on *Looking in Art* at the National Gallery, London, April 1994 which linked art history and appreciation to models of perception.

3. Dürer's misobserved rhinoceros. Equally "anatomically mistaken" was Edward Budd's attempt to extend the Rorschach techniques to the metabolically challenged. See Budd, E. "Rorschach assessment of the 'non living'", *Journal of Polymorphous Perversity*, Vol.9, No.1, Sprıng 1992, pp. 12-16.

4. There is a sustained attempt in the poems of Gerard Manley Hopkins to resolve what are ultimately tensions between faith and experience. See for example Herbert Read, "Creativity and spiritual tension" (1933) in Bottrall, M. (editor) (1975).

5. See Donald McChesney, "The meaning of inscape" (1968) in Bottrall, M. (1975) *op.cit.*.

6. See William Shakespeare, *Macbeth*, New Penguin Edition, pp. 67-68.

7. Similar ambiguities are taken advantage of in Francis Bacon's "Screaming Pope" series, optical ambivalences similar to those that underpin M. C. Escher's disturbing uphill waterflows.

8. My USP editor John Hosack dislikes this formulation since, according to Prior in the *Encyclopaedia of Philosophy* (1967), the concept "distribution" in relation to terms is often ill-defined and too slippery to be useful. But one cannot slough off one's youth that easily.

9. Allegory is probably a fundamental mode of thought in spite of its having thrived in its more elaborate versions in particular cultural contexts. A paradigm example might be the parables of the Kingdom from the *Gospel according to St Matthew*. See Northrop Fry (1957) *Anatomy of Criticism*, Princeton, N.J.

10. As in Shakespeare's *King Lear* where the Gloucester plot revivifies in terms of physical blindness the thematic paradoxes of Lear's spiritual blindness ("I stumbled when I saw").

11. Most empathycally in so-called Sophoclean irony, which adorns accident in the clothing of design.

12. Famously in Samuel Becket's "periodic" *Waiting for Godot*, which is why it has been called a meditation on the Catholic liturgy.

13. Ideas developed by Andreas Dress at the International Symmetry Conference and reflected in this volume. Professor Dress was plenary speaker at the *M. C. Escher: Art and Science Congress* in Rome, 1985.

14. See Gary Kasparov and Raymond Keene, *Batsford Chess Openings* (1982). It is worth noting also that David Hooper and Kenneth Whyld in *The Oxford Companion to Chess* (1984) treat symmetry in part as a theme in composition, giving the example of a symmetrical model mate by Kaminer which won first prize in the 1920 Shakhmaty tourney.

15. See John Nunn, *Secrets of Pawnless Endgames* (1995). The foregoing analysis follows David Norwood's account in his *Daily Telegraph* review, ``The Nunn's story (on zugzwangs - when every move is a bad move)".

16. See also "Female gothic" in Levine, G; and Knoepflmarcher, V. (eds.) *The Endurance of Frankenstein: Essays on Mary Shelley's Novel* (1979). There is a "psychological" as well as a "political" gothic.

17. A relationship demonstrated with examples in a lecture by Carolyn Steedman in the Department of Arts Education, University of Warwick, March 1991.

18. This kind of interpretation is seen as legitimate (almost as a challenge to the autonomy of the text) under "reader response theory", which asserts symmetry between the acts of writing and reading (decoding exactly paralleling encoding). See Jane Tompkins (ed) (1980) *Reader Response Criticism*, and James Kinneary (1983) "The relationship of the whole to the part in interpretation theory and the composing process", *Visible Language* XVII, 2.

19. See for example, Christopher Hussey's *The Picturesque*; Studies in a point of view (1927). There was an attempt to link the landscapes of North Wales and the Lake District emotively and subjectively to sub-alpine aesthetics, a kind of partially tamed version of Burke's sublime.

20. As most matters of format. My University of the South Pacific colleague Liz Todd was instrumental in organising the 1993 *Weaving* exhibition at the Suva Museum and had developed an interest, from an ethnomathematical perspective, on how traditional crafts carry implicit mathematical knowledge. Fijian weaving exhibits complex patterns of partial and broken symmetries.

21. See also Hankin (1925), "The drawing of geometrical patterns in saracenic art" in *Memoirs of the Archaeological Survey of India*, Calcutta. The basic grammar and vocabulary of the patterning is explored in Albarn, K; Smith, J; Steele, S; and Walker, D. (1974).

22. I am aware of remembering an analysis along these lines in a very perceptive quality newspaper review of the exhibition, but unfortunately I do not have the reference. Certainly the account coloured my own viewing of the exhibition.

23. For further exegesis, see Enzo Carli's *Sienese Painting* (1983) p.44.

24. James Gleick (1987) collects a number of specific areas into a general account of Chaos as an interdisciplinary area exhibiting what the cultural anthropologist Clifford Geertz (1983) has called "blurred genres". See also Benoit Mandelbrot's *The Fractal Geometry of Nature* (1982), Peitgen and Richter's *The*

Beauty of Fractals (1986) and Hans Lauwerier's *Fractals* (1991), all dealing with the visual beauty of computer-generated images.

25. In the popular imagination, the Turing test is the ultimate criterion for admitting the arrival of "machines that think", in that their performance on sophisticated tasks cannot be distinguished from those of human beings. The test is named after Alan Turing of Second World War "Bletchey Hall" fame, a locale where chess players and mathematicians were brought together as code-busting cryptographers. See George Atkinson, *Chess and Machine Intuition* (1994). When cybersex passes the Turing test, the pawn's "lust for promotion" will have been fully satisfied.

26. These issues are nicely handled by Jane Prophet (1994) in *Taste Teaching and The Utah Teapot: the Use of Electronic Media in Art and Design Education*. With regard to the apparent authenticity of the image, it is mainly the un-bear-like social psychology of the Coco Cola bears (who are evidently both awed and puzzled by the Northern Lights which they settle down to view as if in a theatre before taking their interval drinks) that persuades us that they are of the digital species *ursor sapiens*, rather than true bears.

27. Although not particularly Elizabethan, comprising in many instance the commonplace speculations of Renaissance humanism and cosmology. See E. M. W. Tillyard, (1943) *The Elizabethan World Picture*.

28. This quest for "generic" features might suggest strong thematic links with traditional Taz'iyeh plays, dealing with the events surrounding the tragedy at Kerbela. See Peter Chelkowski (ed.) (1979) *Taz'iyeh: Ritual and Drama in Iran*, New York, UP.

29. Rosemary Woolf's account of typological interpretation in the *Abraham and Isaac* episode is balanced by that of Martin Stevens (1987) across the cycles as a whole, but with particular reference to N-town as the most overly theological of the cycles.

30. See John Cassian's *Collationes* (XIV, 8, Migne, Vol. 49).

31. See Douglas Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid* (1979). This profoundly serious and playful book is perhaps better reflected in its subtitle, "A Metaphorical Fuge on Mind and Machines in the Spirit of Lewis Carol".

REFERENCES

Albarn, K; Smith, J; Steele, S. and Walker, D. (1974) *The Language of Pattern: an Enquiry Inspired by Islamic Decoration*, London: Thames and Hudson.

Atkinson, G. (1994) *Chess and Machine Intelligence*, Norwood, New Jersey: Ablex Publishing Company.

Auerbach, E. (1957) *Mimesis: the Representation of Reality in Western Literature* [translated by W. Trask], New York: Doubleday.

Beardsley, M. (1966) *Aesthetics from Classical Greece to the Present: a Short History*, New York: Macmillan.

Bethell, S. L. (1948) *Shakespeare and the Popular Dramatic Tradition*, London: Staples Press.

Bootzin, R. Acocella, J. and Alloy, L. (1993) *Abnormal Psychology: Current Perspectives* (Sixth Edition), New York: McGraw Hill.

Bottrall, M., ed. (1975) *Gerard Manley Hopkins: Poems - a Series of Critical Essays, New York:* Macmillan.

Brockett, O. (1969) *The Theatre: An Introduction*, 2nd. ed., New York: Holt, Rinehart and Winston.

Carli, E. (1983) *Sienese Painting*, New York: Scala Books.

Cassirer, E. (1953) *The Philosophy of Symbolic Forms*, 3 Volumes, New Haven: Yale University Press.

Chelkowski, P., ed. (1979) *Taz'iyeh: Ritual and Drama in Iran*, New York: New York University Press.

Cornford, F. (1914) *The Origin of Attic Comedy*, London: E. Arnold.

Devaney, R., ed. (1989) *Chaos and Fractals: The Mathematics Behind the Computer Graphics*, Providence: American Mathematical Society.

Edwards, P., ed. (1967) *Encyclopedia of Philosophy*, New York: Macmillan.

El Said, I., Parman, A. (1976) *Geometric Concepts in Islamic Art*, London: World Islam Festival Publishing Company.

Fish, S. (1980) *Is There a Text in this Class? The Authority of Interpretive Communities*, Cambridge: Harvard University Press.

Fry, N. (1957) *Anatomy of Criticism*, Princeton: Princeton University Press.

Gardner, H. (1982) *Eye, Mind and Brain: a Cognitive Approach to Creativity*, New York: Basic Books.

Geertz, C., (1983) *Local Knowledge*, New York: Basic Books.

Gleick, J. (1987) *Chaos: Making a New Science*, New York: Viking.

Hardy, B. (1975) *Tellers and Listeners: the Narrative Imagination*, London: Athlone Press.

Hassall, W. O., ed. (1954) *The Holkham Bible Picture Book*, (facsimile), London: Dropmore.

Hofstadter, D. (1979) *Gödel, Escher, Bach: an Eternal Golden Braid*, New York: Basic Books.

Hooper, D., Whyld, K. (1984) *The Oxford Companion to Chess*, Oxford: Oxford University Press.

Hussey, C. (1927) *The Picturesque; Studies in a Point of View*, London: G. P. Putnam's Sons.

Kasparov, G., Keene, R. (1982) *Batsford Chess Openings*, London: Batsford.

Kolve, V. (1986) *A Play Called Corpus Christi*, Stanford: Stanford University Press.

Kosco, B. (1993) *Fuzzy Thinking: the New Science of Fuzzy Logic*, New York: Hyperion.

Langer, S. (1942) *Philosophy in a New Key*, Cambridge: Harvard University Press.

Lauwener, H. (1991) *Fractals: Endlessly Repeated Geometric Figures*, Princeton: Princeton University Press.

Levi-Strauss, C. (1963) *Structural Anthropology*, New York: Basic Books.

Levine, G., Knoepfmarcher, V. (eds.) (1979). *The Endurance of Frankenstein: Essays on Mary Shelley's Novel*, Berkeley: University of California Press.

Mandelbrot, B. (1982) *The Fractal Geometry of Nature*, New York: W. H. Freeman.

Moers, E. (1985) *Literary Women*, Oxford: Oxford University Press.

Nunn, J. (1984) *Secrets of Pawnless Endgames*, London: Batsford.

Peitger, H., Richter, P., (eds.) (1986) *The Beauty of Fractals*, Berlin: Springer-Verlag.

Prophet, J. (1994) *Taste, Teaching and the Utah Teapot: The Use of Electronic Media in Art and Design Education* [Ph.D. dissertation], Coventry: Department of Arts Education, University of Warwick.

Rossiter, A. P. (1950) *English Drama from Early Times to the Elizabethans; its Background, Origins and Development*, London: Hutchinson's University Library.

Smith, B. (1982) *European Vision and the South Pacific*, 2nd ed., New Haven: Yale.

Stevens, M. (1987) *Four English Mystery Cycles*, Princeton: Princeton University Press.

Stewart, I., Golubitsky, M. (1993) *Fearful Symmetry*, London: Penguin.

Tent, J. (1993) Phonetic symmetry in sound systems, *Symmetry Culture and Science*, 4, 4

Tillyard, E. M. W. (1943) *The Elizabeth World Picture*, London: Chatto and Windus.

Tompkins, J. (ed.) (1980) *Reader-Response Criticism: from Formalism to Post-structuralism*, Baltimore: Johns Hopkins University Press.

Tucker, P. H. (1989) *Monet in the 90's: The Series Paintings*, New Haven: Yale University Press.

Woolf, R. (1986) *Art and Doctrine: Essays on Medieval Literature*, London: Hambledon Press.

Yeomans, R. (1993) Islam: The abstract expression of spiritual values, in Starkings, D., (ed.) *Religion and the Arts in Education*, Buny St. Edmonds: Hodder and Straughton.

ILLUSTRATIONS

Frontispiece: Sidney Parkinson *Portrait of a New Zealand Man* pen and wash (1769) British Museum Add. Ms, 23920.S4a; and *The Head of a New Zealander*, engraving after Parkinson, in Hawkesworth, *Voyages* (1773) pl.13. Hawkesworth's engraving is derivative, and for self-evident technical reasons ends up virtually as a mirror image (although with slight but interesting differences) of Parkinson's portrait. Noted and commented upon in Bernard Smith (1985).

Figure 1: Welsh flag worn by author as sulu. Photo Media Centre, University of the South Pacific.

Figure 2: Mixtec-Aztec turquoise mosaic pectoral ornament in the form of a double headed serpent from the Museum of Mankind, London (1988) Trustees of the British Museum. Printed by P. J. Graphics Ltd, London.

Figure 3: Romano-British enamelled "dragonesque" brooch, second century AB from the British Museum, London. Length 6.2 *cm*; missing pin. (1992) Trustees of the British Museum. Printed by P. J. Productions Ltd, London.

Figure 4: "Claddyr" ring worn by author in order to give confusing signals in Northern Ireland. Photo Media Centre, University of the South Pacific.

Figure 5: Rorschach inkblot. The authentic inkblots, (to utilise a novel-sounding concept) are subject to copyright and are not available due to the possibility of response-interference. The illustration is of a card of similar design used to discuss projective psychodiagnostic tests in Bootzin, R., Acocella, J., and Alloy, L. (1993), Sixth Edition, p.142.

Figure 6: *Grain Stacks* (End of day; autumn) 1890-91. The Art Institute of Chicago. See Paul Hayes Tucker (1989) *Monet in the 90's: The Series Paintngs*, Yale University Press, plate 21 (cat.18), p.87.

Figure 7: *Last Judgement* from San Gimigniano cathedral.

Figure 8: Johannes de Witt's famous sketch of *The Swan Theatre*. This copy of de Witt's 1596 sketch was made by his friend Arend van Buchell. From Bapst's *Essai sur L'Histoire du Theatre*, Paris, 1893, reproduced in Brockett (1969).

Figure 9: *Reciprocal Zugzwang with Symmetry*. From John Nunn, (1944) *Secrets of Pawnless Endgames*, Batsford, London.

Figure 10: Augustus Earle, *Waterfall in Australia*, canvas 28 x 33 in (c. 1826) Nan Kivell Collection 9, National Library of Australia, Canberra The painting depicts the painter "placing" one of his aboriginal guides who is posed picturesquely before a waterfall. Also in Bernard Smith (1985) *op.cit.*, Plate 157, p.255.

Figure 11. The geometry of Islamic tiles. The illustrations on this page are taken from Plate 1 and Figure 14, and Plate 5 and Figure 16 respectively, of the clear and helpful exposition of the geometry of Islamic decoration offered in Issam El-Said and Ayse Parman, (1976) *Geometric Concepts in Islamic Art*. The first example is from the Masjid-i-Jami, Farumad, Iran (7^{th} Century, Muslim calendar) the second from the Taj Mahal, Agra, India (11^{th} century, Muslim calendar).

Figure 12: Beadwork Pattern from a Charm, Menna, Madagascar. Museum of Mankind, London, cat. 6864. Photograph by P. J. Reproductions Ltd., London.

Figure 13: *The Annunciation* (1344) by Ambrogio Lorenzetti in the Siena Pinacoteca (No.88). Here taken from Plate 63 of Enzo Carli (1983) *Sienese Painting*, Scala Books. Photograph and copyright, Scala, Istituto Fotografico Editoriala, Firenze, Italia

Figure 14: The Mandelbrot set at various scales, Devaney (1989)

Figure 15: Images of the Utah teapot From Jane Prophet's *Taste, Teaching and the Utah Teapot: The Use of Electronic Media in Art and Design Education* (1944). Unpublished Ph.D. thesis, University of Warwick, Figs. 29 and 30. Also in *ACM SIGGRAPH Papers* (1993). The author supervised this dissertation and much of the argument in this section of the paper has been influenced by conversations with Ms Prophet.

Figure 16: Cybersex from *Lawnmower Man* Fig. 17 of Jane Prophet (*op. cit.*). This sequence from the film is in effect a celebration of the cybersex fantasy.

Figure 17: Animated Coco Cola bears. A still from the animation *Polar Bears* by Rhythm and Hues Studios. *ACM SIGGRAPH Papers* (1993).

Figure 18: *The Last Judgement of the Blessed and the Damned* from the *Holkham Bible Picture* Book. The illustration here is a sketch by the author based on Hassall (1954) facsimile copy of the British Museum Manuscript.

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SYMMETRY IN EDUCATION

ESCHER'S WORLD: LEARNING SYMMETRY THROUGH MATHEMATICS AND ART

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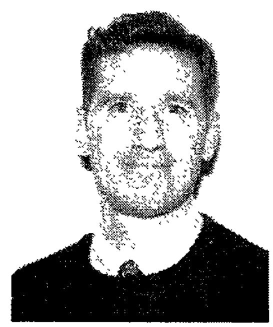


Fields of Interest: Mathematics, Mathematics Education, Computational Media, Design and Design Education.

Publications: Symmetric Intuitions: Dynamic Geometry/Dynamic Art, *Symmetry: Culture and Science*, 6 (1995), 3, 476-479; *Exploring Trigonometry with the Geometer's Sketchpad,* Key Curriculum Press, (1995) 100 pp, Thinking About Two Algebras, Invited paper for National Science Foundation *Algebra for the 21st Century* Conference (1992), Groton, MA, 10 pp.

Abstract: *In recent years, mathematics educators have begun to call for reforms in mathematics learning that emphasize open environments rather than the traditional pedagogy of exposition and drill. This paper explores one example of an open learning environment created by combining mathematics and design activities in a "mathematics studio." The paper looks at: (a) whether students can learn specific mathematics topics in a studio environment, (b) whether learning in such an environment will change the way students solve mathematical problems, and (c) whether learning in such an environment will change students' attitude towards mathematics. Two iterations of the mathematics studio experiment in a project at the MIT Media Laboratory known as Escher's World suggest that: (a) students can learn about the mathematical concept of symmetry in a studio learning environment, (b) students learn to use visual thinking to solve mathematical problems in a studio learning environment, and (c) students develop*

a more positive attitude towards mathematics as a result of working in a studio learning environment.

1 INTRODUCTION

The publication of the National Council of Teachers of Mathematics Standards in 1989 marked the emergence of a coherent reform movement in curriculm (NCTM 1989). Instead of a pedagogy based on exposition by the teacher, reformers propose moving from drill and practice to more open learning environments. They call for the introducion of extended projects, group work, and discussions among students. In many ways, the learning environment these reformers describe seems more similar to a studio course in design or architecture, where students work on extended projects exploring creative solutions to general problems, than to a traditional mathematics class. This paper describes one such mathematics studio, and evaluates the success of this approach in learning some mathematics concepts.

Specifically this paper addresses three aspects of mathematics learning:

- Content knowledge: Can students learn to understand a specific mathematics topic in a design studio?
- Skill acquisition: Will learning in a design studio affect the way students solve mathematics problems?
- Change in attitude: Will students feel differently about mathematics after learning mathematics in a digital studio?

The results suggest that indeed a studio learning environment can be used effectively and profitably in mathematics education.

