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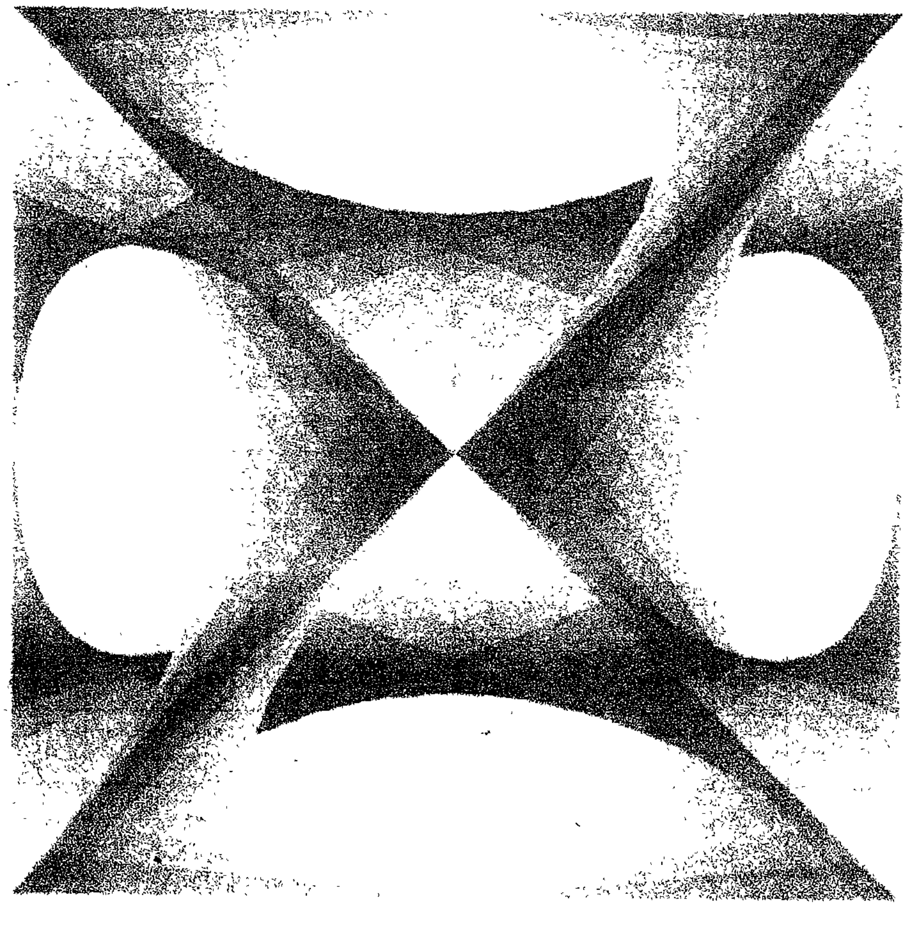

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SYMMETRY: CULTURE AND SCIENCE

GENERATION OF FRACTALS FROM INCURSIVE AUTOMATA, DIGITAL DIFFUSION AND WAVE EQUATION SYSTEMS

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Abstract: *This paper provides modelling tools for formal systems design in the field of information and physical systems. The concept and method of incursion and hyperincursion are firstly applied to the Fractal Machine, an hyperincursive cellular automata with sequential computations with exclusive OR where time plays a central role. Simulations will show the generation of fractal patterns. The computation is incursive, for inclusive recursion, in the sense that an automaton is computed at the future time t+1 in function of its neighbour automata at the present and/or past time steps but also at the future time t+1. The hyperincursion is an incursion when several values can be generated at each time step. External incursive inputs cannot be transformed to recursion. This is really a practical example of the Final Cause of Aristotle. But internal incursive inputs defined at the future time can be transformed to recursive inputs by self-reference defining then a self-referential system. A particular case of self-reference with the Fractal Machine shows a non deterministic hyperincursive field. The concepts of incursion and hyperincursion can be related to the theory of hypersets where a set includes itself. Secondly, the incursion is applied to generate fractals with different scaling symmetries. This is used to generate the same fractal at different scales like the box counting method for computing a fractal dimension. The simulation of fractals with an initial condition given by pictures is shown to be a process similar to a hologram. Interference of pictures with some symmetry gives rise to complex patterns. This method is also used to generate fractal interlacing. Thirdly, it is shown that fractals can also be generated from the digital equations of diffusion and wave, that is to say from the modulo N of their finite difference equations with integer coefficients.*

1. INTRODUCTION

The *recursion* consists of the computation of the future value of the variable vector $X(t+1)$ at time $t+1$ from the values of these variables at present and/or past times, t, $t-1$, $t-2$, ... by a recursive function :

$$X(t+1) = f(X(t), X(t-1), \ldots, p)$$

where p is a command parameter vector. So, the past always determines the future, the present being the separation line between the past and the future.

Starting from cellular automata, the concept of Fractal Machines was proposed in which composition rules were propagated along paths in the machine frame. The computation is based on what I called "INclusive reCURSION", i.e. INCURSION (Dubois, 1992a-b). An incursive relation is defined by:

$$X(t+1) = f(..., X(t+1), X(t), X(t-1), ..., p)$$

which consists in the computation of the values of the vector $X(t+1)$ at time $t+1$ from the values $X(t-i)$ at time $t-i$, $i=1, 2, ...$, the value $X(t)$ at time t and the value $X(t+j)$ at time $t+j$, $j=1, 2, ...$ in function of a command vector p. This incursive relation is not trivial because future values of the variable vector at time steps $t+1$, $t+2$, ... must be known to compute them at the time step $t+1$.

In a similar way to that in which we define hyper recursion when each recursive step generates multiple solutions, I define HYPERINCURSION. Recursive computational transformations of such incursive relations are given in Dubois and Resconi (1992, 1993a-b).

I have decided to do this for three reasons. First, in relativity theory space and time are considered as a four-vector where time plays a role similar to space. If time t is replaced by space s in the above definition of incursion, we obtain

$$X(s+1) = f(..., X(s+1), X(s), X(s-1), ..., p)$$

and nobody is astonished: a Laplacean operator looks like this. Second, in control theory, the engineers control engineering systems by defining goals in the future to compute their present state, similarly to our human anticipative behaviour (Dubois, 1996a-b). Third, I wanted to try to do a generalisation of the recursive and sequential Turing Machine in looking at space-time cellular automata where the order in which the computations are made is taken into account with an inclusive recursion.

We have already proposed some methods to realise the design of any discrete systems with an extension of the recursion by the concept of incursion and hyperincursion based on the Fractal Machine, a new type of Cellular Automata, where time plays a central role. In this framework, the design of the model of any discrete system is based on incursion relations where past, present and future states variables are mixed in such a way that they define an indivisible wholeness invariant. Most incursive relations can be transformed in different sets of recursive algorithms for computation. In the same way, the hyperincursion is an extension of the hyper recursion in which several different solutions can be generated at each time step. By the hyperincursion, the Fractal Machine could compute beyond the theoretical limits of the Turing Machine (Dubois and Resconi, 1993a-b). Holistic properties of the hyperincursion are related to the Golden Ratio with the Fibonacci Series and the Fractal Golden Matrix (Dubois and Resconi, 1992). An incursive method was developed for the inverse problem, the Newton-Raphson method and an application in robotics (Dubois and Resconi, 1995). Control by incursion was applied to feedback systems (Dubois and Resconi, 1994). Chaotic recursions can be synchronised by incursion (1993b). An incursive control of linear, non-linear and chaotic systems was proposed (Dubois, 1995a, Dubois and Resconi, 1994, 1995). The hyperincursive discrete Lotka-Volterra equations have orbital stability and show the emergence of chaos (Dubois, 1992). By linearisation of this non-linear system, hyperincursive discrete harmonic oscillator equations give stable oscillations and discrete solutions (Dubois, 1995). A general theory of stability by incursion of discrete equations systems was developed with applications to the control of the numerical instabilities of the difference equations of the Lotka-Volterra differential equations as well as the control of the fractal chaos in the Pearl-Verhulst equation (Dubois and Resconi, 1995). The incursion harmonic oscillator shows eigenvalues and wave packet like in quantum mechanics. Backward and forward velocities are defined in this incursion harmonic oscillator. A connection is made between incursion and relativity as well as the electromagnetic field. The foundation of a hyperincursive discrete mechanics was proposed in relation to the quantum mechanics (Dubois and Resconi, 1993b, 1995).

