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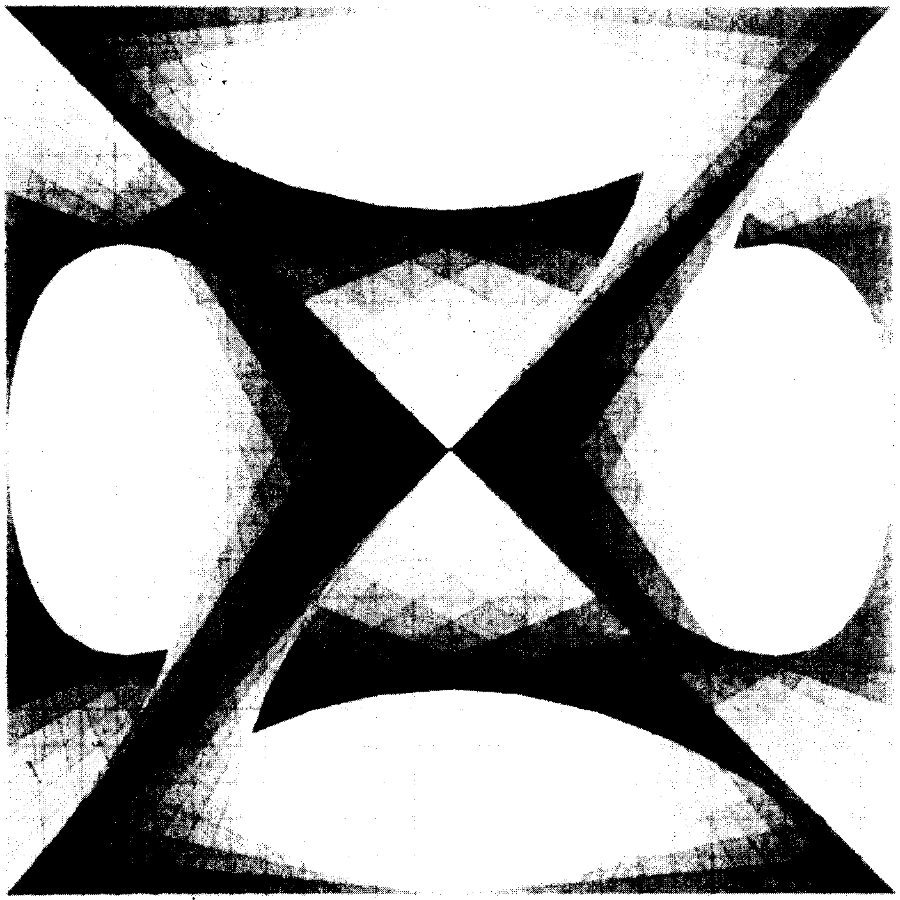

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AN INCORRECT SANGAKU CONJECTURE

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Problem 1.2.7 in Fukagawa and Pedoe (1989) reads as follows. *In Figure 1, which is symmetrical about the diameter AB, if r denotes the radius of the small circle, show that*

$$1/r = 1/FE + 1/AF.$$

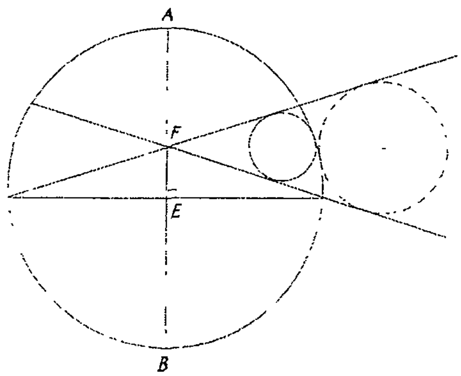

Figure 1

The problem dates from 1878 in a surviving tablet in the Hyogo prefecture, and the result is not difficult to prove using coordinate geometry. It can also be verified that *if* r' *denotes the radius of the broken circle then*

$$1/r' = 1/FE - 1/FB.$$

Hiroshi Okumura has brought to my attention a generalisation of the above problem in *The Sangaku in Gunma* (1987), p. 72. *In Figure 2, if* r *and* s *denote the radii of the two small circles, show that*

$$1/\sqrt{rs} = 1/FE + 1/AF,$$

or equivalently, show that

$$\mathbf{E} = 1,$$

where **E** denotes the expression $\sqrt{rs}\ (1/FE + 1/AF)$.