2 SETTING

This paper reports results from the Escher's World research project at the Massachusetts Institute of Technology Media Laboratory. During the spring and summer of 1995, the project brought high-school students in grades 9 and 10 (age 14-16) from public schools in Boston, Massachusetts to the Media Laboratory. Students attended brief but intensive workshops where they created posters and worked on other design projects using mathematical ideas of mirror and rotational symmetry.

3 METHODS

3.1 Workshop Activities

The Escher's World project ran two workshops for students. In each workshop, six or seven students from Boston public schools came to the Media Lab for twelve hours of workshop activities spread over two or three days. The workshops were divided into two sections, one about mirror symmetry, the other about rotational symmetry. Each section had two sets of activities: investigations and explorations.

3.1.1 Investigations

Investigations lasted approximately one hour. Students worked on a series of short problems relating to symmetry on their own or in small groups, wrote entries in their workshop journals, and discussed their observations with a workshop leader as facilitator. In the first day of the workshop, for example, students began their investigation of mirror symmetry by making name-tags that read normally when viewed in a mirror. This was followed by a search for words that look the same when viewed in a mirror, and from there to the classification of the letters of the alphabet by their mirror lines.

3.1.2 Explorations

Based on their investigations, students spent three to four hours working on extended projects in design on their own or with a partner. Students worked on one shorter project (approximately one hour), and then presented their work to the group for discussion, questions, and comments. Following this "peer review," students began a more ambitious project (approximately two hours), integrating ideas about symmetry, principles of design, and feedback from their presentation. In the first day of the workshop students followed their classification of the alphabet by creating a design of their own choosing that had mirror symmetry. After discussing their designs, students worked for the remainder of the day creating designs that had mirror symmetry but did not place the focus of the composition in the middle of the design (see Figure 1).

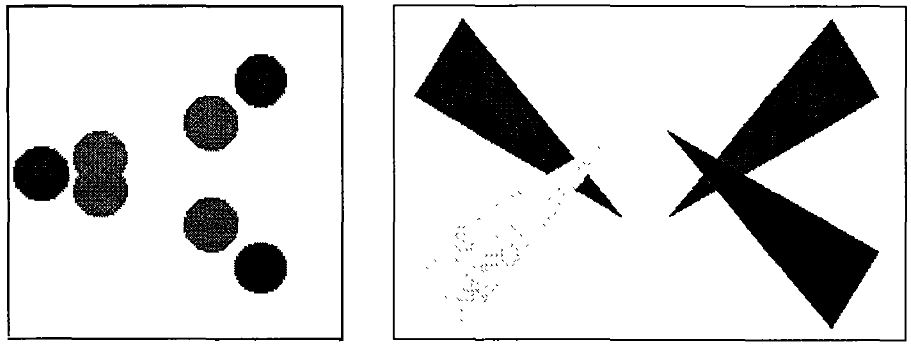


Figure 1: Student work from Escher's World: learning about composition and symmetry

3.2 Workshop Resources

3.2.1 Workshop Space

The workshops took place in a single conference room approximately 15' by 25' that had been modified to resemble an art studio. Works of art by students and professional artists were placed on the walls, and a variety of artistic media available for students' use. In addition to the author, who acted as workshop leader for both workshops, there were one or two other adults in the studio as a resource for students.

3.2.2 Equipment Used

Macintosh computers were available for student use during the workshops, with one computer available for every two or three students. Computers were connected by a network to flatbed scanners, color printers, and a large format color plotter. Computers were equipped with Aldus Superpaint and Adobe Photoshop (commercially-available drawing and image-manipulation programs; Aldus Corp. 1993, Knoll et al. 1993) and with the Geometer's Sketchpad (commercially available educational software for mathematics; Jackiw 1995). During the investigation portion of the workshops, students were introduced to some of the basic functionality of these programs (particularly the Geometer's Sketchpad). Students were able to work on the computers or with traditional materials during their explorations; all of the students chose to use a computer for some portion of their work.

3.3 Data Collection

3.3.1 Kinds of Data Collected

Escher's World uses a qualitative model of research (Geertz 1973, Glesne and Peshkin 1992, Maxwell 1992, Weiss 1994). Qualitative research attempts to understand phenomena by gathering a rich set of data for a limited number of instances. The main source of data for the Escher's World workshops was full, structured pre- and post-interviews conducted with each of the workshop participants immediately before and after the workshop, as well as an additional "affect interview" with each student from two to five months after the completion of the workshop. The format of these interviews and their subsequent analysis was guided by 14 preliminary interviews conducted with students, mathematics and art teachers, and experts in the field of symmetry. Interviews were supported by videotapes of the workshops and field notes from workshop leaders and other facilitators. Student sketches and designs from the workshops and student journals were preserved for review and analysis. Students in the second workshop were also given a brief survey about their feelings towards mathematics, art, and computers immediately before and after the workshop.

3.3.2 Structure of Interviews

The structured pre- and post-interviews were divided into three components. The first component was a series of affect questions about mathematics and art, focusing particularly on attitudes towards these disciplines. The second section of each interview was a detailed discussion of four works of art from a set of 7 images (see Appendix for images). The works of art were reproduced in standard size and format, and students were given three prompts: (1) How would you describe this to someone who had not seen it? (2) Would it be difficult to make something like this? (3) Do you like this piece? The final section of the interviews consisted of two to four mathematics problems from a set of 16 problems (see Appendix for problems). Students were asked to solve the problems, and to describe their thought process as they worked. Affect interviews (conducted two to five months after the workshops) were similar in structure to the first section of pre-and post-workshop interviews. In affect interviews, students described their attitudes towards mathematics and art.

3.3.3 Structure of Survey

The surveys given to students before and after the second workshop asked students to rate how strongly they agreed or disagreed with a series of statements about mathematics, art, and computers (see Appendix for survey questions). Ratings ranged from 5 (agree strongly) to 1 (disagree strongly).

3.4 Data Analysis

3.4.1 Coding of Interviews

Each section of the interviews (affect questions, image descriptions, and word problems) was coded seperately. In order to provide consistency across the interviews, excerpts were mixed randomly before coding, and coding within each secton of the interview questions (affect questions, image descriptions, and word problems) was done by the same person and checked for accuracy.

3.4.1.1 Codes for Image Descriptions

Design texts divide the basics of design education into "elements and principles" (Johnson 1995), where elements generally refer to particular physical portions of the design image and principles refer to formal or systematic relations between elements. Excerpts about images were assigned codes for elements of "form" and "color" and principles of "symmetry" and "composition". These categories were based on preliminary interviews with students, symmetry experts, and mathematics and art teachers, where novices used elements of form and color in their descriptions of images and experts used principles of symmetry and composition.

People tend to go through stages in their development of visual and aesthetic understanding across a range of topics (Parsons 1987). That is, there are stages in their understanding of color, or in their interpretation of forms or composition. Codes for the topics identified in preliminary interviews were subdivided into two stages for the purposes of this analysis: "general" comments and "analytical" comments.

For elements (form and color), general comments referred to excerpts that contained catalogs of shapes or colors ("a lot of circles and lines"). Analytical comments about elements were those that referred to specific relationships between elements of the picture ("the green is too dark for the bright blues and purples"), that distinguished between similar elements based on a specific criteria ("the sun is blue, and then [there is] another sun, smaller—it's red"), or that combined elements into larger descriptive units ("diamond shapes all together combined into like a star").

Similarly, general comments about principles referred to excerpts that contained informal descriptions of formal concepts such as symmetry and composition. For instance, one student's comment that in an Escher print "the heads are all together everywhere you look... so it's like they're standing right beside each other in different [places] " was coded as "symmetry, general. " Analytical comments about principles were those that used formal or mathematical descriptions of symmetry ("it's four time radial symmetric") or composition ("it doesn't really have a focus—or it has multiple focuses").

Thus, the coding matrix for images had 8 cells:

	Elements		Principals	
	Form	Color	Symmetry	Composition
General				
Analytical				

Many excerpts were coded in more than one category. The following description, for example, was coded for "form, analytical," "color, general," and "symmetry, analytical,":

> I noticed these little blue lines coming out of these little red designs, and I realized that it was angular symmetry. It looks like... whoever made this could have just started out with a block that had on two sides the red and blue base, and then on the rest of it just made the yellow, and these blue dots and red dots and the little line right there, and then made part of the circle, or one fourth of those circles—and then... made four versions of it with different angles and... moved them together.

3.4.1.2 Codes for Word Problems

Word problems were coded for students' use of a visual representation during some portion of the problem-solving process (usually some form of sketch of the problem). Following Rieber, the term visual representation was used broadly to refer to "representations of information consisting of spatial, non-arbitrary (i.e., 'picture-like' qualities resembling actual objects or events), and continuous... characteristics," including both internal and external representations (Rieber 1995). Problems were also coded for correct or incorrect answers to the problem, where "correct" answers included answers that fit the stated conditions of the problem even if a student's solution was not the "expected" answer.

3.4.2 Statistical Analysis

When coding was complete, frequencies were tallied for each code. Pre- and post-interview totals were compared overall, between workshops, and for individual students. Student responses for affect interviews were included in the analysis of students' attitudes. The statistical significance of observed changes was computed using a t-test with $n = 12$. Results for the survey from the second workshop (with $n = 6$) were similarly tabulated and analyzed.

4 RESULTS

4.1 Criteria for Understanding

Mathematics education reformers argue that change in mathematics education needs to affect not only the setting in which learning takes places, but also the goals toward which learning is directed. Traditional approaches to education do not place first priority on students understanding mathematical ideas (Brandt 1994, Perkins and Blythe 1994). Tests stress coverage of the "content" of the curriculum, or limited skills acquisition. In contrast, theorists emphasize that "understanding" requires that students develop the ability to use ideas in appropriate contexts, to apply ideas to new situations, to explain ideas, and to extend ideas by finding new examples (Gardner 1991, Gardner 1993, Sierpinska 1994). The goal of the Escher's World project was to help students develop this kind of understanding of mathematics through studio activities.

4.2 Students Learn about Symmetry

4.2.1 Use of Symmetry in Designs

Students were able to use the concept of symmetry to create original designs. During the workshops all of the students (12/12) were able to make designs using mirror symmetry, and 83% of the students (10/12) were able to make designs using rotational symmetry.

4.2.2 Application of Symmetry to Analysis of Images

Students developed their ability to apply the concept of symmetry to the analysis of images. Before the workshop, students made analytical references to symmetry an average of 0.5 times while looking at 4 images in structured interviews. After the workshop, mean analytical references to symmetry rose to 4.3 references over 4 images (see Figure 2, mean change +3.8, $p<0.01$).

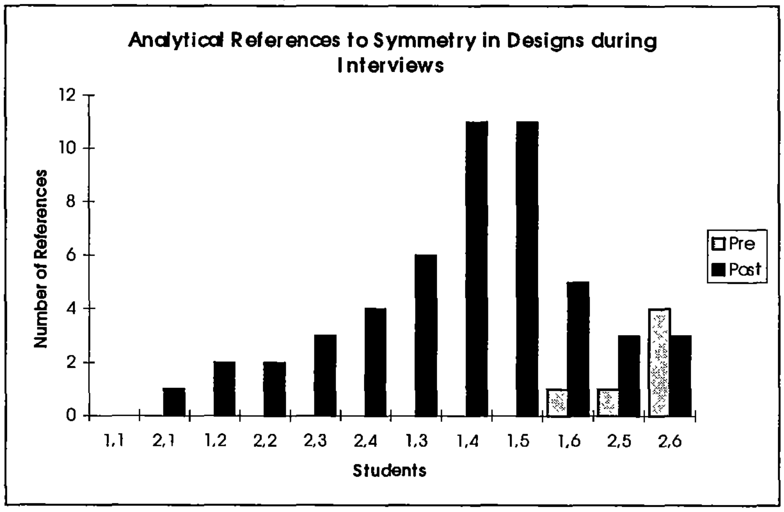


Figure 2· Students learned to use symmetry to analyze images. Students 1.1–1.6 attended the first workshop; students 2.1–2.6 attended the second workshop. Students have been ordered for clarity of presentation.

The small change in the total number of analytical comments (mean change +1.4 references; $p = 0.36$) suggests that students were not becoming more analytical overall; rather, as students began to use the concept of symmetry as a tool for analysis, they replaced some other form of analysis. As shown in Figure 3, change in analytical references to color (mean change -0.8 references; $p = 0.16$) were not statistically significant. Analytical references to composition were too small in pre-interviews (1% overall) to account for the rise in analytical references to symmetry. This suggests that students replaced an analysis in terms of the elements of form with a more mathematical analysis in terms of the principal of symmetry. This hypothesis is supported by the change in percentage of analytical comments made about forms and change in percentage of analytical comments made about symmetry, which have a coefficient of correlation of 0.76 across all students.

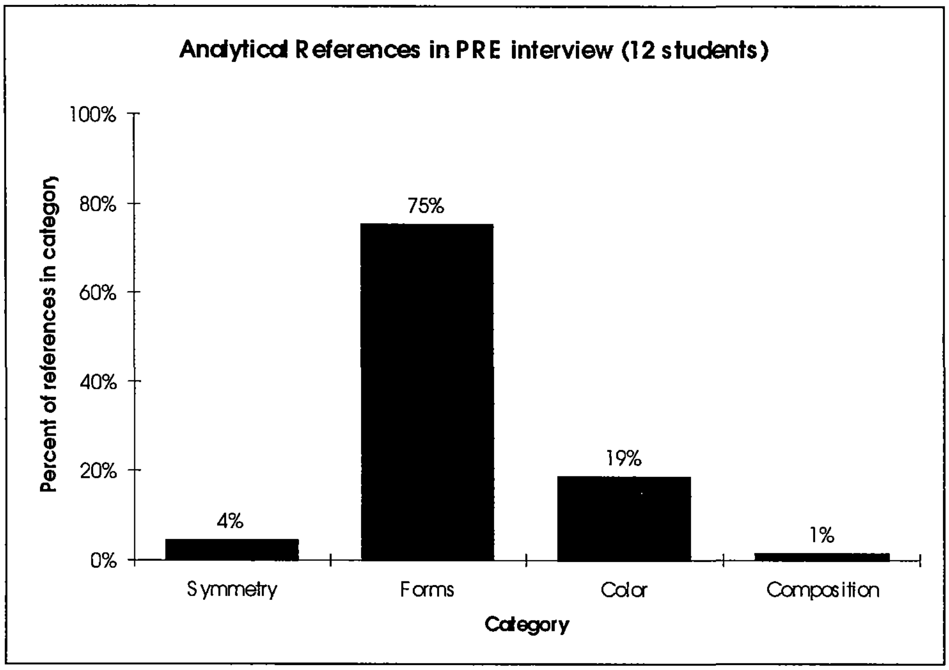
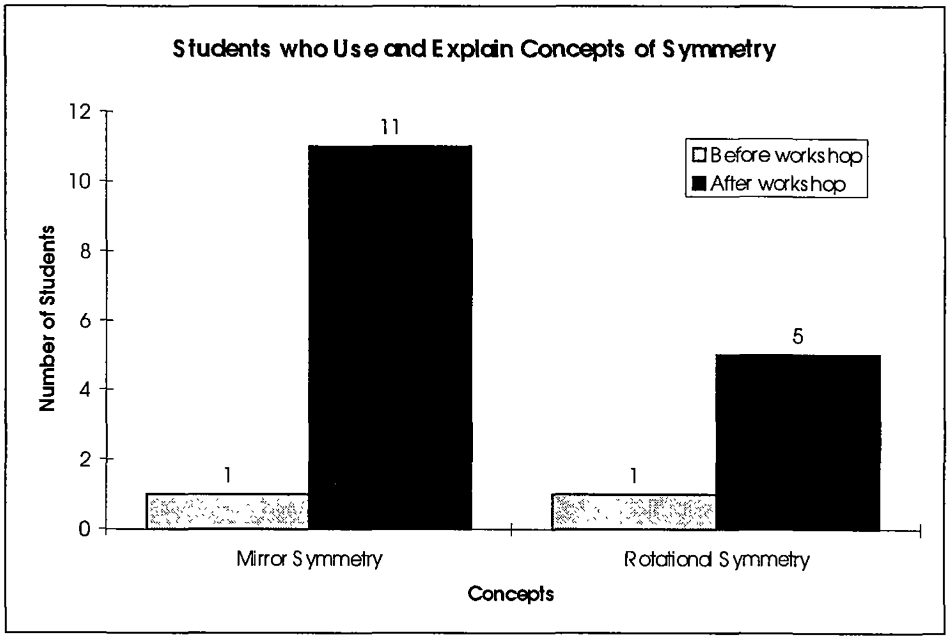


Figure 3: Rise in analytical references to symmetry was related to a drop in analytical references to forms. Graph shows aggregate data for 12 students in 2 workshops. Change in symmetry references is +30%; change in references to forms is –29%.

It should be noted here that students also showed an increase in analytical references to composition, which will be discussed in a later paper about arts learning in Escher's World.

4.2.3 Use and Explanation of the Concept of Symmetry

The number of students who could use and explain formal concepts of symmetry rose dramatically over the course of the workshop (see Figure 4). Some student explanations of symmetry were fairly general descriptions even after completing the workshop, but others are quite specific. For example, one student said: "If you drew a line down the middle—the line of symmetry—the two halves would be identical. They would be exactly the same: mirror images of each other."



4.2.4 Finding New Examples of Symmetry

After the workshop, students started to see symmetry in the world around them: 75% of the students (9/12) reported thinking about symmetry beyond the context of the workshop in post interviews or follow-up interviews. Students reported seeing symmetry in drawings, chairs, wallpaper, rugs, video games, flowers, and clothing.

4.3 Students Learn to Solve Mathematics Problems Visually

4.3.1 Use of Visual Repesentations Shows Mathematical Understanding

The workshop did not deal with mathematics word problems, or explicitly with the use of visual representation as a tool for solving traditional mathematics problems. After the workshop, however, students used visual representations as a successful problem solving strategy. Only 33% of the students (4/12) used visual representations to solve word problems before the workshop, while 75% of students did after the workshop (see Figure 5, $p < 0.06$).