This paper will present new developments and will show that the incursion and hyperincursion could be a new tool of research and development for describing systems where the present state of such systems is also a function of their future states. The anticipatory property of incursion is an incremental final cause which could be related to the Aristotelian Final Cause.

2. INCURSION AND ARISTOTLE'S FINAL CAUSE

Aristotle identified four explicit categories of causation: 1. Material cause; 2. Formal cause; 3. Efficient cause; 4. Final cause. Classically, it is considered that modern physics and mechanics only deal with efficient cause and biology with material cause. Robert Rosen (1986) gives another interpretation and asks why a certain Newtonian mechanical system is in the state (phase) [$x(t)$ (position), $v(t)$ (velocity)]:

1. Aristotle's "material cause" corresponds to the initial conditions of the system [$x(0)$, $v(0)$] at time t=0.
2. The current cause at the present time is the set of constraints which convey to the system an "identity", allowing it to go by recursion from the given initial phase to the latter phase, which corresponds to what Aristotle called formal cause.
3. What we call inputs or boundary conditions are the impressed forces by the environment, called efficient cause by Aristotle.

As pointed out by Robert Rosen, the first three of Aristotle's causal categories are tacit in the Newtonian formalism: "*the introduction of a notion of final cause into the Newtonian picture would amount to allowing a future state or future environment to affect change of state in the present, and this would be incompatible with the whole Newtonian picture. This is one of the main reasons that the concept of Aristotelian finality is considered incompatible with modern science.*
In modern physics, Aristotelian ideas of causality are confused with determinism, which is quite different. ... That is, determinism is merely a mathematical statement of functional dependence or linkage. As Russell points out, such mathematical relations, in themselves, carry no hint as to which of their variables are dependent and which are independent."

The final cause could impress the present state of evolving systems, which seems a key phenomenon in biological systems so that the classical mathematical models are unable to explain many of these biological systems. An interesting analysis of the Final Causation was made by Ernst von Glasersfeld (1990). The self-referential fractal machine shows that the hyperincursive field dealing with the final cause could be also very important in physical and computational systems. The concepts of incursion and hyperincursion deal with an extension of the recursive processes for which future states can determine present states of evolving systems. Incursion is defined as invariant functional relations from which several recursive models with interacting variables can be constructed in terms of diverse physical structures (Dubois & Resconi, 1992, 1993b). Anticipation, viewed as an Aristotelian final cause, is of great importance to explain the dynamics of systems and the semantic information (Dubois, 1996a-b). Information is related to the meaning of data. It is important to note that what is usually called Information Theory is only a communication theory dealing with the communication of coded data in channels between a sender and a receptor without any reference to the semantic aspect of the messages. The meaning of the message can only be understood by the receiver if he has the same cultural reference as the sender of the message and even in this case, nobody can be sure that the receiver understands the message exactly as the sender. Because the message is only a sequential explanation of a non-communicable meaning of an idea in the mind of the sender which can be communicated to the receiver so that a certain meaning emerges in his mind. The meaning is relative or subjective in the sense that it depends on the experiential life or imagination of each of us. It is well-known that the semantic information of signs (like the coding of the signals for traffic) are the same for everybody (like having to stop at the red light at a cross roads) due to a collective agreement of their meaning in relation to actions. But the semantic information of an idea, for example, is more difficult to codify. This is perhaps the origin of creativity for which a meaning of something new emerges from a trial to find a meaning for something which has no a priori meaning or a void meaning.

Mind dynamics seems to be a parallel process and the way we express ideas by language is sequential. Is the sequential information the same as the parallel information? Let us explain this by considering the atoms or molecules in a liquid. We can calculate the average velocity of the particles from in two ways. The first way is to consider one particular particle and to measure its velocity during a certain time. One obtains its mean velocity which corresponds to the mean velocity of any particle of the liquid. The second way is to consider a certain number of particles at a given time and to measure the velocity of each of them. This mean velocity is equal to the first mean velocity. So there are two ways to obtain the same information. One by looking at one particular element along the time dimension and the other by looking at many elements at the same time. For me, explanation corresponds to the sequential measure and understanding to the parallel measure. Notice that ergodicity is only available with simple physical systems, so in general we can say that there are distortions between the sequential and the parallel view of any phenomenon. Perhaps the brain processes are based on ergodicity: the left hemisphere works in a sequential mode while the right hemisphere works in a parallel mode. The left brain explains while the right brain understands. The two brains are complementary and necessary.

Today computer science deals with the "left computer". Fortunately, the informaticians have invented parallel computers which are based on complex multiplication of Turing Machines. It is now the time to reconsider the problem of looking at the "right computer". Perhaps it will be an extension of the Fractal Machine (Dubois & Resconi, 1993a).

I think that the sequential way deals with the causality principle while the parallel way deals with a finality principle. There is a paradox: causality is related to the successive events in time while finality is related to a collection of events at a simultaneous time, i.e. out of time.

Causality is related to recursive computations which give rise to the local generation of patterns in a synchronic way. Finality is related to incursive or hyperincursive symmetry invariance which gives rise to an indivisible wholeness, a holistic property in a diachronic way. Recursion (and Hyper recursion) is defined in the Sets Theory and Incursion (and Hyperincursion) could be defined in the new framework of the Hypersets Theory (Aczel, 1987; Barwise, Moss, 1991).

If the causality principle is rather well acknowledged, a finality principle is still controversial. It would be interesting to re-define these principles. Causality is defined for sequential events. If $x(t)$ represents a variable at time t, a causal rule $x(t+1) = f(x(t))$ gives the successive states of the variable x at the successive time steps t, $t+1$, $t+2$, ... from the recursive function $f(x(t))$, starting with an initial state $x(0)$ at time $t=0$. Defined like this, the system has no degrees of freedom: it is completely determined by the function and the initial condition. No new things can happen for such a system: the whole future is completely determined by its past. It is not an evolutionary system but a developmental system. If the system tends to a stable point, $x(t+1) = x(t)$ and it remains in this state for ever. The variable x can represent a vector of states as a generalisation.

In the same way, I think that determinism is confused with predictability, in modern physics. The recent fractal and deterministic chaos theory (Mandelbrot, 1982; Peitgen, Jürgens, Saupe, 1992) is a step beyond classical concepts in physics. If the function is non-linear, chaotic behaviour can appear, what is called (deterministic) chaos. In this case, determinism does not give an accurate prediction of the future of the system from its initial conditions, what is called sensitivity to initial conditions. A chaotic system loses the memory of its past by finite computation. But it is important to point out that an average value, or bounds within which the variable can take its values, can be known; it is only the precise values at the successive steps which are not predictable. The local information is unpredictable while the global symmetry is predictable. Chaos can presents a fractal geometry which shows a self-similarity of patterns at any scale.

