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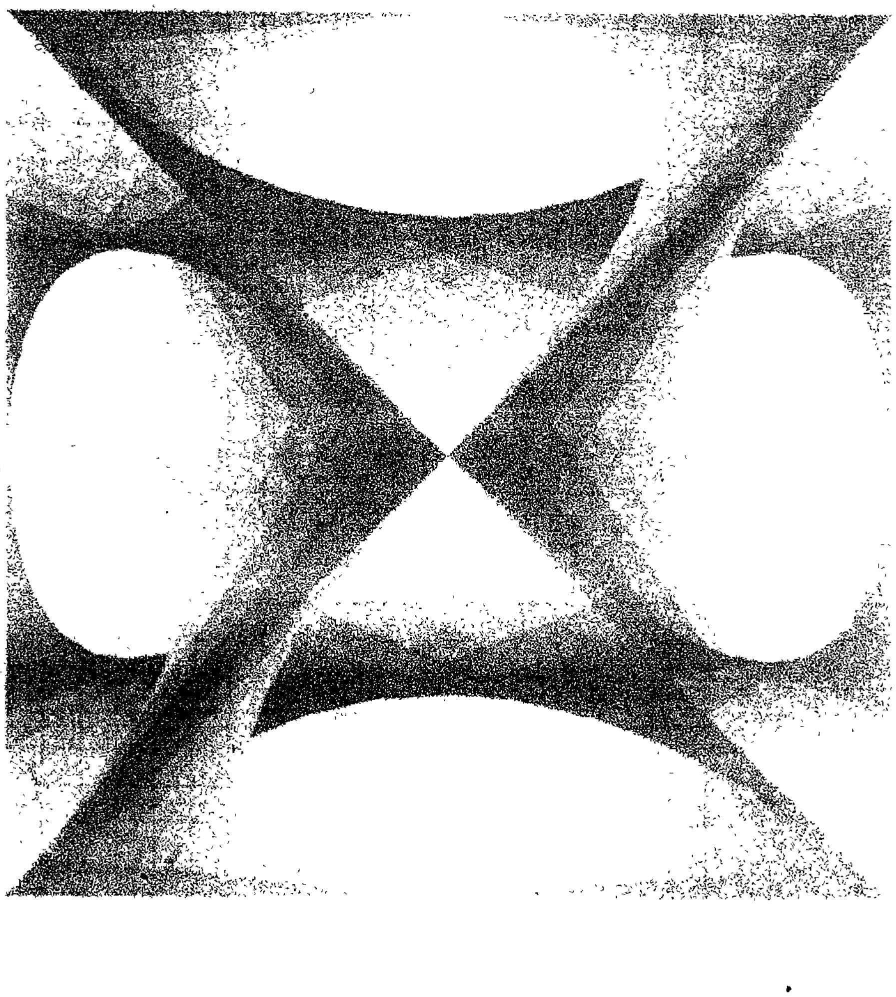

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DIGITAL EVOLUTIONS FOR REGULAR POLYHEDRA

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Visualization of geometric forms has been realized for centuries using two-dimensional representations on paper. This type of representation works well for two-dimensional shapes but is limited for three-dimensional forms. A two-dimensional representation is often only one of the many views necessary for the complete visual understanding of a three-dimensional form. Orthographic projections such as plan and elevation views are not sufficient to visualize complex shapes. Physical models provide a better understanding of three-dimensional forms (Cundy and Rollet, 1961) but they are cumbersome to construct and limited by size; sometimes a form should be seen from its inside for a complete comprehension of its spatial characteristics. Furthermore, the materials used to construct the model often distract from the properties of the "ideal" shape as geometrically defined.

Computer graphics has revolutionized the visualization process and the realization of models. As with physical models, electronic models can be visualized from any viewpoint at any position and distance, but are not limited by size and can be seen from inside as well outside.

The images shown in the each of the illustrations represent top, frontal and perspective views of computer generated models of regular polyhedra as seen from the outside; some of the models are also represented as viewed from the inside.