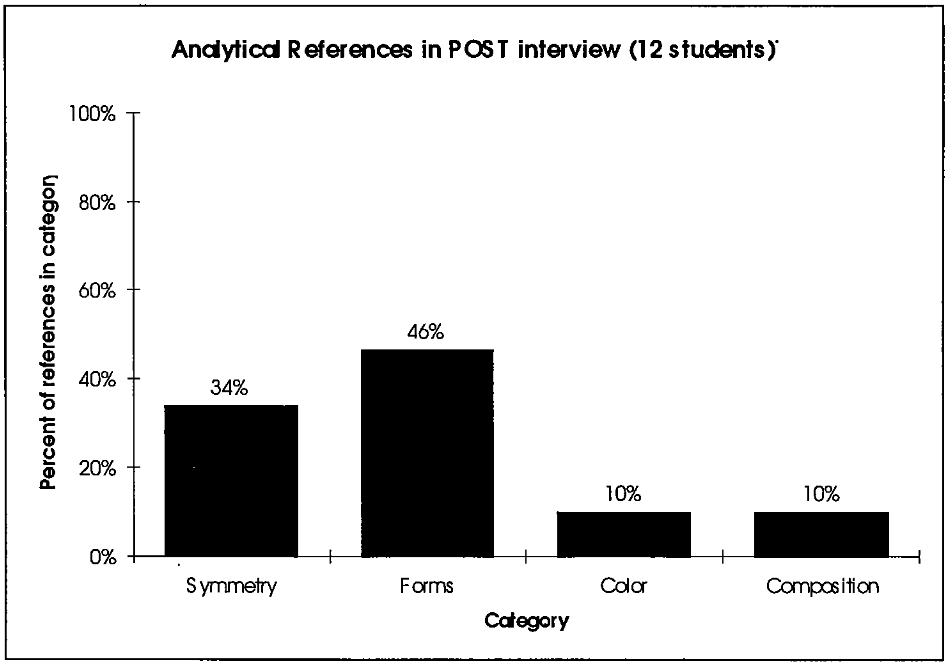
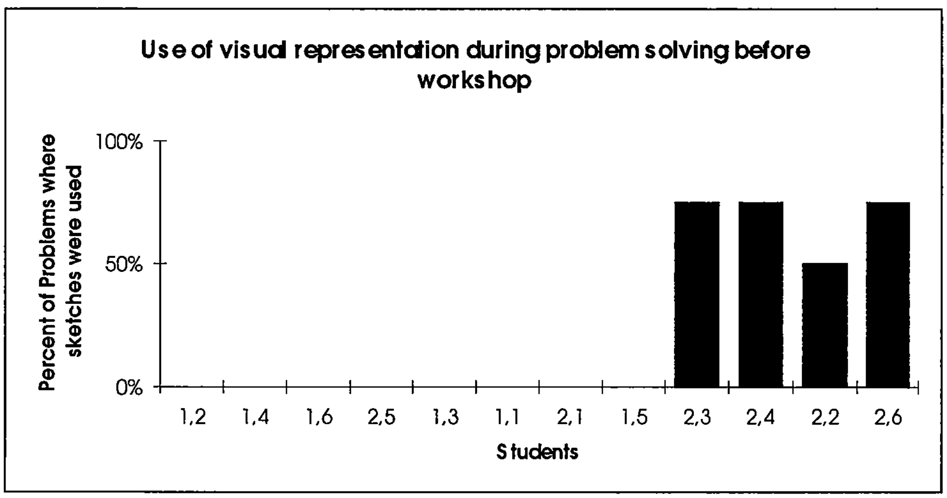


Figure 4: Students learned to use and explain symmetry through design. Graph shows data for 12 students in 2 workshops



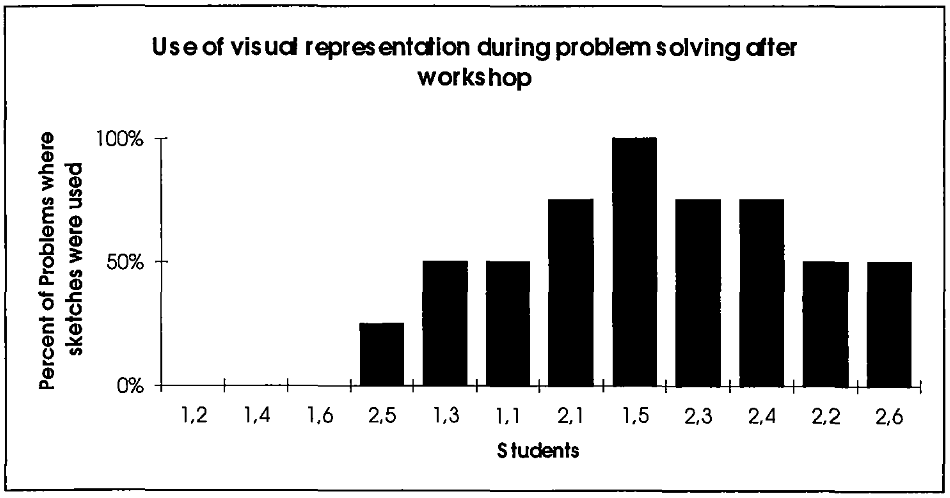


Figure 5: Students learned to use visual representations during problem solving. Students 1.1–1.6 attended the first workshop; students 2.1–2.6 attended the second workshop. Students have been ordered for clarity of presentation.

For example, in Figure 6, the student did not use a visual representation to solve the problem: "One day, Julie decides to go for a walk. She leaves her home and walks for 2 miles due north. Then she turns right and walks for 3 miles due east. After Julie turns right again and walks for another 2 miles, she decides to go home. How far does she have to go to get back to her home?" After the workshop, the same student working on a similar problem used a visual representation of the problem situation.

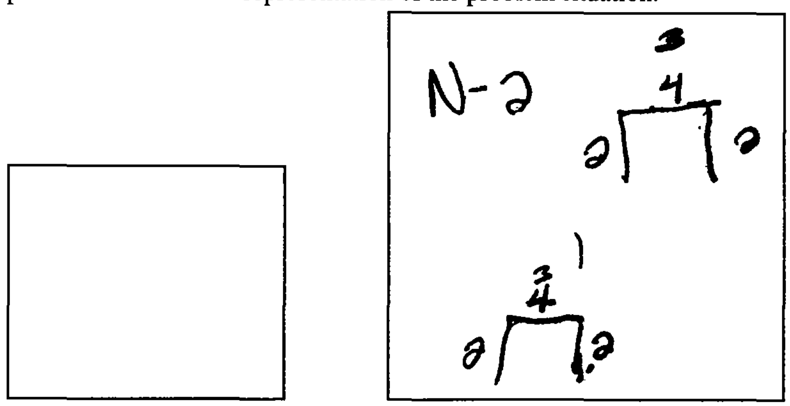


Figure 6: One student's notes while solving a problem during interviews. In the pre-interview (left image), the student did not use a visual representation. While solving a similar problem during her post-interview (right image), the student represented the problem visually andproduced a correct solution.

4.3.2 Visual Representations as a Successful Problem Solving Strategy

Use of visual representations for word problems after the workshop was correlated with success in problem solving during post interview problems (see Figure 7; $r = 0.83$). Some students solved problems without using visual representations, and some students used representations but failed to solve problems; however, no student solved more problems overall than the total number of problems they attempted using visual representations. This is reflected in the absence of data points above and to the left of the dotted line in Figure 7.

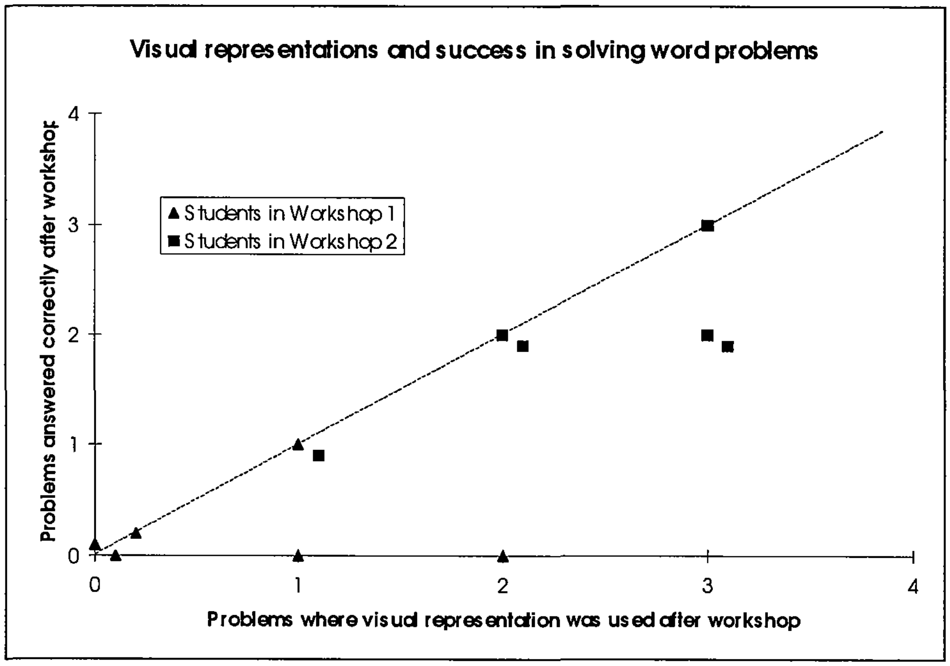


Figure 7: Visual representations helped students solve problems ($r = 0.83$)

4.4 Students Like Mathematics More

In post-interviews and follow-up interviews, 67% of students overall (8/12) reported feeling more positive about mathematics as a result of the workshop. This reported change was supported by data from a written survey. The survey was given to six of the twelve students who participated in the workshops. In the survey, students responded to 4 prompts about mathematics:

"I like math class/I don't like math class."

"I like doing math problems/I don't like doing math problems."

"I like thinking about math/I don't like thinking about math."

"I understand math/I don't understand math."

Students marked a scale from 5 (most positive) to 1 (least positive). As shown in Figure 8, total rating for the 4 mathematics questions went up for two-thirds of students who were given the written survey (4/6). No student's total rating went down from pre- to post-interview survey. Change for the "I like math class/I don't like math class" prompt (mean +0.67, $p < 0.01$) was particularly striking.

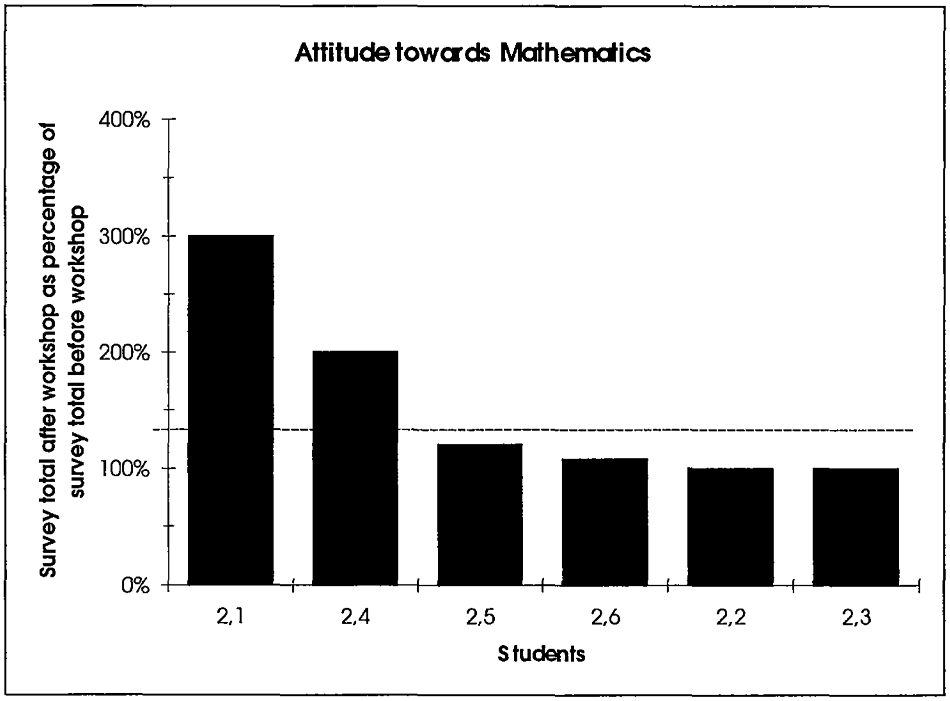


Figure 8: Students felt more positive about mathematics after the workshop. Graph shows data from survey conducted for the second workshop only. Students have been ordered for clarity of presentation.

5 DISCUSSION

5.1 Background Work

5.1.1 Constructionism

The design of the Escher's World workshops was based on the constructionist theory of learning developed by Seymour Papert at the MIT Media Laboratory (Papert 1991a, Papert 1993). There have been a number of attempts since the turn of the century to introduce "learning by doing" into American education (Dewey 1938, Prawat 1995). The learning theory of "constructivism," developed by Jean Piaget and applied to mathematics education in the United States by Les Steffe, Jere Confrey, Paul Cobb, Ernst von Glasersfeld, and others, has been used as a psychological basis for a variety of education reforms in the United States beginning in the 1960s. Constructivism asserts that students must create knowledge for themselves out of their own experiences (von Glasersfeld 1995, Phillips 1995). These two ideas came together in the theory of constructionism, which suggests that building things is a particularly rich context for building understanding. The theory of constructionism has been supported by investigations into the way students learn through the design and construction of real objects and virtual microworlds (Kafai and Harel 1991, Resnick 1991, Resnick and Martin 1991, Resnick and Ocko 1991). By demonstrating that effective mathematics learning takes place in the context of a design studio, Escher's World shows that the visual arts are another potentially effective environment for constructing mathematical understanding by constructing physical and virtual objects.

5.1.2 Visual Art and Mathematics Learning

Escher's World shows that the mathematical concept of symmetry can be explored and learned in an art studio environment. This basic result supports the ideas of numerous theorists who have suggested that learning in traditional academic disciplines can be enhanced or even transformed by the arts (Read 1943, McFee 1961, Arnheim 1969, Field 1970, Silver 1978). In a recent study, for example, Willett showed that mathematics learning was more effective in the context of arts-based lessons than with standard mathematics pedagogy at the elementary school level (Willett 1992). Arthur Loeb's visual mathematics curriculum (Loeb 1993) has not been studied formally, but substantial anecdotal evidence supports his approach to the study of symmetry through a design studio as an effective learning environment for undergraduate students.

5.2 Significant Results of Escher's World

The results reported in this paper extend earlier research in two important dimensions.

5.2.1 A New Mode for Problem Solving

Willet's research established that elementary students can learn mathematics content effectively in an art studio setting. The results of Escher's World support this same conclusion for high-school students, but also show that after exploring mathematical ideas in an art studio setting, students gain access to an additional mode for thinking about mathematics problems. In the Escher's World workshops, learning mathematics in the context of visual arts helped students learn to use visual thinking as part of their mathematics problem solving.

Although a number of researchers argue that visual thinking is related to successful mathematical thinking (Piemonte 1982, Hershkowitz and Markovits 1992), recent work by Campbell et al. suggests that students' visualization ability is not necessarily a factor in their success at solving problems (Campbell et al, 1995). Data from the Escher's World study shows that students' use of visual thinking during problem solving was correlated with their success in solving word problems after the workshop. This suggests that the workshop activities helped students make a more effective connection between visual thinking skills and mathematical problem solving.

5.2.2 Change in Affect

Data from Escher's World also suggests that students' learning of content and skills in an art studio environment is connected to a positive change in attitude toward mathematics. Willett's study of elementary students did not address students' attitude toward mathematics. In another study of the effect of arts activities and mathematics with fourth grade students, Forseth found that students' attitude towards mathematics improved, but that there was no significant improvement in students test scores compared to a control group (Forseth 1976). Results from Escher's World suggest that under the proper circumstances, positive change in students' attitude towards mathematics can be achieved in combination with meaningful changes in the way students approach mathematics problems.

5.3 Limitations of the Study

While the students in the two workshops discussed in this paper showed significant development in their mathematics knowledge, skills, and attitude, it is important to remember that Escher's World represents a very brief intervention. It was necessarily limited in the number of mathematical topics that students could investigate, and in the depth to which students could explore any one topic. It is not clear that the large positive changes seen in this brief but intensive intervention would continue at the same rate over a longer intervention. This fact, combined with the small sample size of the experiment, suggests that some caution should be used in making sweeping claims based on these data.

6 CONCLUSION

The data from Escher's World reported in this paper suggest that a studio setting is productive context for learning mathematics. This result supports other research findings regarding connections between mathematics and the visual arts. However, data from Escher's World goes beyond previous work by suggesting that while learning mathematics in visual arts environment, students not only learn specific mathematical concepts, they also develop the ability to use visual thinking as an effective tool for problem solving.

The limited size and scope of workshop results reported in this paper, combined with the apparent success of the "mathematics studio" as a learning environment suggest that further work should be done to develop the concept of studio mathematics. In particular, it would be useful to have a better understanding of the mechanism by which students develop new modes of mathematical thinking in a studio context. This will be the subject of a future paper on the Escher's World project.

7 REFERENCES

Aldus Corporation (1993) *Aldus Superpaint.* San Diego: Aldus Corporation.

Arnheim, Rudolph (1969) *Visual Thinking.* Berkeley: University of California Press.

Brandt, Ron (1994) On making sense: a conversation with Magdalene Lampert. *Journal of the Association for Supervision and Curriculum Development* 51(5): 26-30.

Campbell, K. Jennifer, Collis, Kevin F., and Watson, Jane M. (1995) Visual processing during mathematical problem solving. *Educational Studies in Mathematics* 28: 177-194.

Dewey, John (1938) *Experience and Education.* New York: Collier Books.

Field, Dick (1970) *Change in Art Education.* London: Routledge & K. Paul.

Forseth, Sonia D. (1976) The effects of art activities on attitudes and achievement in fourth-grade children pertinent to the learning of mathematics and art. Dissertation Abstracts International 37 (12a): 7590-7997.

Gardner, Howard (1991) *The Unschooled Mind.* New York: Basic Books.

Gardner, Howard (1993) Multiple Intelligences: The Theory in Practice. New York, Basic Books.

Geertz, Clifford (1973) The Interpretation of Cultures; Selected Essays. New York, Basic Books.

Glesne, Corrine and Peshkin, Alan (1992) *Becoming Qualitative Researchers.* White Plains, NY: Longman.

Harel, Idit and Papert, Seymour, eds. (1991) *Constructionism.* Norwood, NJ: Ablex Publishing.

Hershkowitz, Rina and Markovits, Zvia (1992) Conquer mathematics concepts by developing visual thinking. *Arithmetic Teacher* 39 (9): 38-41.

Jackiw, Nicholas (1995) *The Geometer's Sketchpad.* Berkeley: Key Curriculum Press.

Johnson, Mia (1995) Elements and principles of design: written in finger Jell-O? *Art Education* 48 (1): 57-61.

Kafai, Yasmin and Harel, Idit (1991) Children learning through consulting: when mathematical ideas, knowledge of programming and design, and playful discourse are intertwined in Harel, Idit and Papert, Seymour, eds. *Constructionism.* Norwood, NJ: Ablex Publishing.

Knoll, Thomas, et al. (1993) *Adobe Photoshop.* Mountain View, CA: Adobe Systems.

Loeb, Arthur (1993) *Concepts and Images: Visual Mathematics.* Boston: Birkhauser.

Maxwell, Joseph A. (1992) Understanding and validity in qualitative research. *Harvard Educational Review* 62 (3): 279-300.

McFee, June King (1961) *Preparation for Art.* San Francisco: Wadsworth Pub. Co.

National Council of Teachers of Mathematics [NCTM], Commission on Standards for School Mathematics. (1989) *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Papert, Seymour (1991a) Situating constructionism, in Harel, Idit and Papert, Seymour, eds. *Constructionism.* Norwood, NJ: Ablex Publishing.

Papert, Seymour (1991b) Perestroika and epistemological politics, in Harel, Idit and Papert, Seymour, eds. *Constructionism.* Norwood, NJ: Ablex Publishing.

Papert, Seymour (1993) The Children's Machine: Rethinking School in the Age of the Computer. New York: Basic Books.

Parsons, Michael J. (1987) How We Understand Art: A Cognitive Developmental Account of Aesthetic Experience. Cambridge: Cambridge Univ. Press.

Perkins, David and Blythe, Tina (1994) Putting understanding up front. *Educational Leadership* 51 (5): 4-7.

Phillips, D.C. (1995) The good, the bad, and the ugly: the many faces of constructivism. *Educational Researcher* 24 (7): 5-12.

Piemonte, Charles (1982) Mathematics: a visual experience. *Curriculum Review* 21 (4): 413-15.

Prawat, Richard S. (1995) Misreading Dewey: reform, projects and the language game. *Educational Researcher* 24 (7): 13-22.

Read, Herbert Edward, Sir (1943) *Education through Art.* London: Faber and Faber.

Resnick, Mitchel (1991) Overcoming the centralized mindset: towards an understanding of emergent phenomena, in Harel, Idit and Papert, Seymour, eds. *Constructionism.* Norwood, NJ: Ablex Publishing.

Resnick, Mitchel and Martin, Fred (1991) Children and Artificial Life, in Harel, Idit and Papert, Seymour, eds. *Constructionism.* Norwood, NJ: Ablex Publishing.