A well-known fractal is the Sierpinski napkin. The self-similarity of patterns at any scale can be viewed as a symmetry invariance at any scale. An interesting property of such fractals is the fact that the final global pattern symmetry can be completely independent of the local pattern symmetry given as the initial condition of the process from which the fractal is built. The symmetry of the fractal structure, a final cause, can be independent of the initial conditions, a material cause. The formal cause is the local symmetry of the generator of the fractal, independently of its material elements and the efficient cause can be related to the recursive process to generate the fractal. In this particular fractal geometry, the final cause is identical to the final cause. The efficient cause is the making of the fractal and the material cause is just a substrate from which the fractal emerges but this substrate doesn't play a role in the making.

3. THE HYPERINCURSIVE FRACTAL MACHINE

A one-dimensional network of cellular automata (Feynman, 1982; Gardner, 1971; Schroeder, 1991; Weisbuch, 1989; Wolfram, 1994, Zuse, 1969) is represented by a vector of automata states, each automaton state having an integer numerical value at the initial time $t=0$. A set of rules defines how the states change at every clock time. A simple rule consists of replacing the value of each automaton by the sum of itself and its left neighbour at each clock time. Figure 1 shows a one-dimensional network of cellular automata giving rise to the Pascal triangle. The recursive model of the Pascal triangle network is

$$(1) \qquad X(n, t+1) = X(n, t) + X(n-1, t)$$

with t=0, 1, 2, ... and n=1, 2, ..., starting with initial conditions $X(n, 0)$, n=1, 2, ... at time t=0 and boundary conditions $X(0, t)$ at each time step t=1, 2, ...

	n=0	1	2	3	4	5	6	7	8
t=0	0	1	0	0	0	0	0	0	0
t=1	0	1	1	0	0	0	0	0	0
t=2	0	1	2	1	0	0	0	0	0
t=3	0	1	3	3	1	0	0	0	0
t=4	0	1	4	6	4	1	0	0	0
t=5	0	1	5	10	10	5	1	0	0
t=6	0	1	6	15	20	15	6	1	0
t=7	0	1	7	21	35	35	21	7	1
t=8	0	1	8	28	56	70	56	28	8

Figure 1: The Pascal triangle generated by the recursive eq. 1

The recursive equation 1 can be reversed in replacing t+1 by t–1

(1a) $X(n, t-1) = X(n, t) + X(n-1, t)$

In making a time translation, replacing t by t+1, one obtains

(1b) $X(n, t+1) = X(n, t)) - X(n-1, t+1)$

This eq. 1b can be computed in an incursive way, that is to say in a sequential order, in giving initial conditions $X(n, 0)$ and boundary conditions $X(0, t+1)$ at the future time t+1, for each time t=0, 1, 2, ... It is absolutely impossible to build a real physical systems governed by such an equation because " How to give to the system the boundary conditions with inputs defined at the future time step t+1 ? ". It will be shown below in this paper that such incursive system can work in practice if the boundary conditions are zero (no inputs) or the system defined itself its boundary conditions in a self-referential way (for example, in defining periodical boundary conditions).

But some problems can appear: uncertainty or indecidability. In such a case, a solution would be to define a purpose to the system so that the future inputs can be replaced by inputs at the present time with a feed-back process as made in cybernetics and control theory (Rosenblueth, Wiener, Bigelow, 1943; Van de Vijver, 1992).

With modulo N, the recursive eq. (1) becomes

(2) $X(n, t+1) = (X(n, t) + X(n-1, t)) \bmod N$ with t=0, 1, 2, ... and n=1, 2, ...,

with initial conditions $X(n, 0)$ and boundary conditions $X(0, t)$. With N=2, the pattern is given by the fractal Sierpinski napkin given in Figure 2a. In this recursion the present time step always determines the next future time step, even for the boundary conditions $X(0, t)$.

In the Fractal Machine (Dubois, 1992), the following incursive digital equation is defined

(3) $X(n, t+1) = (X(n, t) + X(n-1, t+1)) \bmod N$ with n=1, 2, ..., 8, and t=0, 1, 2, ...,7

where $X(n, t)$ is the automaton state at position n and time t. The modulo with N=2 is exclusive OR, XOR. The computation of eq. (3) with N=2, given at Figure 2b, gives rise to a time reverse Sierpinski napkin (Dubois, 1990,1991). Let us remark that with the modulo 2, the negative term in eq. 1b is without importance.

	n=0	1	2	3	4	5	6	7	8		n=0	1	2	3	4	5	6	7	8
t=0	0	1	0	0	0	0	0	0	0		0	1	0	0	0	0	0	0	0
t=1	0	1	1	0	0	0	0	0	0		0	1	1	1	1	1	1	1	1
t=2	0	1	0	1	0	0	0	0	0		0	1	0	1	0	1	0	1	0
t=3	0	1	1	1	1	0	0	0	0		0	1	1	0	0	1	1	0	0
t=4	0	1	0	0	0	1	0	0	0		0	1	0	0	0	1	0	0	0
t=5	0	1	1	0	0	1	1	0	0		0	1	1	1	1	0	0	0	0
t=6	0	1	0	1	0	1	0	1	0		0	1	0	1	0	0	0	0	0
t=7	0	1	1	1	1	1	1	1	1		0	1	1	0	0	0	0	0	0
t=8	0	1	0	0	0	0	0	0	0		0	1	0	0	0	0	0	0	0

Figure 2a-b: Recursive Sierpinski napkin and incursive Sierpinski napkin

3.1. Self-Referential Fractal Machine

In the Fractal Machine, if it is natural to consider the successive time steps t in the increasing order, it is also necessary to consider the successive computations in the increasing order of the number n of automata which can be considered as an internal time.

Explicitly it is possible to define two times: an external time and an internal time. The duration of the external time is the sum of the sequential computational internal times. For n=1, the future inputs $X(0, t+1)$ must be defined at each time step in view of computing the automata X(n, t+1) as a final cause which controls the dynamics of the system. In transforming eq. (3) in a quasi recursive equation system (Dubois, 1996a)

$$
\begin{aligned}
(3a)\quad & X(0, t{+}1) = \textit{external inputs} = \textit{final cause} \\
& X(1, t{+}1) = (X(1, t) + X(0, t{+}1)) \bmod N \\
& X(2, t{+}1) = (X(2, t) + X(1, t) + X(0, t{+}1)) \bmod N \\
& X(3, t{+}1) = (X(3, t) + X(2, t) + X(1, t) + X(0, t{+}1)) \bmod N \\
& \dots \\
& X(8, t{+}1) = (X(7, t) + X(6, t) + \dots + X(1, t) + X(0, t{+}1)) \bmod N
\end{aligned}
$$

it is explicitly seen that the external inputs must be defined in the future time like a final causation which controls completely all the automata at the same time step in a holistic way. Indeed the inputs $X(0, t+1)$ are present in each automata at the same external time. It is impossible to transform external inputs defined in the future time t+1 to inputs defined in the present time t. In this, we can say that we are dealing with a strict incursive system. Thus the final causation is really the 4th causation which must be taken into account in systems modelling as Aristotle had proposed. It seems also impossible to construct a real working engineering system where real working external future inputs would control its current present state. But it is possible to define internal future inputs in considering self-referential systems.

For example, in taking the following initial conditions $X(n, 0)=0$, $n=0, \ldots, 8$ and boundary conditions $X(0, t+1)=X(8, t+1)$, $t=0, 1, 3$, it is shown in Figure 2c that there are two solutions at each time step.

	n=0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
t=0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t=1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
t=2	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1

Figure 2c· Uncertainty in the self-referential fractal machine.