COMPUTER-AIDED DESIGN AND VISUALIZATION

Beyond its potential for visualization, the computer offers the possibility of generating electronic models using computer-aided design (CAD) software. This type of computer graphics application offers a quite specific data structure, characterized by the use of *instances* (Bertol, 1994). An instance can comprise any of the geometric entities, which can be generated in CAD with associated parameters such as position, scale, and rotational angle. These parameters are related to the other main engine for the development of a computer model, that represented by the *geometric transformations* (Bertol, 1994). In a CAD model each element/part subjected to repetition can be purposefully defined as instance. A model made of instances focuses on the simulated shape as a *whole* comprised of *parts.*

The content of an instance can be redefined with different geometric elements while the parameters associated with it are conserved. The change of content in one instance propagates to all the other instances bearing the same name. The redefinition of instances in a model can generate completely different geometric characteristics even if the spatial and symmetry relations between the elements defined as instances are conserved. A model made of instances can therefore *"evolve"* in completely different forms which have in common the symmetry of the initial configuration: e.g. a row of triangles with a side of 5" at a distance of 9" can be transformed into a row of spheres with a radius of 3". The use of instancing is particularly effective in the case of forms strongly characterized by symmetry, such as the regular polyhedra.

REGULAR POLYHEDRA AND SYMMETRY

The number of possible regular convex (two-dimensional) polygons - triangle, square, pentagon, hexagon, etc. - is infinite. Conversely the number of the analogous entities - regular polyhedra - in three-dimensional space is limited to five: tetrahedron, octahedron, icosahedron, cube and dodecahedron. The regular polyhedra are also called the Platonic solids, after the philosopher Plato (IV century B.C.), who thoroughly discussed the characteristics of these solids in his dialogue *Timaeus* (Plato, 1965). Many artists and mathematicians through the centuries have been fascinated with the regular polyhedra.

In each regular polyhedron all the vertices, edges and faces are equivalent, and the faces are regular polygons (Hilbert and Cohn-Vossen, 1952). The name of the polyhedron expresses the number of faces. The definition of each solid is therefore given by the type and the number of *faces*, *edges*, and *vertices* as well as their position and relations in three-dimensional space.

POLYHEDRON	VERTICES	EDGES	FACES	EDGES PER FACE	EDGES PER VERTEX
TETRAHEDRON	4	6	4	3	3
CUBE	8	12	6	4	3
OCTAHEDRON	6	12	8	3	4
DODECAHEDRON	20	30	12	5	3
ICOSAHEDRON	12	30	20	3	5

In the present discussion the emphasis is on the symmetric structure specific to each polyhedron. The regular polyhedra represent the possible regular symmetric configurations in three-dimensional space (Weyl, 1952); rotational symmetry about the center of the polyhedron rules the position of each vertex, edge and face.

The polyhedron significance goes beyond its geometric shape since it becomes a definition of compositional rules. The models shown in the illustrations follow this approach: different types of shapes replace the geometric elements defining the polyhedron, which becomes a schematic diagram embodying its potential evolutions.

POLYHEDRA AND INSTANCES

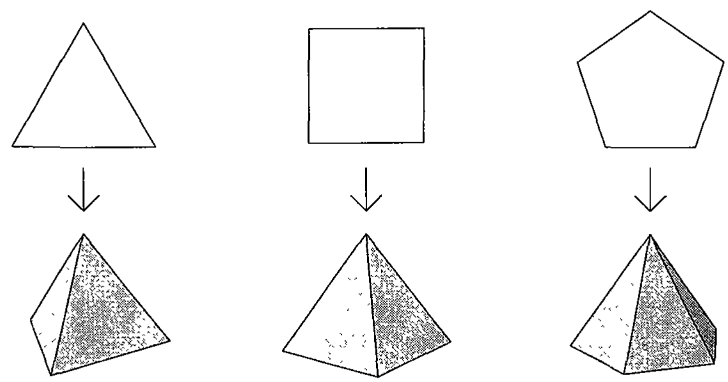
Each of the geometric elements inherent to the architecture of a regular polyhedron - vertices, edges and faces - can be defined as instance, characterized by a *position* and a *rotation* angle relative to the center of the polyhedron. The creation of a CAD database defining a polyhedron exemplifies a situation where the model definition based on the symmetry relations between elements is pregnant of many different possible transformations. The *evolutions* - as defined in previous sections - for each polyhedron depict possible formal transformations involving the change of geometric characteristics while keeping the spatial relations between faces, edges and vertices.