Resnick, Mitchel and Ocko, Stephen (1991) LEGO/Logo: Learning through and about design, in Harel, Idit and Papert, Seymour, eds. Constructionism. Norwood, NJ: Ablex Publishing.

Rieber, Lloyd P. (1995) A historical review of visualization in human cognition. *Educational Technology Research and Development* 43 (1): 45-56.

Sierpinska, Anna (1994) *Understanding in Mathematics.* London: Falmer Press.

Silver, Rawley A. (1978) *Developing Cognitive and Creative Skills through Art.* Baltimore: University Park Press.

von Glasersfeld, Ernst, ed. (1995) *Radical Constructivism: A Way of Knowing and Learning.* Bristol, PA: Falmer Press.

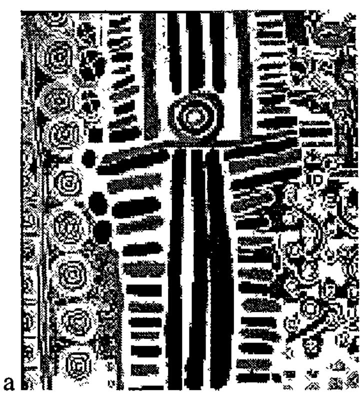
Weiss, Robert S. (1994) Learning From Strangers: The Art and Method of Qualitative Interview Studies. New York: The Free Press.

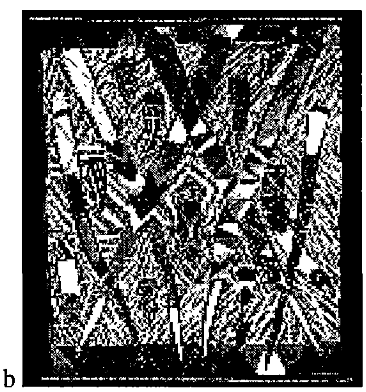
Willett, Leslie V. (1992) The efficacy of using the visual arts to teach math and reading concepts. Paper presented at the 73rd Annual Meeting of the American Educational Research Association, San Francisco, CA.

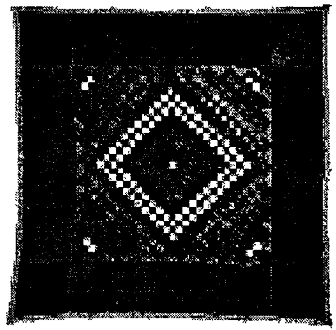
8 APPENDIX

8.1 List of images used in interviews

Students were shown four images in both pre and post interviews. One image was chosen from each of the sets 1-4 below. Note that set 3 contains only one image; all students were shown the same image in both pre and post interviews for this set. All images were resized to 5 inch width and reproduced in color.

1a 

1b 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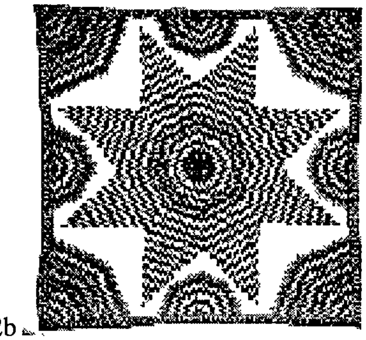
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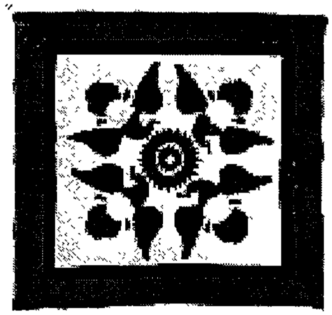
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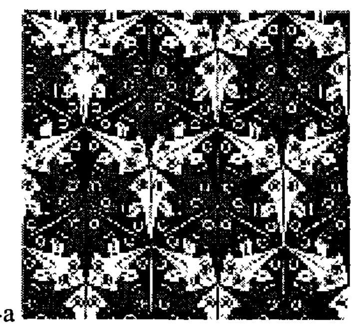
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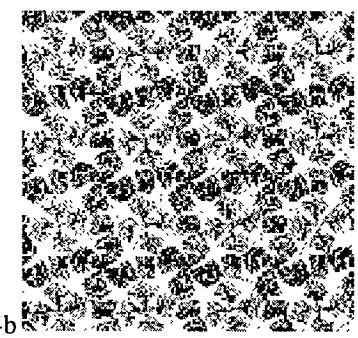
4b

8.2 List of problems used in interviews

During pre and post interviews, students from the first workshop were given two out of four problems 1.1-1.4 below. Students from the second workshop were given 4 problems, one each from sets 2.1–2.4 below. Students were given the problems one at a time, each typed on a separate sheet of paper. Students had unlimited time to work on each problem and were provided with a pad of lined paper and a pen or pencil.

Problem 1.1

Ms. Jones has to 25 pencils and 10 pens to give to her students. She gives each student the same number of pencils. She gives each student the same number of pens. At most how many students does Ms. Jones have?

Problem 1.2

Bob and Tanecka each have a 12-inch pizza pie. Bob cuts his pizza into 8 pieces. Tanecka cuts her pizza into 6 pieces. If you put the pizza's one on top of the other, at most how many cuts in Bob's pizza would be in the same place a cut in Tanecka's pizza?

Problem 1.3

Sally and Juan each have a 14-foot ladder. Sally's ladder has 21 rungs on it. Juan's ladder has 15 rungs on it. If you put the two ladders side by side, how many rungs would be in the same place?

Problem 1.4

A group of students has 12 apples and 15 oranges. They share the apples and oranges so that each students has the same number of whole apples and whole oranges as every other student has. At most how many students could there be in the group?

Problem 2.1a

One day, Julie decides to go for a walk. She leaves her home and walks for 2 miles due north. Then she turns right and walks for 3 miles due east. After Julie turns right again and walks for another 2 miles, she decides to go home. How far does she have to go to get back to her home?

Problem 2.1b

One day, Julie decides to go for a walk. She leaves her home and walks for 2 miles due north. Then she turns right and walks for 4 miles due east. Julie then realizes that she dropped her watch 1 mile back. She turns around and walks until she reaches her watch. After she picks up her watch, Julie turns right again and walks for another 2 miles due south. Now Julie wants to go home. How far does she have to go to get back to her home?

Problem 2.1c

A leaf falls and lands 5 yards east of the tree it was on. A boy picks up the leaf, and walks 10 yards north. The boy sees a swing set on his left, 15 yards to the west. He runs to the swings, but drops the leaf 10 yards before he reaches the swings. How far is the leaf from its tree after the boy drops it?

Problem 2.2a

One border of Theo's backyard is 20 yards long. Theo wants to put up a fence along this border. If he puts up one fencepost every 5 yards, how many posts does Theo need?

Problem 2.2b

One border of Theo's backyard is 20 yards long. Theo wants to put up a fence along this border. If he puts up one fencepost every 5 yards, how many posts does Theo need? Problem Number 2 Theo wants to put up fences along two borders of his backyard. One border is 15 yards long, the other is 20 yards. If he puts up one fencepost every 5 yards, how many posts does Theo need?

Problem 2.2b

Yolanda has a pipe that is 8 feet long. She needs to cut the whole pipe into pieces that are 2 feet long. How many cuts does she need to make?

Problem 2.3a

A snail is stuck on the inside wall of a well, 5 feet down from the top of the well. It moves 3 feet up the wall every day. But every night, the snail slips 1 foot down the wall. After how many days will the snail reach the top of the well?

Problem 2.3b

A snail is stuck on the inside wall of a well, 5 feet down from the top of the well. It moves 2 feet up the wall every day. But every night, the snail slips 1 foot down the wall. After how many days will the snail reach the top of the well?

Problem 2.3c

Yanni goes to his favorite restaurant and orders an 12 ounce soda. Every time he finishes drinking 4 ounces of his soda, a waiter pours 1 more ounce of soda into his cup. How many times will the waiter pour soda into Yanni's cup before Yanni completely empties his cup?

Problem 2.4a

Leo asks Luanda if she will lend him money for a hamburger. Luanda says, "But you borrowed money for a hamburger from me yesterday! And you still owe me 1 dollar for the soda you bought last week!" So Leo says, "Well, okay – lend me money again today, and I'll owe you 4 dollars all together." How much does a hamburger cost?

Problem 2.4b

Leo asks Luanda if she will lend him money for a hamburger. Luanda says, "But you borrowed money for a hamburger from me yesterday! And you still owe me 80 cents for the soda you bought last week!" So Leo says, "Well, okay – lend me money again today, and I'll owe you 4 dollars and 30 cents all together." How much does a hamburger cost?

Problem 2.4c

Two years ago, Janelle was four times as old as Sangita. If Janelle is twenty years old now, how old is Sangita now?

8.3 List of survey questions used in interviews

Students were given the following "sample question":

For each set of statements, circle the number that represents how you feel.
<u>Example:</u>
I like talking on the phone. 5 4 3 2 1 I don't like talking on the phone.

If you really like talking on the phone, circle 5. If you like it most of the time, circle 4. If you like it half of the time and don't like it the other half, circle 3. If you don't like it most of the time, circle 2. If you really don't like talking on the phone, circle 1.

The survey consisted of 15 questions, all in the same format as the example:

1. I like school./I don't like school.
2. I like math class./I don't like math class.
3. I like doing math problems./I don't like doing math problems.
4. I like thinking about math./I don't like thinking about math.
5. I understand math./I don't understand math.
6. I like to make art./I don't like to make art.
7. I like looking at art./I don't like looking at art.
8. I like thinking about art./I don't like thinking about art.
9. I understand art./I don't understand art.
10. I like computers./I don't like computers.
11. I understand computers./I don't understand computers.
12. I learn a lot from listening to the teacher./I don't learn a lot from listening to the teacher.
13. I learn a lot from working on a computer./I don't learn a lot from working on a computer.
14. I learn a lot from working with other students./I don't learn a lot from working with other students.
15. I learn a lot from working by myself./I don't learn a lot from working by myself.

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MATHEMATICAL SYMMETRY: A MATHEMATICS COURSE OF THE IMAGINATION

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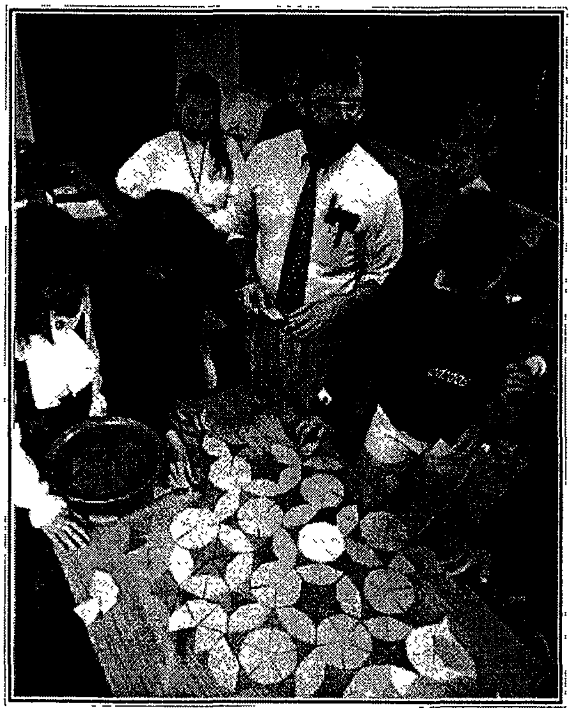
Fields of interest: Combinatorial group theory and topology, Euclidean and non-Euclidean geometry, Mathematical symmetry, Mathematical art, Creating mathematical pottery, Designing and teaching mathematics courses and art courses which involve connections between mathematics and art.

Publications: The swap conjecture, *The Rocky Mountain Journal of Mathematics*, 23, 3, Summer (1992), (with Edward C. Turner); Tessellations with hyperbolic squares, *The Journal of Recreational Mathematics*, 27, 2, (1995). Small study groups in the classroom, *The Canisius College Teaching Quarterly*, 1 1, September (1992), Constructing tessellations and creating hyperbolic art, *Symmetry: Culture and Science*, 3, 4, 367-383, (1992).



Members of the symmetry class at Canisius College, fall semester 1992, discussing a project involving Penrose's nonperiodic "kite and dart" tiles. (Left to right) M. Kearns, F. Bochynski, E. Caldero, J. Trost, Ray Tennant, D. Sherman, J. Krutz, and M. Henry.

Abstract: *Symmetry forms a natural bridge between the worlds of mathematics and art. It is this connection combined with the creative imagination of students that forms the basis for the mathematical symmetry course that is described below.*

THE COURSE - MATHEMATICAL SYMMETRY: A MATHEMATICS COURSE OF THE IMAGINATION

In his classic, *Symmetry*, Hermann Weyl describes symmetry as the "harmony of proportions". It is this notion that is the common thread running through the university mathematics course titled *Mathematical Symmetry: Connections between Mathematics and Art*.

A main perspective of the course holds that students have strong visual senses and that they may be introduced more easily to complex ideas by appealing to their geometric intuition. The students explore about ten major topics, including tiling theory, geometry in nature, group theory, non-Euclidean geometry, 3-dimensional tessellations, and fractals This is done with the aid of straightedge and compass constructions, computer programs, polyhedral models, and examples drawn from the worlds of art and architecture. Each student creates a planar design project and writes a thesis paper involving the imagination. Earlier versions of the symmetry course have been popular with fine arts majors and prospective math teachers as well as students from other disciplines such as psychology, music, history, and engineering. In fact, the classes have been most enjoyable when the students who are participating are drawn from a variety of disciplines.

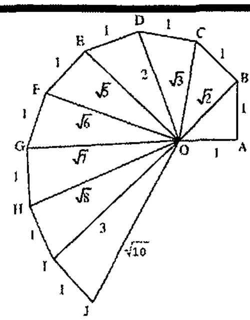
Goal of the course

The symmetry course covers a number of interesting topics from the world of mathematical symmetry. The class is encouraged to explore geometrical ideas using constructions, computer programs, models, and a variety of other investigative techniques. The spirit of the course is based on the Chinese proverb:

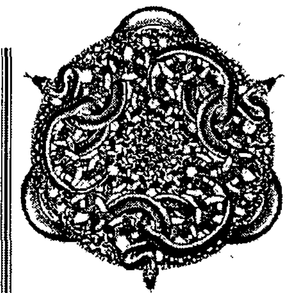
I hear and it helps a little.
I see and ideas begin to form.
I do and ideas become real to me.
I do, see, and hear and I understand.
I talk about it and I understand more.
I apply it and I see it's value.

Picture Syllabus - Mathematical Symmetry

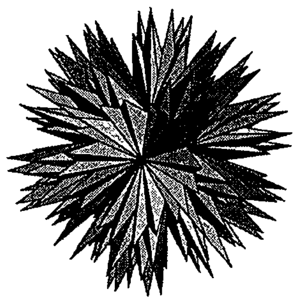
Nature



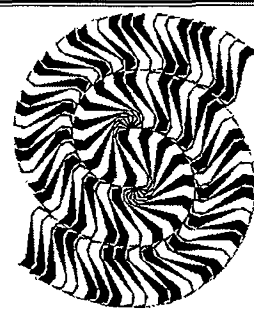
Constructions



Symmetry Groups

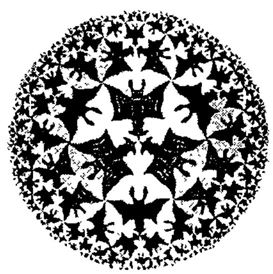


3-Dimensional



Tiling Theory

4-Dimensional



Non-Euclidean Geometry

Fractal Geometry

Topology

Flow of topics

I. Geometry in Nature & Architecture
- The Golden Ratio & Fibonacci Numbers
- Phyllotaxis

II. Geometric Constructions
- Straightedge and Compass Constructions
- Pythagorean Constructions
- Polygon & Star Polygon Constructions
- Geoboard Constructions
- Impossible Constructions

III. Symmetry Groups
- Isometries & Groups of Motions
- Symmetry Group of a Pattern
- Isomorphism of Symmetry Groups
- Cyclic & Dihedral Groups, Group Tables
- Design in Hubcaps & Logos
- Leonardo's Theorem, Frieze Groups

IV. 3-Dimensional Symmetry
- Platonic Solids & Semiregular Polyhedra
- Euler's Formula & Schleigel Diagrams
- Symmetry Motions in Space

V. Tiling Theory
- Regular & Semiregular Tessellations
- Transforming Tilings into "Escher Designs"
- Crystallographic Restriction
- Wallpaper Groups
- Nonperiodic & Aperiodic Tilings

VI. 4-Dimensional Symmetry
- Flatland
- Salvador Dali's "Corpus Hypercubus"
- Hypercubes & N-Dimensional Symmetry

VII. Non-Euclidean Geometry
- The Sphere & the Poincare Disk
- Escher's "Circle Limit IV"
- Hyperbolic Tessellations

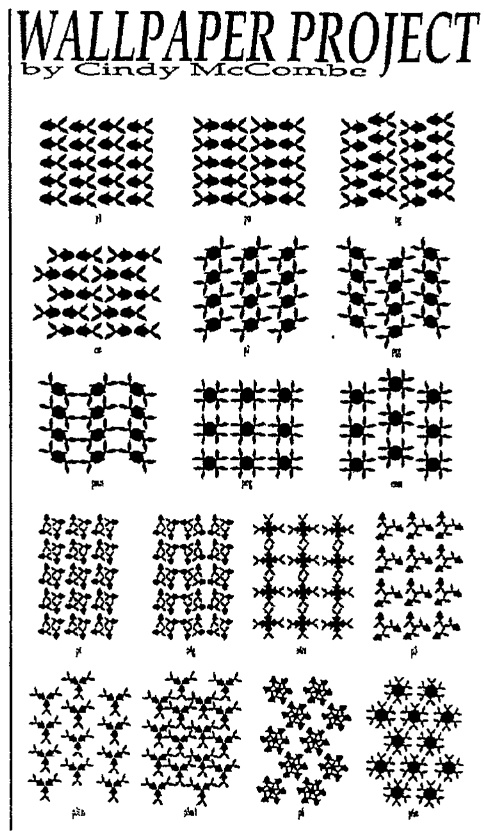
VIII. Fractal Geometry
Von Koch's Snowflake
Computer Generated Fractals
IX. Elementary Topology & Graph Theory Topics
Surfaces & Nonoriented Surfaces
Konigsberg Bridge Problem, 4 Color Problem
X. Geometric Perspective in Art
Albrecht Dürer's "St. Jerome in His Cell"

The planar design project

For many students in the symmetry course, the 2-dimensional design project has proved to be an exciting and challenging endeavor. Students are free to choose a project from the various topics in the course, e.g. tiling theory, wallpaper groups, color groups, fractals, hyperbolic tessellations, etc. As it develops, each student discusses their project with the class.

Past design projects have included:

* Islamic Tiling Constructions
* Interlocking Ceramic Wallhangings
* A Study of Symmetric Music
* Creating Fractals with Mathematica
* Patterns for the 17 Wallpaper Groups, Using NFL Team Logos
* Color Groups in the Plane
* Straightedge & Compass Constructions for High School Teachers
* Poincaré Disk Constructions
* Symmetry Groups of Molecules
* Needlepoint Crystallographic Groups
* Buddhist Mandala Designs
* Symmetry in Flag Design



The design on the right (less than half its original size) was created by Cindy McCombe in a symmetry class at the University of Southern Colorado in 1994. It shows fish patterns for the 17 wallpaper groups and was created using Adobe Illustrator.

The written project of the imagination

Required reading

Flatland: A Romance of Many Dimensions by Edwin A. Abbott, 1884.