Indeed if $X(0, 1)=1$ then $X(8, 1)=1$ and if $X(0,1)=0$ then $X(8,1)=0$. Thus, this is an hyperincursive system because we have two different solutions at each time step. Moreover in some cases, contradiction can appears. For example, starting with the following different initial conditions at time $t=0$ given in Figure 2d in taking $X(0, 1)=1$ then $X(8, 1)=0$ and if $X(0, 1)=0$ then $X(8, 1)=1$. This case could be resolved in deciding that $X(0, t+1)=1-X(8, t+1)$, then the first example will give a contradiction. The Fractal Machine can become non deterministic or non algorithmic, what I suggest to call an HYPERINCURSIVE FIELD where uncertainty (indecidability) or contradiction (exclusion principle) occur (Dubois, 1996a).

	n=0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
t=0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
t=1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Figure 2d: Contradiction in the self-referential fractal machine.

4. GENERATION OF FRACTALS FROM INCURSIVE AUTOMATA

Lets us consider numerical simulations on computer of a few incursive automata.
Figures 3a-b give the simulation of the incursive automata given by eq. (3) for $N=2$ and $N=3$, the initial conditions and boundary conditions are the same as in Figure 2-b.

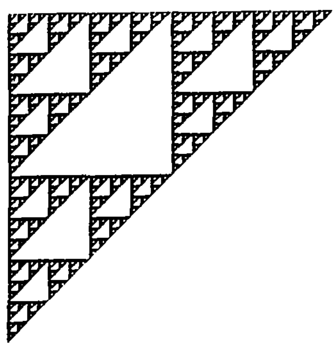


Figure 3a: Simulation of hyperincursive equation (3) with N=2.

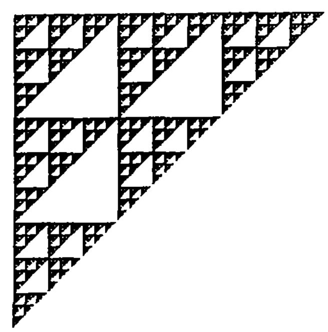


Figure 3b: Simulation of hyperincursive equation (3) with N=3.

Figures 4a-b show an other Sierpinski napkin and a fractal pattern from the incursive relation

(4) $X(n, t+1) = (X(n-1, t-1) + X(n-2, t+1)) \bmod 2$

where n=2, 3, .. and t=1, 2, ... The boundary conditions are 0's. Two different initial conditions at time t=1 are considered:

(4a) $X(n, 1)=0$ for n=0,1,2,...

(4b) $X(11.p,1)=1$ for p=0,1,2,...

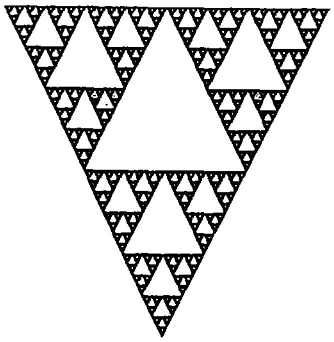


Figure 4a: Simulation of hyperincursive equation 4 with initial condition 4a.

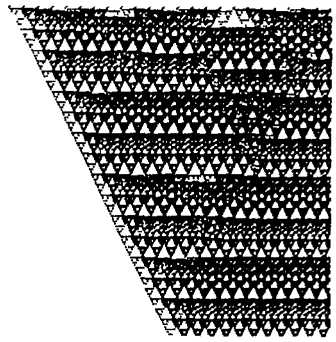


Figure 4b: Simulation of hyperincursive equation 4 with initial condition 4b.

The following incursive relation depending on three automata

(5) $X(n, t+1) = (X(n, t) + X(n-1, t+1) + X(n-1, t)) \bmod 3$

with n=1, 2, ... and t=0, 1, 2, ... generates a square fractal given in Figure 5a with the initial condition $X(1, 0)=1$ and the boundary conditions $X(0, t)=0$ for t=0, 1, 2, ...

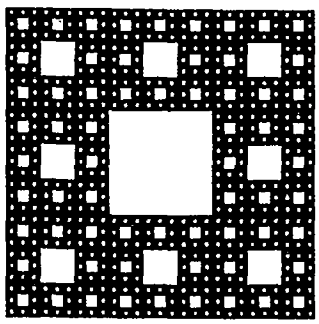


Figure 5a: Simulation of hyperincursive equation (5) giving a square fractal.

Similarly with the Pascal Triangle, an incursive square with modulo 3 can be generated as follows:

	n=0	1	2	3	4	5	6	7	8	9
t=0	0	0	0	0	0	0	0	0	0	0
t=1	0	1	1	1	1	1	1	1	1	1
t=2	0	1	0	2	1	0	2	1	0	2
t=3	0	1	2	1	1	2	1	1	2	1
t=4	0	1	1	1	0	0	0	2	2	2
t=5	0	1	0	2	0	0	0	2	0	1
t=6	0	1	2	1	0	0	0	2	1	2
t=7	0	1	1	1	2	2	2	1	1	1
t=8	0	1	0	2	2	0	1	1	0	2
t=9	0	1	2	1	2	1	2	1	2	1

Figure 5b: Incursive square fractal generated by eq. 5, starting with X(1,1)=1. This fractal pattern is similar to the Sierpinski carpet in considering the 0's and non 0's pixels.

The time reverse of the eq. 5 is given by

(5a) $X(n, t+1) = (X(n, t) - X(n-1, t+1) - X(n-1, t)) \bmod 3$

which is again an incursive equation with the same terms with different signs. Contrary to the Sierpinski gasket, the time reverse of this square fractal is not a recursive equation, but another incursive one (see Figure 5c). Let us recall that a definition of modulo 3 for negative values is: $(-1) \bmod 3=2$; $(-2) \bmod 3=1$.

	n=0	1	2	3	4	5	6	7	8	9
t=0	0	0	0	0	0	0	0	0	0	0
t=1	0	1	2	1	2	1	2	1	2	1
t=2	0	1	0	2	2	0	1	1	0	2
t=3	0	1	1	1	2	2	2	1	1	1
t=4	0	1	2	1	0	0	0	2	1	2
t=5	0	1	0	2	0	0	0	2	0	1
t=6	0	1	1	1	0	0	0	2	2	2
t=7	0	1	2	1	1	2	1	1	2	1
t=8	0	1	0	2	1	0	2	1	0	2
t=9	0	1	1	1	1	1	1	1	1	1

Figure 5c: Incursive Sierpinski carpet starting from X(1,1)=1. The 0's are exactly at the same position in the direct and reverse time incursive process.

The direct and reverse times patterns show a reflection symmetry for each space-time (n, t) value. The symmetry of the whole pattern is the symmetry of the elementary part (the generator), that is to say a self-similarity as shown in Figure 5d. Indeed, the symmetry is space-time scale invariant in considering units in 3^m, $m=1, 2, 3, \ldots$

	n=1	2	3			n=1	4	7
t=1	1	2	1		t=1	1	2	1
t=2	1	0	2		t=4	1	0	2
t=3	1	1	1		t=7	1	1	1

Figure 5d: Elementary square fractal as the generator and self-symmetry at space-time scale Dn=3 and Dt=3, giving a self-similar pattern characterizing a fractal.