In the evolutions of the models of polyhedra shown in the illustrations, the geometric entities which make the original polyhedra are completely transformed. The evolved model, in some cases, does not resemble a polyhedron any longer. While the contents of the instances are changed, the symmetry relations are kept. The numbers of faces, edges and vertices defining each polyhedron are conserved as well.

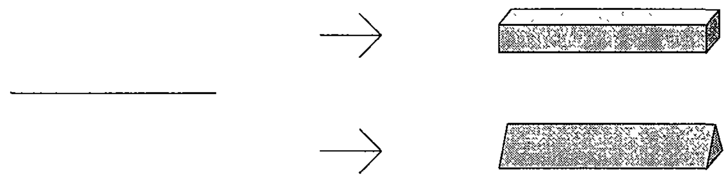
The definition of the elements which make a polyhedron in terms of instances bridges the realm of ideal geometry, made of mere geometric entities, such as points, lines and planes, to the realm of constructable realizations, made of physical materials which are subject to many constraints. For example, the polygons defining the faces of the polyhedra can be assigned a thickness, or similarly, the edges can evolve from a line to a cylinder or prism, changing their geometry characteristics of two-dimensional and uni-dimensional entity to three-dimensional objects assimilable to physical materials.

EVOLUTIONS OF THE REGULAR POLYHEDRA

One of the simplest evolutions is the transformation of each polyhedron into its stellated match (Kepler, 1619). In this transformation the faces defining the polyhedron are replaced by pyramids whose base is identical with the polygon which define each face of the polyhedron.



The "open" polyhedra, already illustrated by Leonardo da Vinci (Pacioli, 1509), can easily be evolved from the basic polyhedra by replacing the edges with prismatic elements. In this type of polyhedra the definition of enclosure, present in the regular polyhedra, no longer exists.



Further evolutions are possible from the basic models of the regular polyhedra by replacing vertices, edges and faces, with a variety of forms resembling natural and man-made elements. A well known example is Haeckel's *Kunstformen der Natur*, where many illustrations of natural forms, such as protozoa, sponges, starfishes and several plants, clearly resemble the regular polyhedra (Haeckel, 1974). In this type of "evolution", vertices, edges and faces completely loose their geometric meaning, assuming complex formal characteristics.

Beyond the stellated and open polyhedra evolutions, the models shown in the illustrations include two additional types of evolutions. In some of the polyhedra evolutions, such as those shown for the tetrahedron, cube and octahedron, the redefined geometric elements are assimilated to physical elements: vertices are replaced by joints and edges by bars. This type of model can bring insights about their physical construction and can be purposefully used to represent and investigate space frame structure.

In the other models derived by the cube and octahedron, vertices are replaced by revolution surfaces, generated by the rotation of a circular arc around the axis defined by the original vertex and the center of the polyhedron. The re-defined faces connecting the revolution surfaces create interesting enclosures: the relationship between inside and outside is completely transformed from that of the original polyhedron.

Other shapes represent ideal, gravityless structures, such as the model evolved from the dodecahedron, where vertices have been replaced by space frame towers. Both of the additional models derived by the icosahedron resemble spaceship architectures: in one model trussed arches replaces the edges while in the other the original vertices evolve in helices.

The digital models shown in the illustrations represent only some of the unlimited number of models which can be derived from the regular polyhedra. The potential presented by electronic models can offer insights in the investigation of symmetric configuration, offering a contemporary interpretation of the regular polyhedra, which have continued fascinating scientist and artist for thousands of years.

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