Other possible reading (discuss with instructor)

Sphereland, a Fantasy about Curved Spaces and an Expanding Universe by Dionys Burger and Cornelie Reinboldt, 1965.

Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions by Thomas Banchoff, 1990.

Hyperspace: A Scientific Odyssey through Parallel Univrerses, Time Warps, and the 10th Dimension by Michio Kaku, 1994.

The Mathematical Tourist: Snapshots of Modern Mathematics by Ivars Petersen, 1990.

Goal of the Assignment

To use writing and the imagination as a way of gaining insight into an abstract land in the world of mathematics (e.g., 4-dimensionland, or even n-dimensionland, the land of fractals, non-Euclidean worlds, topology landscapes, and others.

The research for the paper may involve reading, discussion, the study of a painting, a computer investigation or some other method of discovery. The student will then use imagination to describe an aspect of their abstract world or to create a fictitious story which extends their new world in some way.

Assignment Steps

1. All students read Flatland and are given written assignments to describe aspects of Flatland and then to use their imagination to extend Flatland to areas which are not covered in the book.

2. Each student chooses an abstract world that they wish to investigate.

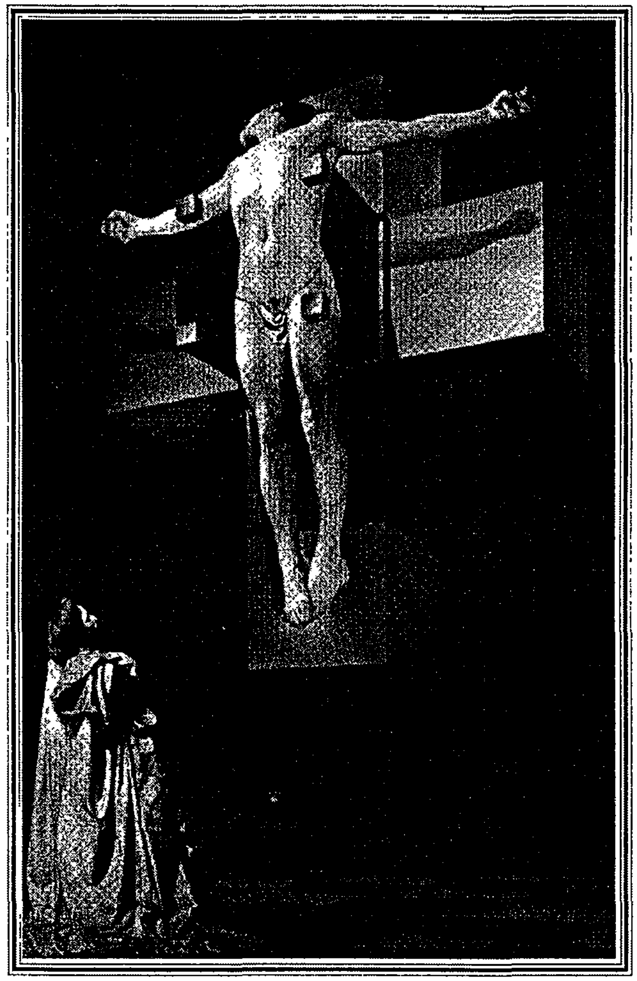
3. Each student finds a resource (possibly a book, a painting, a computer program, etc.) as the motivating thesis for their paper.

4. Through discussions and rewrites, each student writes a thesis paper about their abstract land. The paper can range anywhere from being scientific to being pure fantasy.

5. The writing projects evolve over the semester and students discuss their progress during class.

Some writing projects of the imagination from past symmetry classes are listed below.

* A Flatland Person Discovers "Up" or "Upward not Northward".
* Police and the Law in Flatland.
* Salvadore Dali's "The Crucifixion", subtitled "Corpus Hypercubus" and Hyperspace.
* Salvadore Dali's "Clocks", and Mappings in Topology.
* Can 4D-man turn 3-D man inside out?
* The Poincare Pool and Hyperbolic Fish.
* Picasso, Cubism, and the 4th Dimension.
* Fractal Dimension: Between Lines and Planes.
* Magic Squares and Polyhedra in Albrecht Dürer's "Melancholia"



Salvadore Dali's *Corpus Hypercubus* 1954

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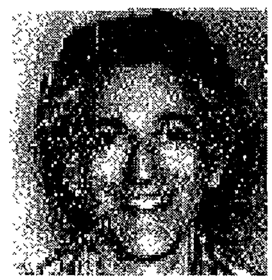
SEEING THROUGH SYMMETRY - AS SEEN THROUGH ITS LABS

Laurence I. Gould



Physicist

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Fields of interest: Mathematical physics, science education, interdisciplinary symmetry.

Publications: (a) What Is Symmetry That Educators and Students Should Be Mindful Of It?, Chapter for the book, *Interdisciplinary General Education: Questioning Outside the Lines*, Seabury, M. B., ed., NY: College Entrance Examination Board (1999). (b) Faraday's Legacy: The Joys of Scientific Methodology, Guest Editorial for *QUANTUM* (November/December 1998), pp. 2-3. (c) Reflections of the Relevance of Nonlocality to Cognitive Science and the Philosophy of Mind, from *New Directions in Cognitive Science*, Proceedings of the International Symposium (Saariselka, 4-9 August 1995, Lapland, Finland). (d) Nonlocal Conserved Quantities, Balance Laws, and Equations of Motion, *International Journal of Theoretical Physics* 28, 335 - 363 (1989) (e) Audio-Visual Aids in Materials Science and Engineering: A Current Overview, *Journal of Materials Education* 11, 169 - 180 (1989).

Abstract: *Seeing Through Symmetry is a one-semester course that has been created for non-science majors. Here symmetry is utilized as what could be called a "hub concept". It is as if it stood for the axis of a cylinder consisting of a multidisciplinary world with ties to disciplines constituting the surface, and with the disciplines, as a consequence, tied to each other. Through this hub many aspects of the scientific and artistic worlds can be better understood and appreciated. In this paper that is explained primarily through the description of a highly graphical laboratory experience. The labs utilize computers in order to explore the many facets of symmetry (in such areas as geometry, the arts, biology, music, and physics) and enable students to create patterns through the use of hardware/software packages and elementary programming.*

1 INTRODUCTION

Seeing Through Symmetry is a one-semester course that has, on the average, run every fall and spring for the past six years. I have created this course in order to enable students explicitly to develop their quantitative abilities and analogical thinking by using concepts of symmetry both as a method within a discipline and as a bridge between disciplines. Starting with topics of symmetry in the ancient world, we then go on to broaden, refine (through a precise definition of "symmetry"), and interrelate those topics to the symmetry in areas such as art, poetry, music, mathematics, physics, chemistry, biology, and cosmology. Thus students "see through" (i.e., understand) the concept of symmetry as well as "see the world" *through* (i.e., by means of employing) the concept of symmetry. They develop interrelational and scientific abilities, in part, through the medium of a highly graphical laboratory experience; this utilizes computers in order to explore the many facets of symmetry (in such areas as geometry, the arts, biology, music, and physics) including the generation of their own patterns through the use of hardware/software packages and elementary programming. [There are, as the readers of this journal know, innumerable references for symmetry in education. A particularly good description of symmetry's value for education in general and for science education in particular is given by (Klein, 1990, p. 86).]

2 LECTURES

These are multimedia interactions: Two films are shown: One of these is an easily understood general introduction to a variety of symmetries (Robinson, 1970); the other stresses the glories of learning as seen through the mind of the prototype "Renaissance man", Leonardo da Vinci (Bobker, 1992). An audiotape is employed to demonstrate the use of the Golden Ratio in Bartok's "Divertimento for Strings". A laser is used to exhibit both the wave interference phenomenon called "diffraction" and the manner in which light waves can interfere to produce a hologram. A special feature of the course is a computer-animation-and-sound show that illustrates symmetry in art, geometry, geophysics, and both cellular and molecular biology (including the dynamically "broken" symmetry through a portion of a motion picture showing the "sickling" of a circular red blood cell into one shaped like a crescent moon, a signature of the disease called "sickle cell anemia").

Over the years the subject matter of the course has been broadened. Some of the more recent topics which Seeing Through Symmetry includes are: the structure and function of DNA, the nature of the chemical bond through explanations of electric forces, scientific bases for common uses of electricity, an introduction to the mathematical theory of groups with applications to computer algorithms, the application of waves to some aspects of the strange world of quantum physics, and the very important issue of scientific methodology (as exhibited in part through the use of Venn diagrams to illustrate class inclusions and methods of concept formation).

We were able to outfit a multimedia classroom, in part as a result of a 1993 NSF Instrument and Laboratory Improvement grant (No. DUE-9352670) for the course. The room contains 12 nodes, each with a Mac computer (plus desk lamp) in a space that easily accommodates two students, an instructor node with another Mac tied to a projection system at the front center of the classroom, and a laser printer on the opposite side of the room from this node; all are connected via an Ethernet network. There is also a VCR player connected to a large monitor at the front right side of the room, a non-chalk board (which has its own lighting) at the front center, and an overhead projector at the front left side. The instructor node permits access to each of the student nodes (e.g., for loading software or transferring files). The equipment is completed by a variety of hardware and software packages used both in teaching and in the laboratory portion of the course.

3 LABS

The laboratory experience gives the student a more complete sense of what is discussed in lecture. The labs, meeting for two hours once each week, have extensive student interactions with discussion being continually encouraged among the participants. Although students work in pairs, an individual typed report is required for each lab. Earlier labs feed into later ones as lower-level abstractions feed into higher-level ones. The labs are briefly described below in relation to the question: What is the activity and how does it connect to other areas of the course so as to convey the interdisciplinary experience? (Subjects referred to are covered earlier during lecture.)

3.1 Computer Drawing: Reflections, Rotations, and Designs

The object of this lab is to: (a) introduce students to the computer, and (b) enable participants to create and understand designs that have reflectional or rotational symmetry. Students can test for those symmetries by using the software package to compare an "image" of the design with the design itself. They can also do a comparison by getting a printout of their design on which can be laid a transparency for tracing the design and then folding or rotating that tracing. In addition they can create their own colored designs using a variety of reflectional and rotational symmetries. In this way the student sees a relation between symmetry and art.

3.2 Learning Algorithms through the Language of Logo

The object of this lab is to enable students to learn about a computer algorithm. What they see is how a repetitive command, or "algorithm", can be used to create certain simple figures. Hence they experience "time-translational" symmetry which they can, for example, interrelate to rhyme schemes of poetry.

3.3 Drawing and Hearing Patterns: Polygonal Symmetry and Fibonnacci Tones

In this lab (a continuation of 3.2) the student: (a) employs algorithms (via Logo) to simplify the creation of regular polygons that have both reflectional and rotational symmetry, and (b) uses the Fibonnacci sequence, an example of an algorithm discussed in lectures, to hear tones and thus learn more about the musical scale. Hence students can see connections between music, design, and computers; for example, by using the concept of an algorithm one may create certain musical patterns.

3.4 Intricacies of Coloration: Coordinates, Translations, Groups and Tessellations

In this lab (a continuation of 3.3) the student sees: (a) how coordinates of points can be represented by the computer, a prelude to (b) how different colors are determining factors for the nature of translationally symmetric designs, (c) an instance from the mathematical theory of groups, and (d) how algorithms can be created to tessellate the screen. Hence students learn (or re-learn) the analytical-geometry basics of locating objects in space; see how this is related to design, to poetry, and to music through the creation of symmetries and "broken" symmetries; understand how art can be a manifestation of group theory in mathematics and how the latter can be used to create art; and glimpse "infinity" through symmetries that can go on and on through time and space.

3.5 Patterns in Music: Sound and Sight

This lab takes place at our School of Music (each pair of students is in a small room containing a piano). Its purpose is to serve as a literal "hands-on" introduction to elementary ideas in music and music symmetry through exploration of the visual and aural aspects of the keyboard. It does this by: (a) asking students to learn about symmetrical patterns that can be associated with the keyboard, both visual and (what I would call) "aural temporal"; (b) teaching them to relate the psychophysical concepts of pitch and frequency to each other and to the Fibonnacci sequence; and (c) giving them the experience of performing the broken symmetry of musical "rounds". As a result of this lab students can interrelate music, mathematics, art, and physics, with even a little psychoacoustics!

3.6 Experiencing Motion in Space and Time

The purpose of this lab, and the ones that follow, is to see the value of technology for investigating aspects of certain natural phenomena including those, which exist beyond the range of human vision and hearing. The computer with auxiliary devices attached to it is used to "extend" our visual and auditory senses. This in turn makes it possible for us to understand the manner in which symmetric aspects of nature contribute to our sense of the world of motion, sound, and light. In this lab students build on their experience of graphical representations (as introduced in Lab 4) through the visualization of scientific data obtained via the simultaneous monitoring and display of different phenomena by computer equipment. This enables them to: (a) see the value of technology's omnipresent concept of *voltage* through the display of battery outputs as a function of time; (b) understand aspects of *motion* through experiencing the movement of their hand, using an ultrasonic motion detector; (c) conceptualize some details of a periodic phenomenon that has time-translational invariance, called "simple harmonic motion", using a mass-on-a-spring connected to a force probe. By means of this lab students can see the obvious tie to mathematics, as well as to epistemology (i.e., how we acquire knowledge about the world; in this case, scientific knowledge). They can also see connections to art, music, and poetry. For example, the stressed-and-unstressed pattern in poetry's sonorous iambic pentameter may be represented through the use of visual voltage steps exhibiting a similar periodicity in time.

3.7 Symmetry of Oscillations: Sine Waves and Sound

This lab, drawing from concepts learned in the previous one, serves as an introduction to some elementary properties of sound by using the computer to measure periods that are thousands of times shorter than one second. The student learns: (a) how to measure the frequency and amplitude of periodic sinusoidal signals produced by tuning forks and then apply that knowledge to the observation of periodic *non*-sinusoidal signals; (b) what gives rise to the remarkable symmetrical patterns, called "Lissajous figures", that result from combining 2 sinusoidal signals whose frequencies stand in a simple numerical relation to each other. Ties to epistemology are seen through consistency arguments: the participant must compare the measured value of frequency with the "standard" value stamped on the tuning fork (is the standard value the "correct" one? what do we mean by "correct"? do we use similar methods to acquire knowledge in the arts?) Through the use of (b) can be seen obvious ties to mathematics as well as feedback to the visual arts; and with a little more imagination one can see how both music and poetry can benefit from such insights. An example is the tuning of a piano string via visual comparisons between a given input frequency and that from the string.

3.8 Building Symmetry from Symmetry: The Fourier Spectrum

This lab serves as an introduction to the addition of symmetrical wave patterns. The student learns about: (a) "Fourier synthesis", as manifested in complex periodic sounds, with their corresponding shapes, occurring when two or more simple periodic sounds with their corresponding shapes, resembling "sine wave" shapes, are combined, or "synthesized" (although true sine-wave shapes are referred to as "Fourier components", our simple approximate shapes will also be so referred to); (b) "Fourier decomposition", a method showing how complex periodic shapes can be broken down (or "decomposed") into their Fourier components, enabling students to look at the "shape" of their voice and of their heartbeat; and (c) how to synthesize their voice and heartbeat from the Fourier components. There are a variety of interconnections gleaned from this lab. Ties to mathematics are obvious. But in addition, connections can be made to the technology of "electronic" music and voice production, the visual arts, biology (e.g., how does the structure and function of the vocal chords and heart relate to the nature of the waves they produce?), and, most remarkably, to the limitations of a certain type of knowledge through the Heisenberg Uncertainty Relation in quantum physics (e.g., why is it that to be able to precisely locate a particle's position is to be unable to precisely locate its velocity?).

3.9 Waviness: In Water and Light

This lab serves as an introduction to the addition of wave patterns — a far-ranging investigation, as it applies to almost all waves in the universe. Here the focus is on the visual as students learn about: (a) brightness and light waves through the experience and measurement of the intensity of light as well as through the observation of a spatial and temporal invariance (each a symmetry property) of that intensity under certain conditions. By modeling light as a transverse wave and using polarizers they are also able to demonstrate and explain how such polarizers work in altering the intensity of light; and (b) waviness in water by the computer simulation of waves, through which students can observe the wavelength of waves, measure their speed, see how waves interfere with each other, and observe the effects of their interference. The connections are of course to mathematics (and once more emphasize an important epistemological point about methodology in the physical sciences), but also to the visual arts, to music (with the phenomenon of "intensity of light" being analogous to the "loudness of sound"), and to cosmology (e.g., how do we know the distance to stars and what they are made of?!).

Other labs in the planning stage are titled: "Slipping" Symmetry: Crystals, Fractals, and Chaos; Bounced, Flipped, Rotated, and Decomposed Light: From Mirrors to Spectra; and On Balls and Bombs: The Geometry of Projectile Motion.

4 OTHER ASPECTS OF THE COURSE

These are: (a) 2 closed-book exams; (b) a visit to a science museum: students submit a paper describing their observations and making remarks critiquing (pro or con) exhibits they found notable; (c) a term project: the capstone experience of the course, this consists, in order, of a report on their preliminary idea, a progress report, a class presentation, and the final report; (d) two types of (anonymous) course evaluations: my own mid-semester evaluation is designed to let me take steps to immediately modify aspects of the course; another, end-of-term All University Curriculum questionnaire has questions related to the ones given at mid-semester. Although we are using supplementary materials plus portions of one text (O'Daffer and Clemens, 1992) I am currently writing a book, instructor's manual, computer demonstration files, and updated laboratory manual.

5. CONCLUDING REMARKS

I hope that students carry away from the course the lesson that science *begins* from an individual's observations of the world as a highly imaginative probing into the workings of nature (not just a rigid compilation of facts and formulas). Seeing Through Symmetry is designed to help show students that the world is of a piece. It is to convey the sense that where the human mind journeys there are no barriers.

REFERENCES

Bobker, L. R. (1992) *Leonardo: A Journey of the Mind,*. ©Vision Associates, Inc. for IBM. VHS. 40 minutes.

Klein, P. (1990) On symmetry in science eucation, *Symmetry: Culture and Science* 1, 1, 77-91.

O'Daffer, P. G., Clemens, S. R. (1992) *Geometry: An Investigative Approach*, 2nd ed,, New York: Addison-Wesley.

Robinson, P. (1970) *Aspects of Symmetry.* Polytechnic Institute of Brooklyn. 16 mm. 15 minutes. (I am grateful to A. Rosenberg, Emeritus professor from Swarthmore College, for bringing this film to my attention.)

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SYMMETRY AND RECREATION

SYMMETRY IN PRACTICE – RECREATIONAL CONSTRUCTIONS

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1. INTRODUCTION (RECREATIONAL CONSTRUCTIONS)

The purpose of this article is to show how fascinating geometrical ideas can be by introducing the reader to some polyhedral puzzles. Our intent is to present the material in much the order in which we ourselves discovered it. We would like you to experience some of the joy of discovery we have had, which means, of course, that you risk experiencing some frustration along the way before you finally achieve success in assembling your puzzles. However, don't despair since we will give you many hints along the way and, eventually, more complete instructions for the details involved in assembling the puzzles.

In Section 2 we will describe how to fold the tape required to make your puzzles. In Section 3 we will explain how to make the puzzle pieces for 9 models and challenge you to construct some of them without any further information. We also include in Section 3 one intriguing example of how the braided models may be used to visualise the answer to a combinatorial question in geometry. In Section 4 we give either more hints or complete instructions on how to actually assemble the remaining models. In Section 5 we suggest some variations on certain models you will already have built that, for one reason or another, seem to lack the symmetry you would expect them to have. In this last section we will challenge you to build the more perfect tetrahedron, octahedron and icosahedron without any further information other than the description, two illustrations and a picture.