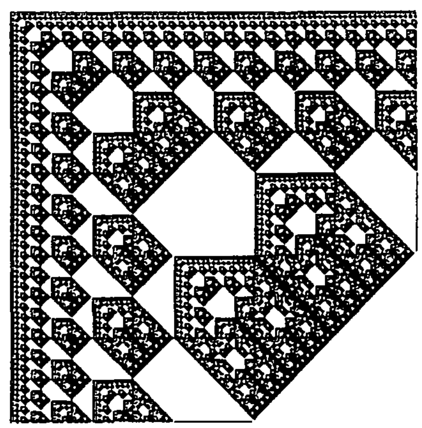
The two diagonals are identical and represent the Cantor set in one dimension:

1 0 1 0 0 0 1 0 1

Figure 5e: Cantor set in one dimension given by the diagonals of the square fractal.

Let us remark that the square fractal is similar to the Sierpinski carpet in the sense that the zeros are at the same places. In the classical Sierpinski carpet all non zero values are given by the same value 1. In this incursive square fractal, it is necessary to consider 1 and 2, that is to say the modulo 3. With modulo 2, equation 5 gives rise to a uniform Euclidean pattern of dimension $D=2$.

Figure 6 shows a Pentagon fractal generated by the incursive equation with four automata



Figures 6: Simulation of hyperincursive equation (6) showing a pentagon fractal.

$$X(n, t+1) = (X(n, t) + X(n-1, t+1) + X(n-1, t-1) + X(n-2, t)) \bmod 2 \quad (6)$$

with $n=1, 2, \ldots$ and $t=0, 1, 2, \ldots$ with the initial condition $X(1, 0)=1$ and the boundary conditions $X(0, t)=0$ for $t=0, 1, 2, \ldots$

5. HOLOGRAPHIC GENERATION OF FRACTALS AT DIFFERENT SCALES

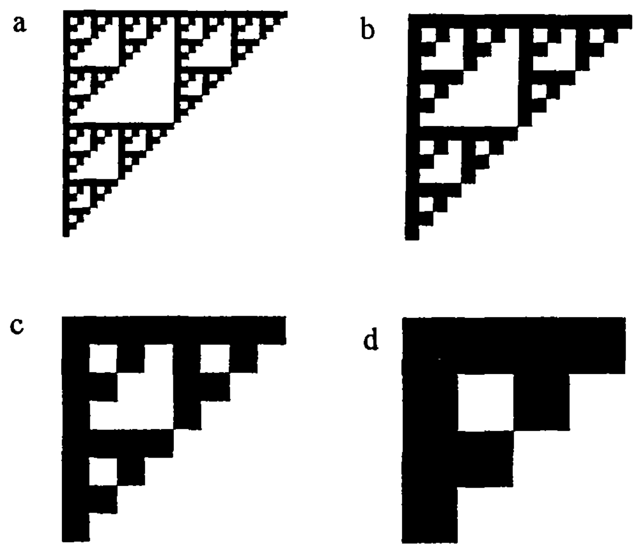
Different patterns at different scales can be generated in the following way. Let us take the following equation which will gives the Sierpinski gasket (N=2) in different space scales:

(7) $X(n, m, t+\Delta t) = (X(n, m, t) + X(n-\Delta n, m, t+\Delta t) + X(n, m-\Delta m, t+\Delta t)) \bmod N$

with the first space parameter n=1, 2, ..., for each successive second space parameter m=1, 2, ... in function of the time t=0, 1, 2, ... The space steps Δn and Δm define the scale at which the fractal is generated, starting with an initial condition given by a picture. If the picture is a black square

(7a) $X(n, m, 0) = 1$ for n=1 to Δn and m=1 to Δm

different Sierpinski gaskets are generated in one time step, t=0, with space steps given, by example, by $\Delta n=2^p$, p=0, 1, 2, 3 and $\Delta m=2^q$, q=0, 1, 2, 3, as shown in Figures 7a-b-c-d.

a 

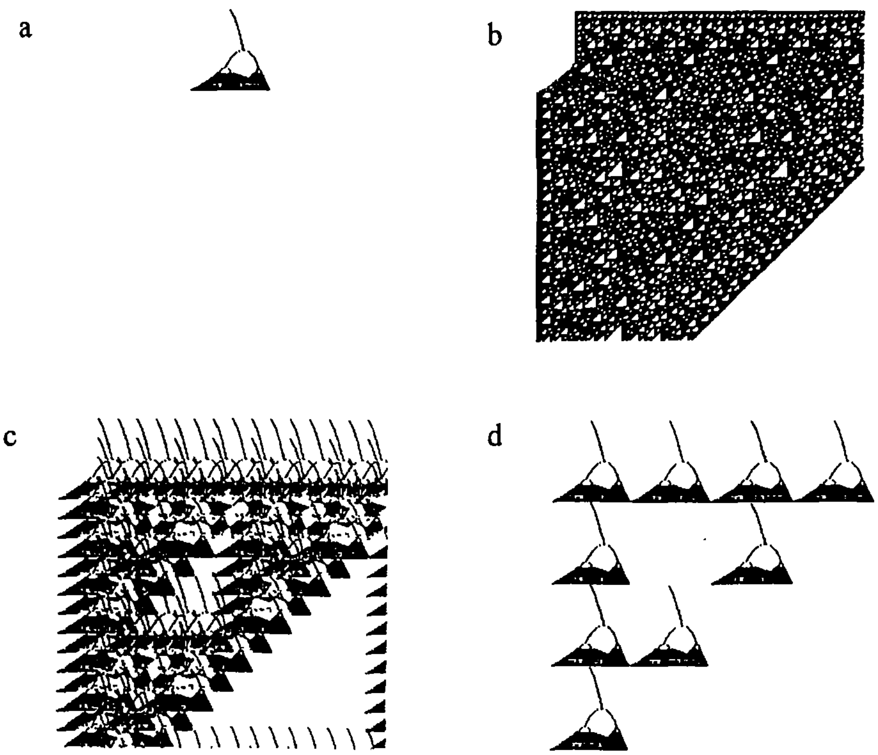
Figures 7a-b-c-d

With this incursive process 7, the fractal dimension is easy to compute by "Box counting" (see for example Peitgen, Jürgens, Saupe, 1992). When $p=q$, with the same initial condition 7a, the same fractals can be generated from

(7b) $X(n, m, t+\Delta t) = (X(n, m, t) + X(n-1, m, t+\Delta t) + X(n, m-1, t+\Delta t)) \bmod N$

in several time steps given respectively by t=0 to 2^p, p=0, 1, 2, 3. This is possible by the symmetry property of this Sierpinski fractal, that is to say the self-similarity at scales 2^p.

Starting with an initial condition given a particular picture, instead of the black square, this three-dimensional equation 7b was already simulated (see Dubois, 1992, 1995). The basic evolution of the system is the multiplication of the initial picture given as initial condition through the whole frame with order/chaos transitions (the chaos transitions are obtained for odd time steps) and then their fusion by interference. The process is similar to a hologram but with the interference of the multiple identical pictures. Lets us show that the same holographic effect can be obtained directly in one time step from eq. 7 in taking, for example, the space steps Δt=1, Δt=15 and Δt=64 as shown in Figures 8b-c-d, the Figure 8- giving the initial picture.

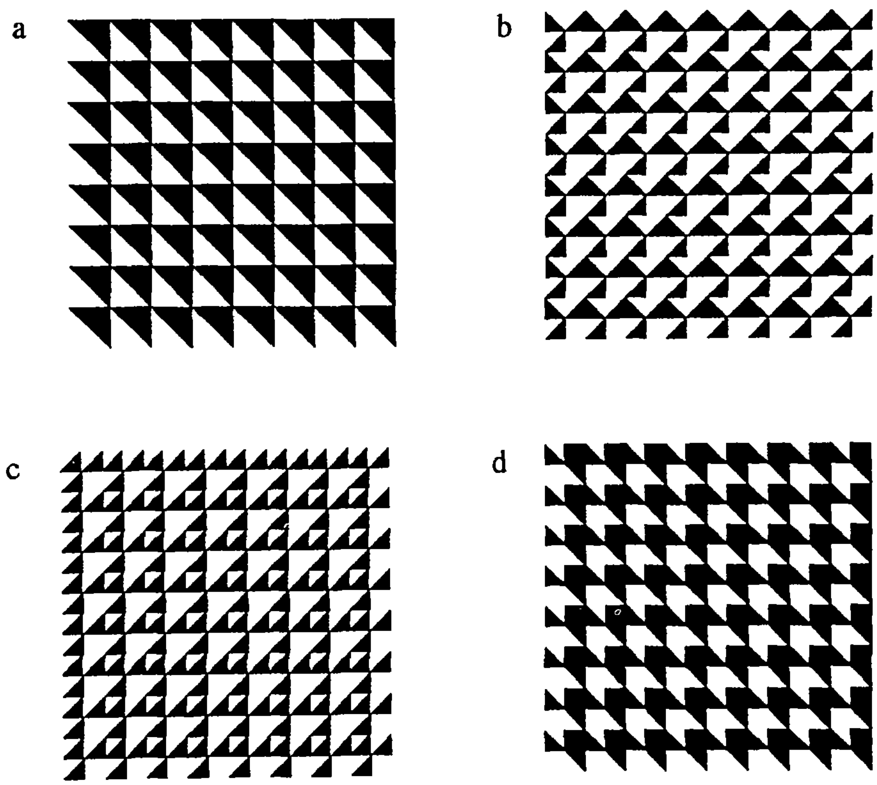


Figures 8a-b-c-d: Holographic generation of fractals with chaos-order transitions

The same method can be used for any other fractals. For example, from the following equation

(8) $X(n, m, t+\Delta t) = (X(n, m, t) + X(n-\Delta n, m, t+\Delta t) + X(n, m-\Delta m, t+\Delta t) + X(n-\Delta n, m-\Delta m, t+\Delta t)) \bmod N$

the square fractal is obtained for N=3. For N=2, with a black square as initial condition, the pattern is a *uniform* pattern of *dimension* D=2. With an initial condition given by a black and white pattern with a certain symmetry, the Figures 9a-b-c-d give several simulations.



Figures 9a-b-c-d: Different patterns from eq. 8 with symmetrical initial conditions.

6. GENERATION OF FRACTAL INTERLACING

Let us consider again eq. 7

(9) $X(n, m, t+\Delta t) = (X(n, m, t) + X(n-\Delta n, m, t+\Delta t) + X(n, m-\Delta m, t+\Delta t)) \bmod N$

with the following initial condition

(9b) $X(n, 1, 0)=1$ for $n=\Delta n+1, \Delta n+2, \ldots$ and $X(1, m, 0)=1$ for $m=\Delta m+1, \Delta m+2, \ldots$

a Sierpinski interlacing can be generated as shown in Figure 10a, for $p=q=3$.

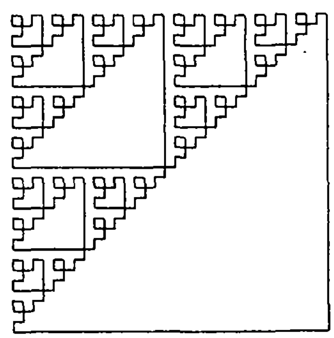


Figure 10a: Generation of a Sierpinski interlacing.

This is really an interesting result because a Sierpinski gasket can be generated with a unique line (this line is given by successive 1's and has sometimes one value 0 in changing of direction) travelling in the space (n,m). The fractal dimension of this interlacing is $D=1$ for space scales $(n, m) < (2^p, 2^q)$, and $D=\log3/\log2$ for larger scales.

With the same method, many other fractal interlacing can be generated.

Figure 10b was generated from the equation

(10) $X(n, m, t+1) = (X(n, m, t) + X(n, m-4, t+1) + X(n-4, m-4, t+1) + X(n-8, m, t+1)) \bmod 2$
(10a) $X(n, 1, 0)=1$ for $n=16, 17, 18, \ldots$ and $X(1, m, 0)=1$ for $m=16, 17, 18 \ldots$

and Figure 10c, from the equation

(11) $X(n, m, t+1) = (X(n, m, t) + X(n,m-16, t+1) + X(n-12, m-16, t+1) + X(n-16, m-12, t+1) + X(n-16, m, t+1)) \bmod 2$
(11a) $X(n, 1, 0))=1$ for $n=16, 17, 18, \ldots$ and $X(1, m, 0)=1$ for $m=16, 17, 18, \ldots$

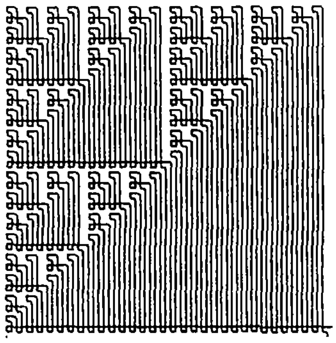


Figure 10b: Fractal interlacing generated from equation 10 with initial condition 10a

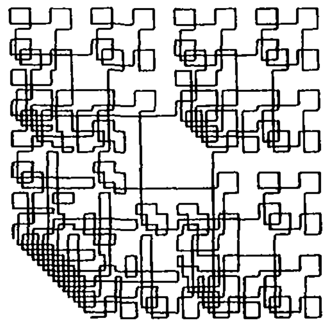


Figure 10c: Pentagon fractal interlacing generated from equation 11 with initial condition 11a

The conclusion of this section on interlacing is that the local rule given by incursive equations give rise to a global pattern exhibiting only one simple line travelling in a very complex way through the whole space.

7. FRACTALS FROM DIGITAL DIFFUSION EQUATION

Many physical, chemical or biological systems deal with diffusive reactions systems. Let us show that fractal can be generated from a digital diffusion equation and that the incursive diffusion equation can be algorithmically deterministic with a self-reference for the inputs defined at future time. Indeed, let us consider the one dimension diffusion equation

(12) $\partial x(s, t)/\partial t = a.x(s, t) + D.\partial^2 x(s, t)/\partial s^2$

With a discrete forward time derivative,

(13a) $D_f(x(s, t))=((x(s, t+\Delta t) - x(s, t))/\Delta t$

the discrete diffusion equation

(12a) $x(s, t+\Delta t)-(s, t) = -a.\Delta t.x(s, t) + D.\Delta t.[x(s+\Delta s, t)-2.x(s, t) + x(s-\Delta s, t)]/\Delta s^2$

gives unstable solution for integer parameters. In defining a diffusion difference equation modulo N, with a=0, D=1 and Δt=Δs=1, the following "digital" diffusion equation gives a fractal pattern shown in figure 11a (Dubois, 1996a)

(12b) $x(s, t+1) = [-x(s, t) + x(s+1, t) + x(s-1, t)] \bmod N$

The term "digital" was proposed by Konrad Zuse (1982). With a=1, D=1 and Δt=Δs=1, the following digital equation gives the Sierpinski fractal pattern given in Fig. 11b (Dubois, 1996a)

(12c) $x(s, t+1) = [x(s+1, t) + x(s-1, t)] \bmod N$

```
   s= 01234567...            01234567...

t=0   00000000100000000    00000000100000000
t=1   00000001110000000    00000001010000000
      00000010101000000    00000010001000000
      00000110101100000    00000101010100000
      00001000100010000    00001000000010000
      00011101110111000    00010100000101000
      00101000100010100    00100010001000100
      01101101110110110    01010101010101010
      10000000100000001    10000000000000001
```

Figure 11a-b: Generation of fractal patterns by eqs. (10b-c) with N=2.