Although this may be viewed as purely recreational mathematics, knowledge of the symmetry group for each model may be helpful in solving the puzzle. We ourselves are very much in favor of exploiting the mathematics connected with these fascinating models[1] and we are delighted that the editor of this journal has suggested that our article concerning some of the mathematics connected with these models should appear as a companion piece to the present article, in this same issue. That related article, entitled "Symmetry in Theory – Mathematics and Aesthetics", abbreviated in this article to [Math], contains a fairly comprehensive list of references which you may consult if you wish to build other models. Of particular relevance is [HP5] of [Math]. We also include at the end of our companion article a brief history of how some of the folding procedures evolved, and how our friendship with the great mathematician and teacher, George Pólya, and with each other, resulted in mathematical collaborations concerning these models (and other topics).

We will concentrate on the directions for constructing polyhedral puzzles (including the Platonic solids) which have regular triangles, squares, or pentagons for faces. In order to be able to carry out the instructions and build these puzzles you will need

(1) some ordinary *unreinforced* gummed tape about 2 inches wide (a minimum of 30 to 40 running feet);

(2) some large sheets of paper such as gift-wrapping paper, or brightly colored lightweight construction paper (six different designs or colors is the most that will be required for any of these models);

(3) a pair of scissors;

(4) some paper clips (maximum 30);

(5) some bobby pins (maximum 6);

(6) a sponge, a bowl and some water;

(7) some rags and a flat place to work.

[1] On one occasion we were horrified to hear a mathematics teacher answer the question, "What did you do with the models you had the students build?" with "Oh, we hung them up!"

2. HOW TO FOLD THE TAPE

We describe, in this section, two iterative folding procedures that may be used on your gummed tape to fold strips of equilateral triangles and strips which can be used to construct regular pentagons. In both of these cases the procedure is a convergent one so that the angles you produce on the tape become more and more regular as you continue to fold. You will also need to fold a strip of consecutive squares (but this is an exact construction that we feel confident you can do on your own).

To fold the equilateral triangles simply follow the instructions in the numbered frames of Figure 1.

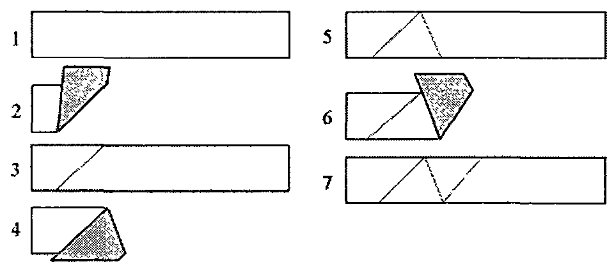

Figure 1: 1) Begin with a long strip of gummed tape; 2) Fold UP - any way will do; 3) Unfold; 4) Fold DOWN - now you must do it exactly as shown; 5) Unfold; 6) Fold UP, exactly as shown; 7) Unfold

Now go to frame 4, and keep repeating the folding in frames 4 through 7, to make a string of triangles as long as you need. Notice two things. First, the folding procedure, after your initial fold, goes DOWN, UP, DOWN, UP,... We will abbreviate this folding procedure by D^1U^1 and call the tape produced the D^1U^1-tape.[2] Second, you can readily see that, as we claimed, the triangles become more and more regular as you fold. Thus, if you wish to use this tape to construct models requiring equilateral triangles, all that you need to do is compare successive triangles, beginning with the first one formed, until it is not possible to detect any difference between them - and then throw away the defective ones and continue to fold, in the prescribed manner, to obtain the equilateral triangles you need for the constructions in Section 3. (We show in [Math] how to prove that all angles on this tape do, in fact, approach $\pi/3$.)

[2] Of course, one could adopt the 'systematic' folding procedure in which 'DOWN' and 'UP' are interchanged. The procedure would then be written U^1D^1.

To fold tape from which you can construct regular pentagons simply follow the instructions in the numbered frames of Figure 2.

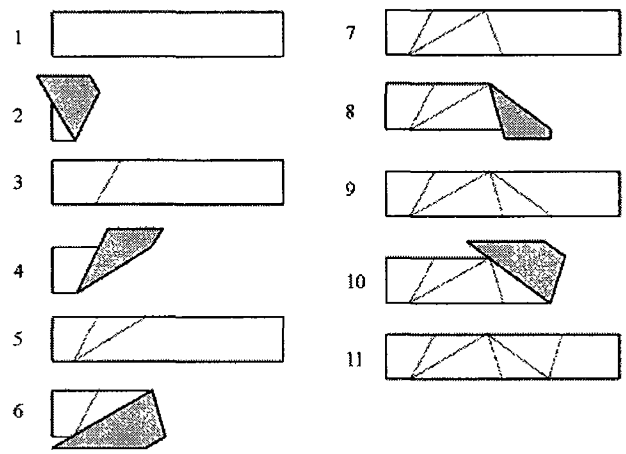

Figure 2: 1) Begin with a long strip of gummed tape; 2) Fold UP - any way will do; 3) Unfold; 4) Fold DOWN - now you must do it exactly as shown; 5) Unfold; 6) Fold UP, exactly as shown; 7) Unfold; 8) Fold DOWN, exactly as shown; 9) Unfold; 10) Fold UP, exactly as shown; 11) Unfold.

Now go to frame 4, and keep repeating the folding in frames 4 through 11, to make a string of tape from which you will be able to construct regular pentagons. First, notice that the folding process, after the first two initial folds, goes DOWN, DOWN, UP, UP, DOWN, DOWN, UP, UP,... We will abbreviate this folding procedure by D^2U^2 and call the tape produced the D^2U^2-tape. Second, notice that this tape has two kinds of crease lines, which we will refer to as *short* and *long* crease lines. Third, it is evident that the configuration formed by these crease lines is becoming more and more regular, reproducing the same angles at each edge of the tape. (We show in [Math] how to prove that the smallest angles on this tape do, in fact, approach $\pi/5$.)

This is the tape that you will use to make regular pentagons and models with regular pentagonal faces. To see how this works throw away the first few triangles you have folded (10 will be very safe) and continue to fold, in the prescribed manner, to obtain the tape you need to produce the constructions in Section 3. Just to practise now, cut off a piece of tape and make the pentagon shown in Figure 3. Notice that when you constructed this pentagon you cut the D^2U^2-tape along a short crease line, and folded on short crease lines.

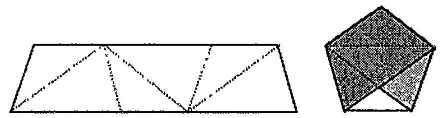

Figure 3: Make a section of tape that looks like this (1), into a pentagon that looks like this (2) (Shading indicates the other side of the tape).

What about the long crease lines? Try cutting along a long crease line and folding on successive long crease lines to construct the pentagon shown in Figure 4 (of course you will also have finally to cut along another long crease line to complete the model as it is shown).

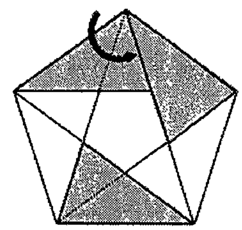

Figure 4: The end of the tape must be tucked in here (arrow).

3. HOW TO MAKE THE PUZZLE PIECES OR STRIPS

This section describes how to make the puzzle pieces for the following 9 models which naturally divide themselves into three types.

A pentagonal dipyramid with a single strip of 31 equilateral triangles.
A triangular dipyramid with a single strip of 19 equilateral triangles.
The Platonic[3] Puzzles:

- A tetrahedron with 2 strips
- A hexahedron (cube) with 3 strips
- An octahedron with 4 strips
- An icosahedron with 5 strips
- A dodecahedron with 6 strips

(Do you notice any interesting pattern here? Do you see why we have written the dodecahedron last, although it's usually written before the icosahedron?)

- A diagonal cube with 4 strips.
- A golden dodecahedron with 6 strips.

There are some general comments that apply to each of these 9 models. In each case you should first make the required pattern pieces (or strips), by folding the gummed tape. Then glue the strips onto colored paper (when more than one strip is involved glue each strip onto paper of a different color).

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[3] As any Greek scholar will tell you, the names of the Platonic solids are designed to show that they have 4, 6, 8, 20, 12 faces, respectively.

When gluing the strips onto the colored paper, make certain the paper you plan to use for each strip is large enough. Then place a sponge (or washcloth) in a bowl and add water to the bowl so that the top of the sponge is very moist (squishy). Next moisten one end of the strip by pressing it onto the sponge and then, holding that end, pull the rest of the strip across the sponge (This part of the process is often messy!)[4] Make certain the entire strip gets moistened and then place it on the colored paper. Use a hand towel (or rag) to wipe up the excess moisture and to press the tape into contact with the colored paper.

Put some books on top of the tape so that it will dry flat. When the tape is dry, cut out the pattern piece, trimming off a small amount of the gummed tape (about 1/16 to 1/8 of an inch) from the edge as you do this. This trimming procedure serves to make the model look neater and, more importantly, it allows for the increased thickness produced by gluing the strip to another piece of paper. Refold the piece (only along the lines you need for your particular model) so that the raised (mountain) folds are on the colored side of the paper. You will then have your puzzle pieces and can proceed to construct your model.

We begin the actual puzzles with the two dipyramids to let you get a feel for the way your materials behave. We will be fairly explicit here, to get you started, but we'll be less detailed when we come to the Platonic Puzzles.

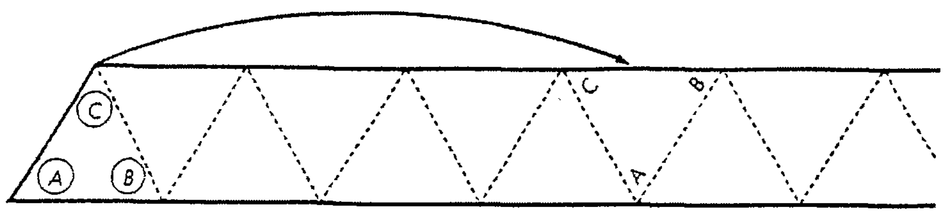


Figure 5: (a) Left-hand end of pattern piece

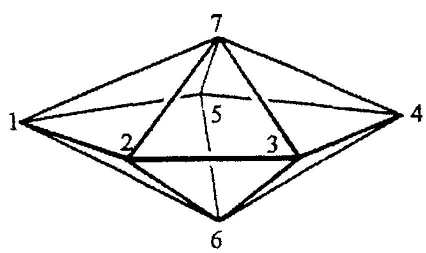
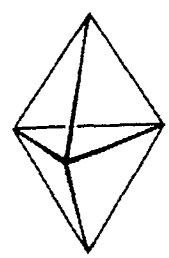


Figure 5: (b) Pentagonal dipyramid



(c) Triangular dipyramid

[4] Perhaps we should have included some very old clothes as optional (or even essential) materials.

Figure 5(a) shows the left-hand end of the 31-triangle strip used to construct the pentagonal dipyramid. You should mark the first and eighth triangles *exactly* as shown (note the orientation of each of the letters within their respective triangles). To assemble the model place the first triangle *over* the eighth triangle so that the circled letters *A*, *B*, *C* are over the uncircled letters *A*, *B*, *C*, respectively. Holding those two triangles together in that position, you will notice that you have the frame of a double pyramid for which there will be five triangles above and five triangles below the horizontal plane of symmetry (the plane containing the vertices 1, 2, 3, 4, 5 in Figure 5(b)). Now hold this configuration up and turn it so that the long strip of triangles falls around this frame. If the creases are folded well, the remaining triangles will fall into place. When you get to the last triangle there will be a crossing of a strip that the last triangle can tuck into, and your model will be complete and stable.

If you have trouble because the strip doesn't fall into place there are two frequent explanations. The first (and most likely) reason is that you have not folded the crease lines firmly enough. In that case all you need to do is crease them again with more gusto! The second possible reason is that the strip seems too short to reach around the model and tuck in. This may be remedied by trimming a tiny amount from each edge of the strip.

An analogous construction may be made for the triangular dipyramid shown in Figure 5(c). This model can be made from a strip of 19 equilateral triangles. Knowing what the finished model should look like and that you should begin by forming the *top* three faces with one end of the strip should be sufficient hints.

You may discover that you can construct each of these dipyramids with fewer triangles, but we chose the construction that produces the most *balanced* model. You will note that both of these constructions place *precisely* three thicknesses of paper on each face, except where the last triangle tucks in (producing four thicknesses). In both cases, you could remedy this small defect by cutting off half of the first and last triangle on the strip.

Now let us turn to the Platonic Puzzles.

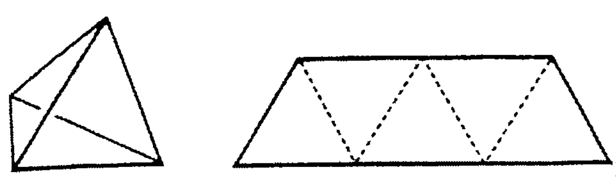
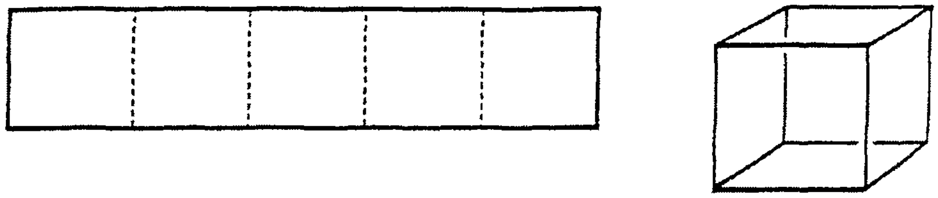
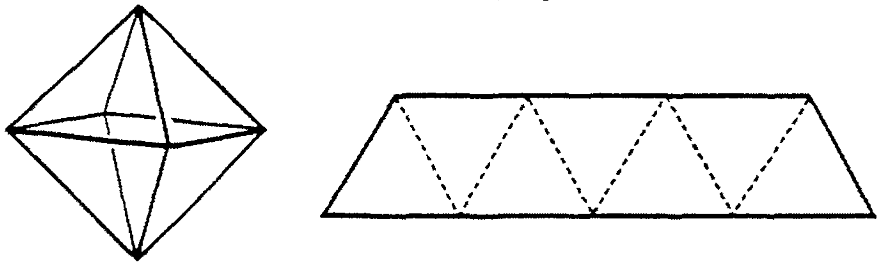


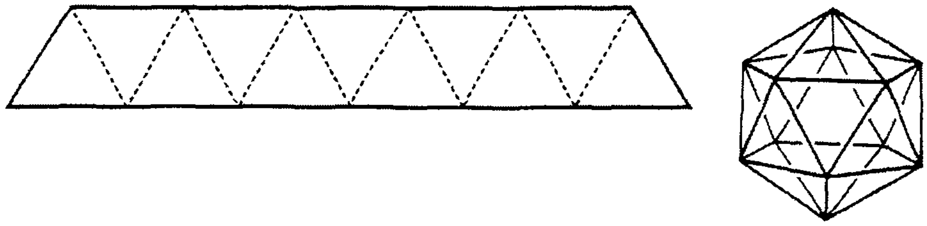
Figure 6: 1) Tetrahedron (2 strips)



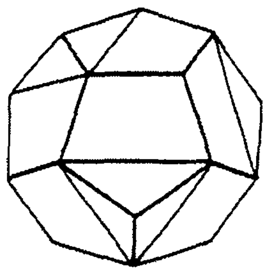
2) Hexahedron (Cube) (3 strips)



3) Octahedron (4 strips)



4) Icosahedron (5 strips)



5) Dodecahedron (6 strips, 3 of each kind).

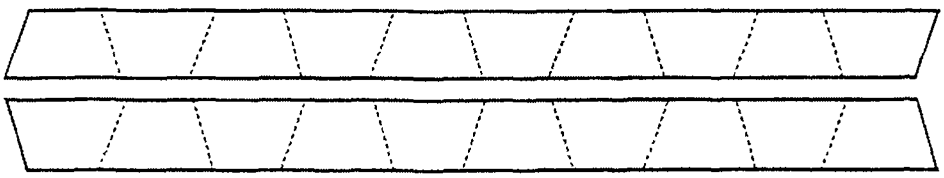


Figure 6 shows a typical puzzle piece (or strip) next to each solid, and tells you how many are needed. In each case the puzzle is this: take the required strips and braid them together to from the required solid in such a manner that

(a) the same area is visible on each strip, and
(b) all ends are tucked in.

The *tetrahedron, octahedron,* and *icosahedron* all involve strips obtained from the D^1U^1-folding procedure. All you need to do is prepare the pattern pieces as we described above and try to assemble the models. You may note that, on all of these models, if you take into account the coloring of the surface, they will have lost some of the symmetry you would expect to find on Platonic Solids; that is, not all edges will look the same. For some edges the two adjacent faces will have the same color but, for other edges, the two adjacent faces will have different colors. (We will propose another type of construction that corrects this defect in Section 5). If you manage to get all three of these together without any hints you are really an *expert*! If you have trouble getting your models together check the hints given in Section 4.

The *hexahedron (cube)* pattern pieces may be made by making exact folds on the tape. All that you need to remember is that if you fold the tape directly back on itself you will produce an angle of precisely $\pi/2$, and if you bisect that angle you will know exactly where to fold the tape back on itself to produce a square. Once you have one square on the tape you may then simply fold the tape back and forth, accordion style, on top of this square to produce the required number of squares. There are actually two ways to braid these three pieces together to satisfy the conditions for the puzzle. One of these ways produces a cube with opposite faces the same color and the other way produces a cube with certain pairs of adjacent faces the same color. From the point of view of symmetry the first is more symmetric because, on that model, all edges abut two faces of different colors. If you have trouble assembling this model check the hints given in Section 4.

The *dodecahedron* involves strips obtained from the D^2U^2-folding procedure. But notice on the final pattern piece you should only fold the pattern piece firmly along the *short* crease lines (ignoring the long lines) after you have cut out each piece. We should tell you that on this model *four* sections of each strip will overlap (for stability). It may also be helpful to let you know that the strips go together in pairs and the construction is then similar to that of the more symmetric cube you have constructed above – and, if coloring is taken into account, the completed model loses a lot of the symmetry you expect to see on a dodecahedron. You may now have enough hints, but if you have difficulty consult Section 4.

The *diagonal cube* involves four strips each containing 7 right isosceles triangles as shown in Figure 7(a). The strips for these pieces may be folded by the exact procedure similar to that described above for the cube in the Platonic Puzzles. Just remember that this time you want to emphasize those crease lines that make an angle of $\pi/4$ with the edges of the tape. To assemble the cube you begin by laying out the four pieces as shown in Figure 7(b), with the colored side of the paper not showing. You may wish to put a small piece of tape in the exact center to hold the strips in position. Now, thinking of the dotted square surrounding the center as the base of your cube, you begin to *braid* the strips to make the vertical faces, remembering that each strip should go successively over and under the strips it meets as it goes around the model. When you get to the top face you will find that all the ends will tuck in to produce a very beautiful and highly symmetric cube; indeed, *none* of the symmetry of the cube has been lost. You will notice that every face has a different arrangement of four colors and that every vertex is surrounded by a different arrangement of three colors.