The numerical simulations of eq. 12b with N=2 and N=3 are given in Figures 12a-b. They give very special symmetries.

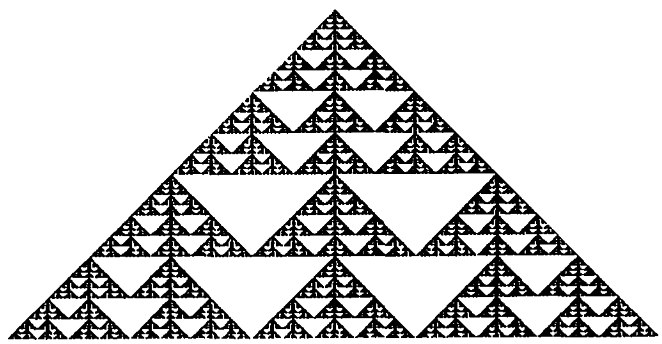


Figure 12a: Fractal generated from digital diffusion eq. 12b with N=2.

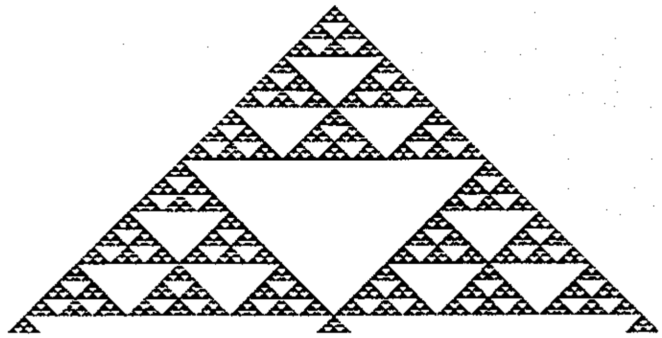


Figure 12b: Fractal generated from digital diffusion eq. 12b with N=3.

Another way to avoid numerical instabilities is to use the implicit finite difference method (see for example Scheid (1986)) in considering the discrete backward time derivative:

(13b) $\Delta_b(x(s, t)) = (x(s, t) - x(s, t-\Delta t))/\Delta t.$

Let us remark that the backward derivative is the time reverse of the forward derivative in replacing Δt by $=\Delta t$.

With a backward time derivative, the finite difference equation obtained from equation 12, with $a=0$, is then given by the incursive equation (Dubois, 1996a)

(14) $x(s, t+1) = x(s, t) + D.[x(s-1, t+1) - 2.x(s, t+1) + x(s+1, t+1)]$

or

(14b) $x(s, t+1) = x(s, t)/[1+2.D] + [D/[1+2.D]][x(s-1, t+1) + x(s+1, t+1)]$

To compute the value of $x(s, t+1)$, we must know the value of $x(s-1, t+1)$ as in the Fractal Machine, but also $x(s+1, t+1)$.

(14c)
$$\begin{aligned}
&[1+2.D].x(1, t+1) - D.[x(0, t+1) + x(2, t+1)] = x(1, t)\\
&[1+2.D].x(2, t+1) - D.[x(1, t+1) + x(3, t+1)] = x(2, t)\\
&[1+2.D].x(3, t+1) - D.[x(2, t+1) + x(4, t+1)] = x(3, t)\\
&\ldots\\
&[1+2.D].x(S, t+1) - D.[x(S-1, t+1) + x(S+1, t+1)] = x(S, t)
\end{aligned}$$

that is to say with the matrix form $A.x(t+1)=x(t)$. It looks like the Fractal Machine where each iterate $x(s, t+1)$ is propagated to its two space adjacent iterates $x(s-1, t+1)$ and $x(s+1, t+1)$. In the Fractal Machine, the iterates are only propagated in one direction. In inverting the matrix A, we obtain a recursive system $x(t+1)=A^{-1}.x(t)$. In the literature, this procedure is called "implicit recursion" which justifies the name of "incursion for inclusive or implicit recursion" I proposed. Indeed, it is the inclusion of each iterate in the others which defines them in a self-reference way. In taking periodic boundary conditions $x(0, t+1)=x(S, t+1)$ and $x(S+1, t+1)=x(1, t+1)$, the system defines itself the values of the boundaries. For example, with a constant diffusion $D=1$ and $S=3$, we obtain, in inverting the matrix A, the equation system 14d. The simulation of this system in Figure 13 shows that the convergence is very rapid, a few time steps. I think that this example is a good one to explain "holistic" properties of self-referential system.

$$
\begin{aligned}
&x(1, t+1) = x(1, t)/2 + x(2, t)/4 + x(3, t)/4 \\
(14d)\quad &x(2, t+1) = x(2, t)/2 + x(3, t)/4 + x(1, t)/4 \\
&x(3, t+1) = x(3, t)/2 + x(1, t)/4 + x(2, t)/4
\end{aligned}
$$

	s= 1	2	3
t=0	1	0	0
t=1	½	¼	¼
t=2	6/16	5/16	5/16
t=3	22/64	21/64	21/64

Figure 13. Simulation of self-referential diffusion equation system 14d

Each automaton at time $t+1$ is related to itself at the preceding time t and at the future time $t+1$ and to its direct space adjacent neighbours at the future time $t+1$. Due to the self-reference of each automaton with its neighbours, it is possible to compute a new transformed recursive system where now each automaton is computed in function of itself only at the preceding time step but in function of all the automata of the system at the preceding time.

This is really an important result which shows that an incursive holistic non-local property comes from local interaction dealing with a recursive system depending on future states.

With boundary conditions as external inputs $x(0, t+1)$ and $x(4, t+1)$, the system becomes (Dubois, 1996a)

$$
\begin{aligned}
&x(1, t+1) = [8.[x(1, t) + x(0, t+1)] + 3.x(2, t) + [x(3, t) + x(4, t+1)]]/21 \\
(11e)\quad &x(2, t+1) = [3.x(2, t) + 9.[x(3, t)+x(4, t+1)] + 3.[x(1, t) + x(0, t+1)]]/21 \\
&x(3, t+1) = [[x(3, t) + x(4, t+1)] + 3.[x(1, t)+x(0, t+1)]+8.x(2, t)]/21
\end{aligned}
$$

We remark that the inputs are defined at the future time $t+1$ and they are present in the three equations. It means that the inputs at the boundaries are transmitted instantaneously, that is to say during the time step 1, to each automata, and that their effect are immediate because it is the same time step that the equations are computed. Why this phenomenon? As the movement equations are defined in the future time by the backward derivative, the inversion of the matrix A has the effect of mixing all the automata together at the present time t (and the inputs at the future time $t+1$) to compute their future values at time $t+1$. The inversion of the matrix A transforms a local incursive system to a non-local recursive system, that is to say a folding of each automaton to the other ones from the future time $t+1$ to the present time t.