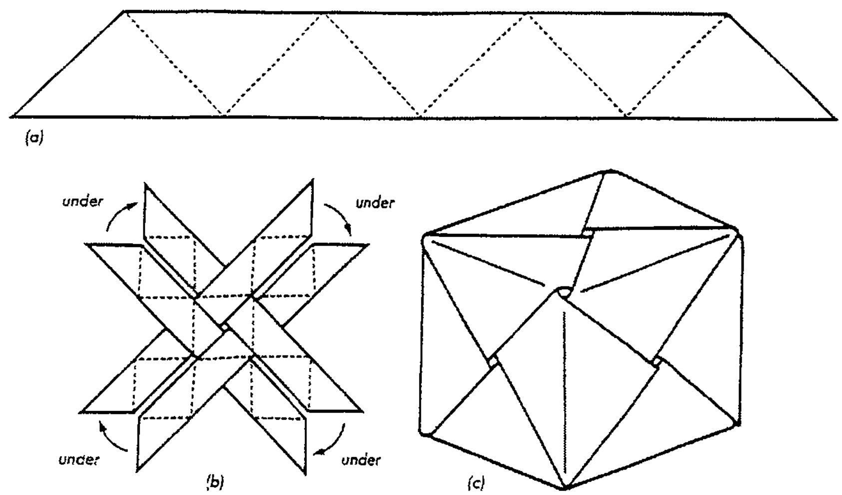


Figure 7

The *golden dodecahedron* involves strips obtained from the D^2U^2-folding procedure. But notice on the pattern pieces you should only fold the pattern piece firmly along the *long* crease lines (ignoring the short lines) after you have cut out each piece. To complete the construction of this model, begin by taking five of the strips and arranging them, with the colors showing, as shown in Figure 8(b), securing them with paper clips at the points marked with arrows. View the center of the configuration as the North Pole. Lift this arrangement and slide the even-numbered ends clockwise over the odd numbered ends to form the five edges coming south from the arctic pentagon. Secure the strips with paper clips at the points indicated by crosses. Now weave in the sixth (equatorial) strip, shown shaded in Figure 8(c), and continue braiding and clipping, where necessary, until the ends of the first five strips are tucked in securely around the South Pole. Above all, *keep calm*, you can even take a break – the model will wait for you! Just make certain that every strip goes alternately over and under each strip it meets all the way around the model. When the model is complete (with the last ends tucked in) you may remove all the paper clips and the model will remain stable. We notice that this constructed dodecahedron is aesthetically very satisfying – more so than the dodecahedron previously described. This is due to its amazing symmetry – *none* of the possible symmetry has been lost.

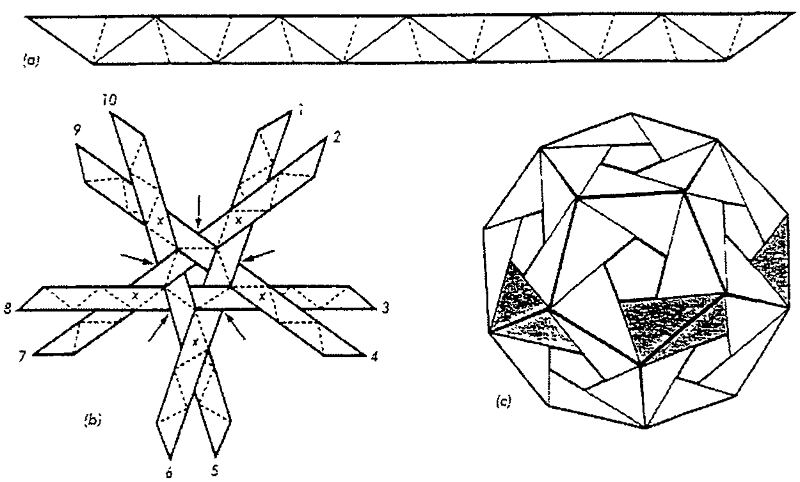


Figure 8

Before giving you the hints for the remaining models we cannot resist showing you a lovely use for the cube with three strips, the diagonal cube and the golden dodecahedron. An interesting question[5] that has been asked by geometers (see [KW]) is "How many disjoint pieces – both finite and infinite (or unbounded) – are formed by the extended face planes[6] for each Platonic solid?" As it turns out we can use our braided models to answer some of these questions in the case of the unbounded regions.

Let us use the ordinary cube to show how the braided models are useful. Notice that the edges of the three strips used to create the braided model lie in 6 planes which interesect each other to form a cube. Figure 9, suitably interpreted, shows that the extended face planes of a cube partition space into 27 pieces. There is, of course, the cube itself, which is bounded. Then come the unbounded regions. There are

[5] A question is always interesting to mathematicians if the answer is not obvious but they can see a possible way to answer it.

[6] The planes in which the faces of a polyhedron lie are called the extended face planes of the polyhedron.

(a) 6 unbounded square prisms from its faces,
(b) 12 unbounded wedges from its edges and
(c) 8 unbounded trihedral regions from its vertices.

Now if you examine your braided cube you will see that there are
(a) 6 square regions that are covered with precisely two thicknesses of paper,
(b) 12 small slits along the edges where there is just one thickness of paper and
(c) 8 tiny triangular holes at the vertices.

This observation gives us the clue as to how braided models may be useful more generally in answering our question.

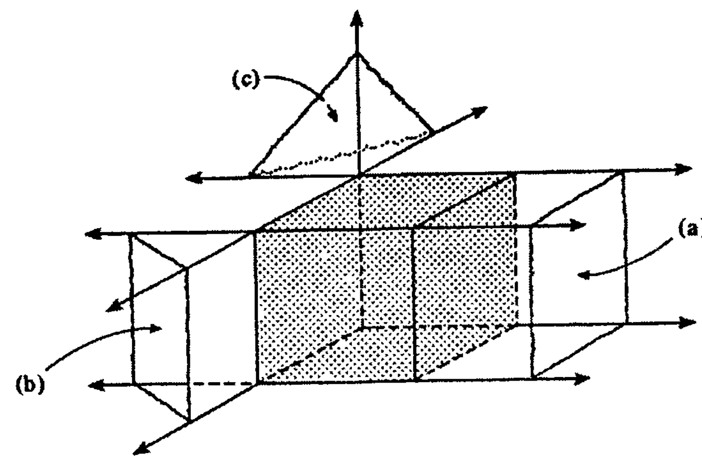


Figure 9: Using the braided cube to count the unbounded regions created by the extended face planes of the cube.

What turns out to be true is that the braided models partition the surface of the polyhedron into mutually disjoint sets of 'polygons' where each polygon is covered by 0, 1 or 2 thicknesses of paper. The polygons where there are holes (0-thickness) define unbounded polyhedral regions, the polygons which are narrow slits (1-thickness) define unbounded wedges, and the polygons where the strips actually are crossing each other (2 thicknesses) define unbounded prisms. The shapes of these unbounded regions may vary with the braided model, but these general statements always hold.

If we are to be able to answer our question for the octahedron we need a braided model with 4 strips so that their edges will define 8 face planes – and, of course, the braided model should also have the same symmetry group as the octahedron. Fortunately the diagonal cube satisfies our conditions (see [Math] concerning the duality of the cube and octahedron). Figure 10 shows the octahedron with some of its face planes extended so that you can see a typical finite region and typical unbounded regions of each type.

The braided models don't help to count the bounded regions (in this case, however, we can see from the part of Figure 10 labeled (a) that there is a tetrahedron on each face of the original octahedron). The rest of the labels in Figure 10 indicate unbounded regions and Figure 11 reproduces those regions as they are associated with the surface of the diagonal cube. Thus, using Figure 11, we may now count the unbounded regions. They are (using the labels in Figure 11):

(b) 6 unbounded tetrahedral regions from the holes in the center of the faces,
(c) 24 unbounded wedges from the 4 slits on each of the 6 faces,
(d) 8 unbounded trihedral regions from the vertices,
(e) 12 unbounded, prism-like regions from the crossings of the strips on the edges comprising a total of 50 unbounded regions.

So the golden dodecahedron must also be useful. In fact it is composed of six strips and the planes defined by the edges of those strips intersect inside this model to form a dodecahedron. Thus the surface of the golden dodecahedron can be used to see that there are 122 unbounded regions created by the extended face planes of the dodecahedron. You might like to try to count them yourself using your golden dodecahedron (or see [P] for more details).

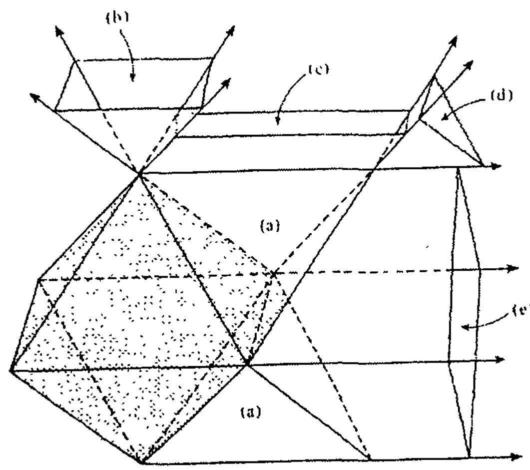


Figure 10: Extending the face planes of an octahedron.

What about the icosahedron? Figure 11 shows how a model, braided from 10 straight strips, may be made from the D^2U^2-tape that can be used to count the 362 unbounded regions created by the extended face planes of the icosahedron (see [P] for more details). We have not written down anywhere how to make the model shown in Figure 11, but we're sure the interested reader will be able to figure it out from what we have said and the illustration. You may be asking yourself why we have slighted the tetrahedron. The answer is that the tetrahedron does not have faces lying in opposite parallel planes, so our models are not useful here. However, it is not difficult to imagine extending the face planes of the tetrahedron and seeing that you have one finite region (the tetrahedron) and 14 unbounded regions (4 from vertices, 6 from edges and 4 from faces).

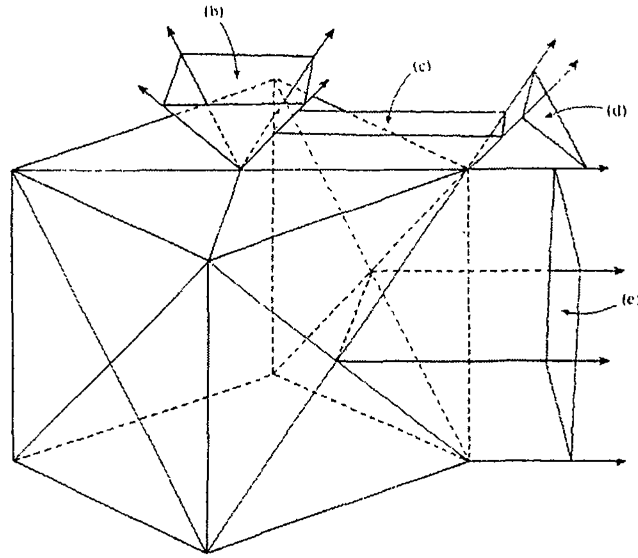


Figure 11: The use of the diagonal cube to visualise the unbounded regions formed by the extended face planes of the regular octahedron.

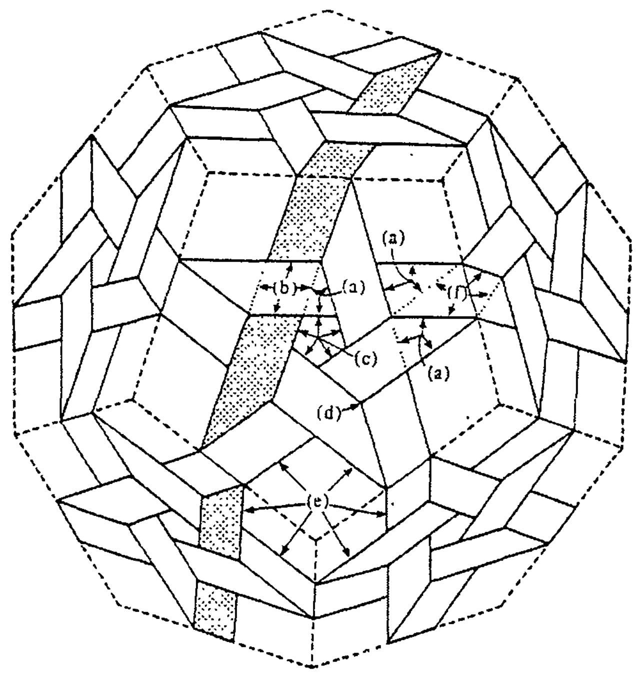
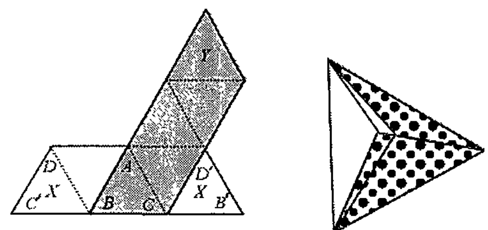


Figure 12: Strips whose edges define the extended face planes of an icosahedron lying on the surface of a phantom dodecahedron.

4. FURTHER HINTS FOR CONSTRUCTING THE PLATONIC PUZZLES

Tetrahedron: Lay one strip over the other strip (with the colors not showing) exactly as shown in Figure 13(a). Think of the triangle *ABC* as the base of the tetrahedron; for the moment the triangle *ABC* remains fixed. Then fold the bottom strip into a tetrahedron by lifting up the two triangles labeled *X* and overlapping them so that *C*' meets *C*, *B'* meets *B*, and *D'* meets *D*. Don't worry about what is happening to the other strip as long as it stays in contact with the bottom strip where the two triangles originally overlapped. Now you will have a tetrahedron with three triangles sticking out from one edge. Complete the model by wrapping the protruding strip around two faces of the tetrahedron (with the color showing) and tucking in the last triangle so that it looks like Figure 13(b).



(a) (b)

Figure 13

Hexahedron (cube): First take one strip and clip it together so that the color is outside and the end squares fit over each other. Do the same with a second strip. Slip one of these over the other so that the holes of the cube are all covered and so that the overlapping squares of the second strip do not cover any squares from the first strip, and so that the paper clip on the first strip is covered as shown in Figure 14(a). Now slide the third strip underneath the top square so that two squares from the third strip stick out on both the right and left sides of the cube, as shown in Figure 14(b). Turn the model upside down and tuck in the ends of this strip to form Figure 14(c). You may remove the paper clips before you complete the construction; but, when you become really adept, you'll find you don't need the paper clips at all.

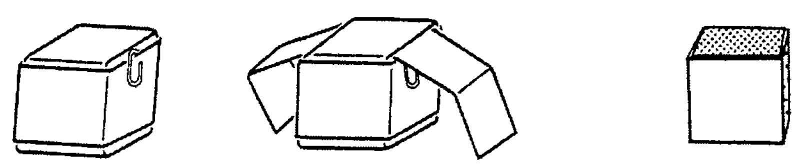
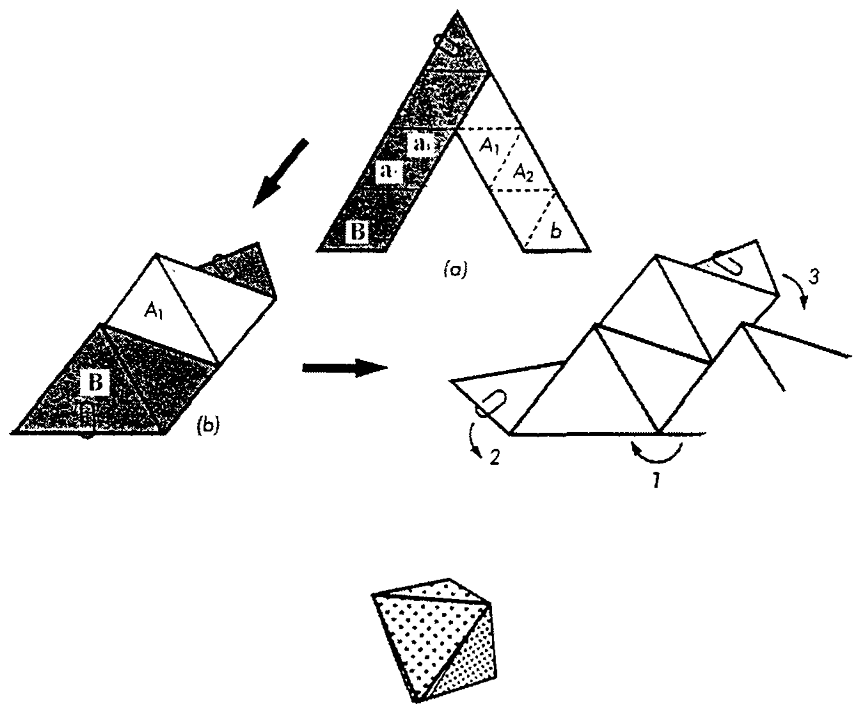

(a) (b) (c)

Figure 14

Octahedron: Begin with a pair of overlapping strips held together with a paper clip, as indicated in Figure 15(a) (with the color visible). Fold these two strips into a double pyramid by placing triangle a_1 under triangle A_1, triangle a_2 under triangle A_2, and triangle b under triangle B. Secure the overlapping triangles b, B with a paper clip to produce the configuration shown in Figure 15(b). Repeat this process with the other two strips. Then place the second pair of braided strips over the first pair, as shown in Figure 15(c). When doing this, make certain the flaps with the paper clips are oriented precisely as shown in the figure. Now, pick up the entire configuration and complete the octahedron by moving the pyramids together as shown by the arrow marked 1. Performing step 2 simply places the flap with the paper clip on it against a face of the octahedron. In step 3 you wrap the remaining portion around the octahedron and tuck the last flap (with a paper clip on it) *inside the model.* Again, when you become adept at this process you will be able either to do it without paper clips, or, at least, to slip the paper clips off just before you perform the last three steps. Actually this is just an aesthetic consideration, since the paper clips won't be visible on the completed model.



(d)

Figure 15

Icosahedron: Label each of triangles on one of the strips with a 1 on the uncolored side of the tape. Then label the next strip with a 2 on each of its triangles, the next with a 3 on each of its triangles, the next with a 4 on each of its triangles, and, finally, the last with a 5 on each of its triangles.

Now lay the 5 strips out so that they overlap each other *precisely* as shown in Figure 16(a), making sure that the center 5 triangles form a shallow cup that points *away* from you. You may wish to use some transparent tape to hold the strips in this position.

Now study the situation carefully before making your next move. You must bring the 10 ends up so that the part of the strip at the tail of the arrow goes *under* the part of the strip at the head of the arrow (this means "under" as you look down on the diagram, because we are looking at what will become the inside of the finished model). Half the ends wrap in a clockwise direction, and the other end of each strip wraps in a counterclockwise direction. What finally happens is that each strip overlaps itself at the top of the model. In the intermediate stage it will look like Figure 16(b). At this point it may be useful to put a rubber band (not too tight) around the emerging polyhedron just below the flaps that are sticking out from the pentagon. Then lift the flaps as indicated by the arrows in Figure 16(b) and bring them toward the center so that they tuck in as shown in Figure 16(c).

The model is completed by first lifting flap 1 and smoothing it into position. Then you should do the same with flaps 2, 3 and 4. Finally, flap 5 will tuck into the obvious slot and you will have produced the model shown in Figure 16(d).

This model is, in the view of the authors, the most difficult of the 9 puzzles to construct and it is not very stable. You might want to put a couple of lightweight rubber bands around it to prevent it from falling apart when it is handled.

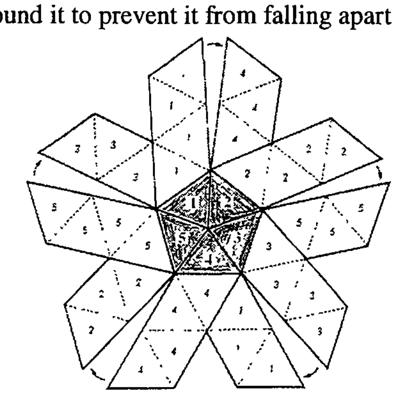
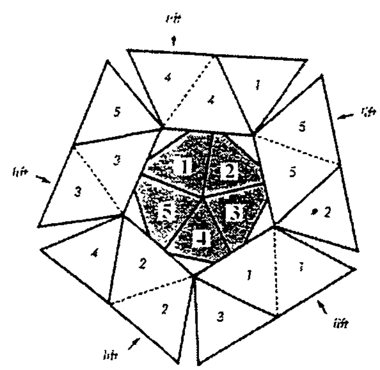


Figure 16 (a)



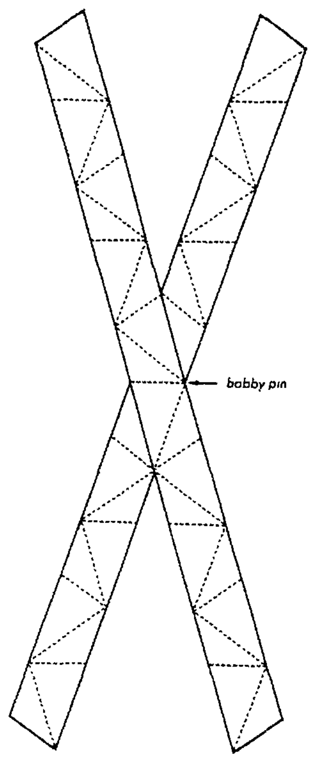
(b)

Dodecahedron: Take two of the strips and secure them with a bobby pin as shown in Figure 17(a) (with the colored side visible)[7]. Then make a bracelet out of each of the strips in such a way that

(a) four sections of each strip overlap, and
(b) the strip that is *under* on one side of the bracelet is *over* on the other side. (This will be true for both strips.)

Use another bobby pin to hold all four thicknesses of tape together on the edge that is opposite the one already secured with a bobby pin.

Repeat the steps above with another pair of strips. You will then have two identical bracelet-like arrangements. Slip one inside the other one as illustrated in Figure 17(b), so that it looks like a dodecahedron with triangular holes on four of its faces.

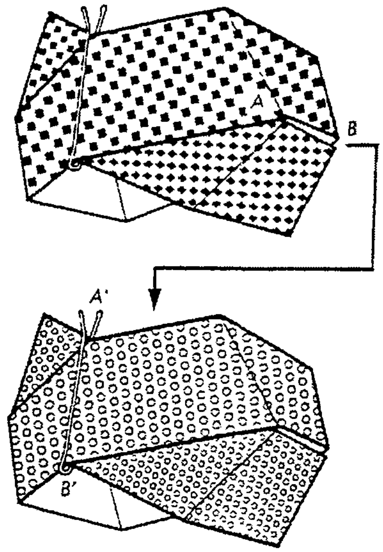


Next take the last two strips and cross them precisely as you did in Figure 17(a) (to do otherwise would destroy some of the symmetry); then secure them with a bobby pin. Carefully put two of the loose ends (either the top two or the bottom two) through the top hole and pull them out the other side so that the bobby pin lands on *CD*. Then put the other two ends through the bottom hole and pull them out the other side. Now you can tuck in the loose flaps, but make certain to reverse the order of the strips – that is, whichever one was on the bottom at *CD* should be on the top when you do the final tucking.

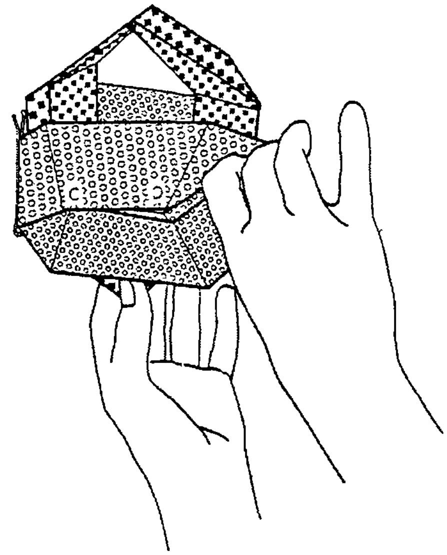
After you have mastered this construction you may wish to try to construct the model with tricolored faces, shown in Figure 17(e), which illustrates, rather vividly, exactly how to inscribe the cube symmetrically inside the dodecahedron. You may also note a similarity between this construction and the cube of Figure 10.

Figure 17 (a)

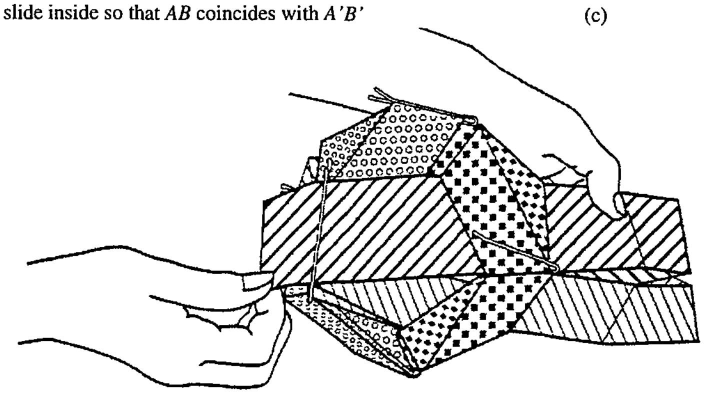
[7] Notice that the long lines are shown in this figure but, as we said earlier, your strips should only be creased on the short lines.



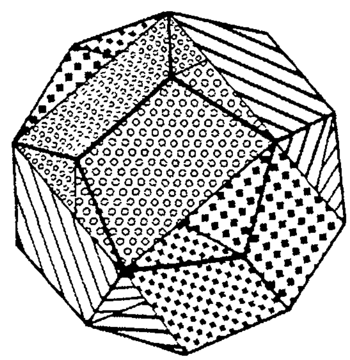
(b) Turn and slide inside so that AB coincides with $A'B'$



(c)



(d)



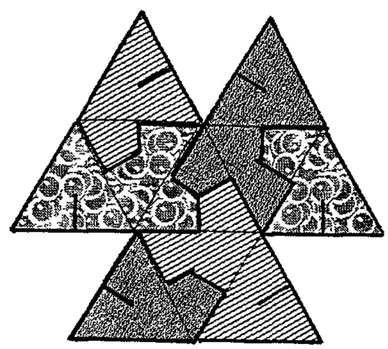
(e)

5. CONSTRUCTING THE PERFECT TETRAHEDRON, OCTAHEDRON AND ICOSAHEDRON

Recall how much pleasure we took in the fact that the diagonal cube and the golden dodecahedron retained all their inherent symmetry. Generally speaking, braided models lose some of the symmetry of the underlying geometric figure; indeed, our braided tetrahedron, octahedron and icosahedron all lost some of the underlying geometric symmetry. Thus it is natural to ask "Is it possible to braid the tetrahedron, octahedron and icosahedron in such a way as to retain all the symmetry of the original polyhedron?" We have recently discovered a way to do this. The problem was to design strips so that three strips cross over each other to form each (triangular face) in a symmetric way.

Figure 18(b) shows a typical straight strip of 5 equilateral triangles with a slit in each triangle from the top (or bottom) edge to (just past) the center[8]. The perfect tetrahedron is constructed out of Figure 18(a) where you will see how the 3 strips are interlaced initially. We leave the completion of the model as a challenge to you.

Figure 19(a) shows the layout of 3 strips for the beginning of the construction of the perfect octahedron. We'll give you one more hint. When you use Figure 19(a) remember that the strip shown below it in Figure 19(b) has to be braided into the figure above it.



(a)

(b)

Figure 18

[8] Theoretically the slit could go just to the center, but the model is then impossible to assemble. You need to have some leeway for the pieces to be free to move during the process of construction – although they will finally land in a symmetric position so that it looks as though the slit need not have gone past the center.

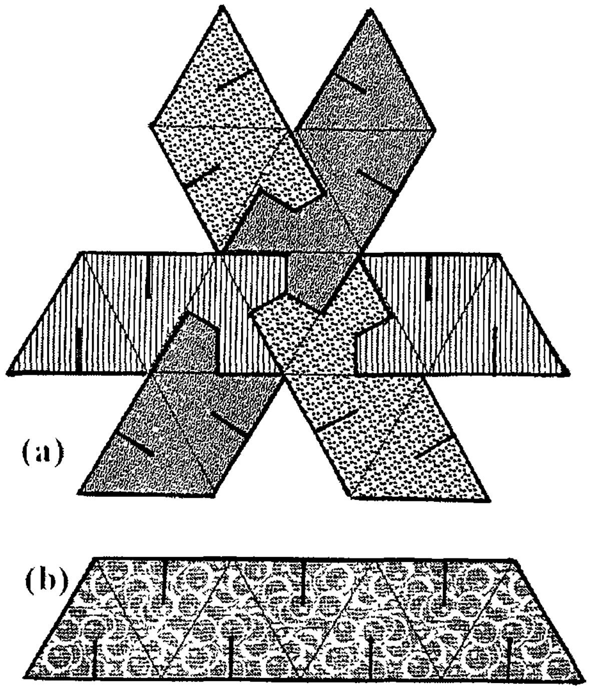
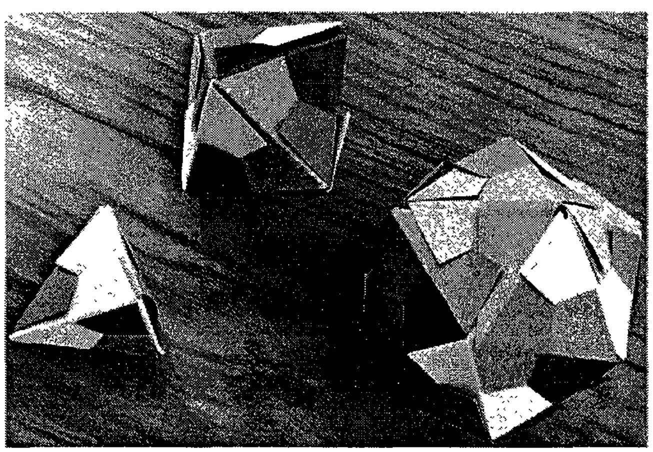


Figure 19



A perfect icosahedron may be constructed from 6 strips of this type having 11 triangles on each strip. Over to you! But take heart - these models take several hours to construct. Just to prove that they really do exist we show the photo of them in Figure 20.

Figure 20

REFERENCES

[Math] Hilton, Peter, and Jean Pedersen, Symmetry in Theory - Mathematics and Aesthetics.

[KW] Kerr, Jeanne W., and John E. Wetzel, Platonic divisions of space, *Mathematics Magazine*, 51 (1978), 229 - 234.

[P] Pedersen, Jean, Visualising parallel divisions of space, *Math. Gaz* 62 (1978), 250 - 262.

[P] Pedersen, Jean, Visualising parallel divisions of space, *Math. Gaz.* 62 (1978), 250 - 262.

Contributions to *SYMMETRY: CULTURE AND SCIENCE* are welcomed from the broadest international circles and from representatives of all scholarly and artistic fields where symmetry considerations play an important role. The papers should have an interdisciplinary character, dealing with symmetry in a concrete (not only metaphorical!) sense, as discussed in "Aims and Scope" on the next page. The quarterly has a special interest in how distant fields of art, science, and technology may influence each other in respect of symmetry. The papers should be addressed to a broad non-specialist public in a form which would encourage the dialogue between disciplines.

Manuscripts may be submitted directly to the editors, or through members of the Board of ISIS-Symmetry.

Contributors should note the following:

- All papers and notes are published in English and they should be submitted in that language. The quarterly reviews and annotates, however, non-English publications as well.

- In the case of complicated scientific concepts or theories, the intuitive approach is recommended, thereby minimizing the technical details. New associations and speculative remarks can be included, but their tentative nature should be emphasized. The use of well-known quotations and illustrations should be limited, while rarely mentioned sources, new connections, and hidden dimensions are welcomed.

- The papers should be submitted either by e-mail attachment, or on (PC formatted) computer diskettes (rtf format preferred). Typewritten texts will not be rejected in the evaluation process, but the preparation of these items takes longer; reviewed and accepted papers, however, will be requested in electronic form. For any method of submission, three hard-copies of the text are also required, where all the necessary editing is marked in red. No manuscripts, diskettes, or figures will be returned, unless by special arrangement.

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- Only black-and-white, camera-ready illustrations (photos or drawings) can be used. Scanned illustrations inserted in the electronic version are preferred. The required (approximate) location of the figures and tables should be indicated in the main text by typing their numbers and captions (Figure 1: [text], Figure 2: [text], Table 1: [text], etc.), as new paragraphs. The figures, which may be slightly reduced in printing, should be enclosed on separate sheets. The tables may be given inside the text or enclosed separately.

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Sample of heading (Apologies for the strange names and addresses)

SYMMETRY IN AFRICAN ORNAMENTAL ART
BLACK-AND-WHITE PATTERNS IN CENTRAL AFRICA
Running head: Symmetry in African Art
Section: Symmetry: Culture & Science

Susanne Z. Dissymmetrist	and	Warren M. Symmetrist
8 Phyllotaxis Street		Department of Dissymmetry, University of Symmetry
Sunflower City, CA 11235, U.S.A.		69 Harmony Street, San Symmetrino, CA 69869, U.S.A.
		E-mail: symmetrist@symmetry.edu

Abstract: *The ornamental art of Africa is famous ...*

Sample of references

In the following, note punctuation, capitalization, the use of square brackets (and the remarks in parentheses). There is always a period at the very end of a bibliographic entry (but never at other places, except in abbreviations). Brackets are used to enclose supplementary data. Those parts which should be italicized - titles of books, names of journals, etc. - should be underlined in red on the hard-copies. In the case of non-English publications both the original and the translated titles should be given (cf., Dissymmetrist, 1990).

Asymmetrist, A. Z. (or corporate author) (1981) *Book Title: Subtitle*, Series Title, No. 27, 2nd ed., City (only the first one): Publisher, vii + 619 pp.; (further data can be added, e.g.,: 3rd ed., 2 Vols., *ibid.*, 1985, viii + 444 + 484 pp. with 2 CD-s; Reprint, ibid., 1988; German trans., *German Title*, 2 Vols., City: Publisher, 1990, 986 pp.; Hungarian trans.)

Asymmetrist, A. Z., Dissymmetrist, S. Z., and Symmetrist, W. M. (1980-81) Article or e-mail article title: Subtitle, Parts 1-2, *Journal Name Without Abbreviation*, [E-Journal or Discussion Group address: journal@node (if applicable)], B22 (volume number), No. 6 (issue number if each one restarts pagination), 110-119 (page numbers); B23, No. 1, 117-132 and 148 (for e-journals any appropriate data).

Dissymmetrist, S. Z. (1989a) Chapter, article, symposium paper, or abstract title, [Abstract (if applicable)], In: Editorologist, A.B. and Editorologist, C.D., eds., *Book, Special Issue, Proceedings, or Abstract Volume Title*, [Special Issue (or) Symposium organized by the Dissymmetry Society, University of Symmetry, San Symmetrino, Calif., December 11-22, 1971 (those data which are not available from the title, if applicable)], Vol. 2, City: Publisher, 19-20 (for special issues the data of the journal).

Dissymmetrist, S. Z., ed. (1990) *Dissimmetriya v nauke* (title in original, or transliterated, form), [Dissymmetry in science, in Russian with German summary], Trans. from English by Antisymmetrist, B. W., etc.

[Symmetrist, W M.] (1989) Review of *Title of the Reviewed Work*, by S. Z. Dissymmetrist, etc. (if the review has an additional title, then it should appear first; if the authorship of a work is not revealed in the publication, but known from other sources, the name should be enclosed in brackets).

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Sample of biographic entry

Name: Warren M. Symmetrist, Mathematician, (b. Boston, Mass., U.S.A., 1938).

Address: Department of Dissymmetry, University of Symmetry, 69 Harmony Street, San Symmetrino, Calif. 69869, U.S.A. *E-mail:* symmetrist@symmetry.edu .

Fields of interest: Geometry, mathematical crystallography (also ornamental arts, anthropology - non-professional interests in parentheses).

Awards: Symmetry Award, 1987; Dissymmetry Medal, 1989.

Publications and/or *Exhibitions*: List all the symmetry-related publications/exhibitions in chronological order, following the conventions of the references and annotations. Please mark the most important publications, not more than five items, by asterisks. This shorter list will be published together with the article, while the full list will be saved in the data bank of ISIS-Symmetry.

There are many disciplinary periodicals and symposia in various fields of art, science, and technology, but broad interdisciplinary forums for the connections between distant fields are very rare. Consequently, the interdisciplinary papers are dispersed in very different journals and proceedings. This fact makes the cooperation of the authors difficult, and even affects the ability to locate their papers.

In our 'split culture', there is an obvious need for interdisciplinary journals that have the basic goal of building bridges ('symmetries') between various fields of the arts and sciences. Because of the variety of topics available, the concrete, but general, concept of symmetry was selected as the focus of the journal, since it has roots in both science and art.

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(1) to bring together artists and scientists, educators and students devoted to, or interested in, the research and understanding of the concept and application of symmetry (asymmetry, dissymmetry);
(2) to provide regular information to the general public about events in symmetrology;
(3) to ensure a regular forum (including the organization of symposia, congresses, and the publication of a periodical) for all those interested in symmetrology.

The Society organizes the triennial Interdisciplinary Symmetry Congress and Exhibition (starting with the symposium of 1989) and other workshops, meetings, and exhibitions. The forums of the Society are *informal* ones, which do not substitute for the disciplinary conferences, only supplement them with a broader perspective.
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(1) they describe concrete interdisciplinary 'bridges' between different fields of art, science, and technology using the concept of symmetry;
(2) they survey the importance of symmetry in a concrete field with an emphasis on possible 'bridges' to other fields.

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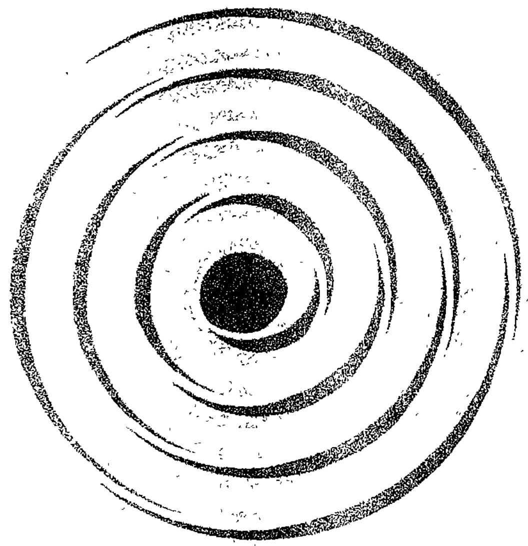

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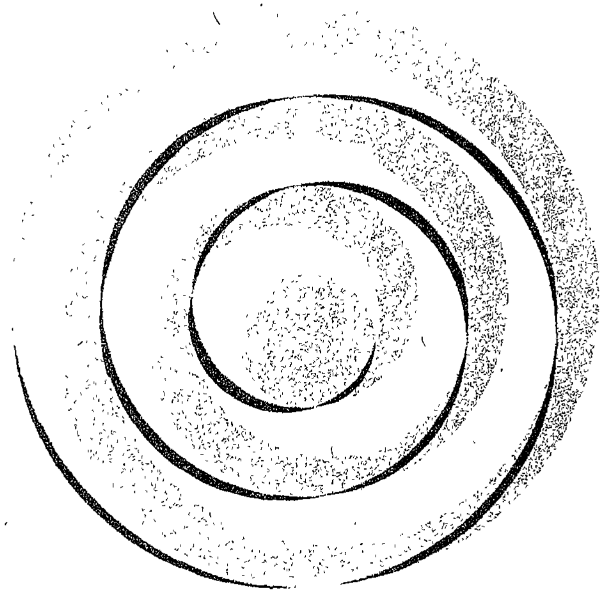

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