It must be pointed out that classical mathematical analysis deals with derivatives defined for a unit time interval Δt tending to zero, so the backward and the forward derivatives are identical for derivable continuous systems. But the majority of physical systems seem discrete and not continuous, for example, the molecules of all actual systems are discrete entities and not a continuous medium. Newtonian mechanics is called the rational mechanics of the point, i.e. the dynamics of a Newtonian system are described by differential equations where all the objects are represented by sets of points without spatial dimension. It is well-known that Physicists have many mathematical problems with classical mathematical analysis due to the appearance of infinities. Maurice Jessel (1993) believed in the Non-Standard Analysis of Abraham Robinson as a first attempt to avoid this problem. He thought also that a Finality Principle should be established after the proposition of the French Nobel Prize winner Alfred Kastler.

So any evolving system can be defined at any time step t by its backward derivative $\Delta_b(X(t))$ or its forward derivative $\Delta_f(X(t))$: the forward derivative is related to the formal causation of Aristotle and the backward derivative, to the final causation, because the resulted equation takes into account the future time step $t+\Delta t$ (Dubois, 1995). Final causation is a potential causation because it is not yet realised. Let us remark that we defined here an incremental final causation, changing at each time step. The principle of finality or teleonomy defined classically deals with the final value of a variable when the system reaches its stationary state, i.e. when it is no longer evolving but only developing.

A mathematical model theory of evolving systems would be helpful. If evolving systems reach a stationary state (which can show a complex behaviour like a living system), the mathematical model can be simplified to deal only with its stationary state: in this case, no more information exists to explain how the system evolved to this state. It will be a developing system rather than an evolving system. The development of organisms from their birth to their death is a sub-class of the evolution of these organisms viewed at the level of species evolution.

The interaction between a system and its environment can be viewed as a whole system (Bohm, 1987). The environment views the system and the system views the (external) environment. Anticipation seems to be the rule for physical structures in living systems: how can we justify the fact that mathematical models are recursive processes based only on past events? The anticipative nature of evolving systems is difficult to observe and to model because it is included in the recursive model of the system (some incursive control can be transformed to a recursive control, although not always), i.e. in the formal cause. It was shown with the hyperincursive field that it is not always possible to change, mathematically, a final cause to a formal or efficient cause (Dubois, 1996a).

8. FRACTALS FROM DIGITAL WAVE EQUATION

Let us consider the one dimension wave equation

(15) $\partial^2 x(s, t)/\partial t^2 = -w^2.x(s, t) + c^2\partial^2 x(s, t)/\partial s^2$

where $x(s, t)$ is the value of the wave at position s at time t, c the velocity and w is the pulsation of oscillators. This differential equation can be replaced by the finite difference equation:

(15a) $x(s, t+\Delta t) - 2.x(s, t) + x(s, t-\Delta t) = -w^2.\Delta t^2.x(s, t) + c^2.\Delta t^2.[x(s+\Delta s, t) - 2.x(s, t) + x(s-\Delta s, t)]/\Delta s^2$

In taking Δs=1, Δt=1, c=1 and w=1, the following digital wave equation is obtained [a similar equation was studied with w=0 in Dubois, 1996b]:

(15b) $x(s, t+1) + x(s, t-1) = -x(s, t) + x(s+1, t) + x(s-1, t)$

In taking the modulo N of this equation, the following digital wave equation is obtained:

(15c) $x(s, t+1) = [-x(s, t) - x(s, t-1) + x(s+1, t) + x(s-1, t)] \bmod N$

The Figures 14a-b show the numerical simulations of this equation for N=2 and N=3, with the initial condition

(15d) $x(255, 0) = +1$ and $x(257, 0) = -1$

where s=1 to 512 and t=1, 2, ... The first one gives a particular new fractal and the second one is similar to a square fractal as shown previously in this paper.

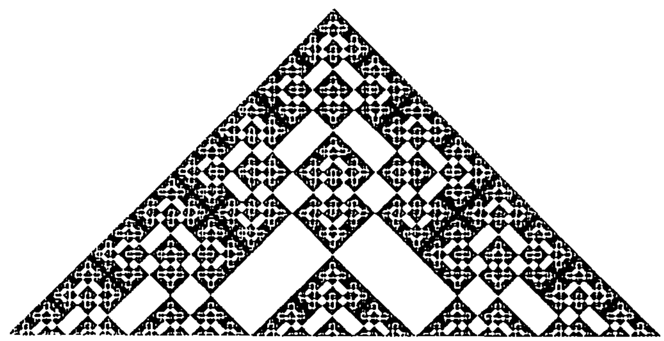


Figure 14a: Fractal generated from the digital wave equation 15c with initial condition 15d for N=2 A new type of fractal is generated.

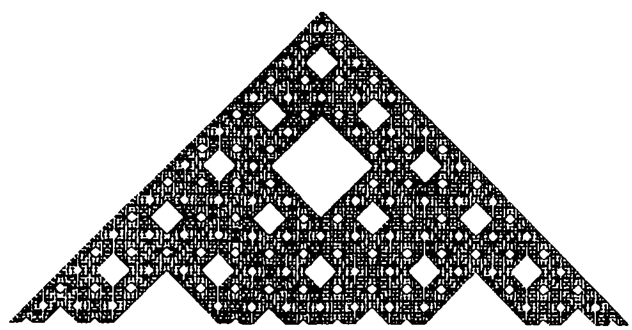


Figure 14b: Fractal generated from the digital wave equation 15c with initial condition 15d for N=3. The fractal is similar to a square fractal, that is to say a Sierpinski carpet.

9. CONCLUSION

This paper deals with the concept and method of incursion and hyperincursion to model discrete systems. These are an extension of recursive processes where the computation of future time steps only depends on present and past steps. With incursion and hyperincursion, future time steps can be introduced to compute these future steps. Hyperincursion is an incursion when several future values can be generated at each time step. These can be interpreted as an incremental anticipation similar to the Aristotelian Final Causation. They are related to the definition of backward and forward time derivatives.

The method of incursion is a powerful tool for modelling discrete systems. With the Fractal Machine, it is explicitly seen that the external inputs must be defined in the future time like a final causation which controls completely all the automata at the same time step in a holistic way. Indeed the inputs are present in each automata at the same external time. It is impossible to transform external inputs defined in the future time t+1 to inputs defined in the present time t. In this, we can say that we are dealing with a strict incursive system. Thus the final causation is really the 4th causation which must be taken into account in systems modelling as Aristotle had proposed. It seems also impossible to construct a real working engineering system where real working external future inputs would control its current present state. But it is possible to define internal future inputs in considering self-reference systems. The Fractal Machine can become non deterministic or non algorithmic, what I suggest to call an HYPERINCURSIVE FIELD where uncertainty (indecidability) or contradiction (exclusion principle) occur. It was shown that the incursive diffusion equation can be algorithmically deterministic with a self-reference for the inputs defined at future time by space periodic conditions. As the movement equations are defined in the future time by the backward derivative, the recursive transformation has the effect of mixing all the automata together at the present time t in view of computing their future values at time t+1. The transformation of a non-local incursive system to a local recursive system leads to a folding of each automaton to the other ones from the future time to the present time. Several fractals were simulated from incursive automata giving rise to square and pentagon symmetries. An interesting case was given by the generation of fractal interlacing represented by only one travelling line in the space.

It was shown that fractals can also be generated from the digital diffusion and wave equations in using the modulo N of the finite difference equations.

Finally, the concepts of incursion and hyperincursion can be related to the theory of hypersets which are defined as sets containing themselves. This theory of hypersets is an alternative theory to the classical set theory which presents some problems as the incompleteness of Gödel: a formal system cannot explain all about itself and some propositions cannot be demonstrated as true or false (undecidability). Fundamental entities of systems which are considered as ontological could be explain in a non-ontological way by self-referential systems.

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