Any investigation of the lengths occuring in this expression involves awkward calculations, but I eventually began to suspect, for various reasons, that the result is not true. To verify this we need only find a single counterexample, but we shall go further and show that **E** *can take all positive values.*

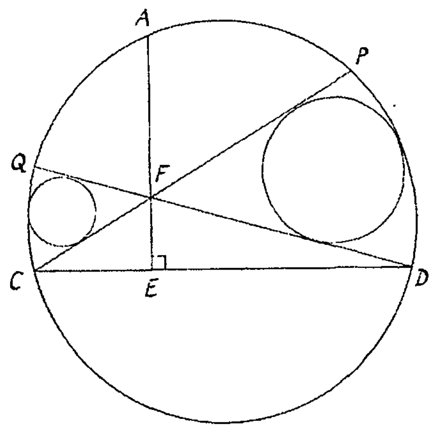

Figure 2

I first considered the case in which CP is a diameter, with DQ perpendicular to it, as in Figure 3. Take DQ and CP as cordinate axes; assume that the large circle has radius 1 and centre O $(0, -\cos2\theta)$. Calculations using coordinate geometry and trigonometry show that the circles α and β have radii $r = 2(\sin\theta - \sin^2\theta) = 2\sin\theta\,(1 - \sin\theta)$ and $s = 2(\cos\theta - \cos^2\theta) = 4\cos\theta\,\sin^2(\theta/2)$; also $FE = \sin2\theta\,\cos\theta = 2\sin\theta\,\cos^2\theta$. The line AB has equation $y = -x\tan\theta$, and if we investigate where this line meets the circle $x^2 + (y + \cos2\theta)^2 = 1$,

$$AF = \sin\theta(\sqrt{4\cos^4\theta + 1} - \cos2\theta).$$

Consider now what happens as θ varies between $\pi/4$ and 0. We see that **E** can be written in the form $\mathbf{E} = \sin(\theta/2)f(\theta)/\sqrt{\sin\theta}$, where $f(\theta)$ is a continuous positive function of θ that tends to a finite limit as $\theta \to 0$. Hence $\mathbf{E} \to 0$ as $\theta \to 0$. But we know that

$$\mathbf{E} = 1$$

in the symmetrical case when $\theta = \pi/4$. Hence **E** takes all positive values ≤ 1 as θ varies between $\pi/4$ and 0.

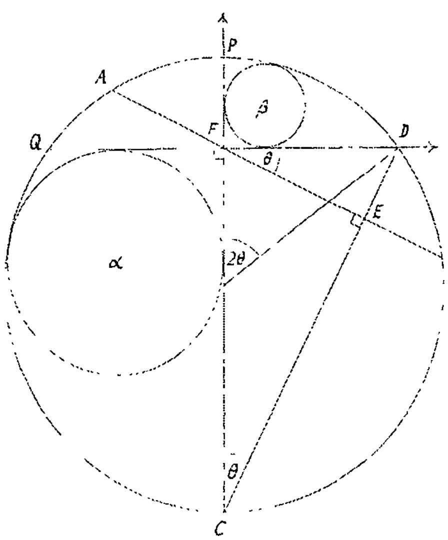

Figure 3

A simpler special case is shown in Figure 4, in which CQ and XY are perpendicular diameters. Suppose the radius of the large circle is 1. Then

$$r(1 + \sec\phi) = XK + KF = 1 + \tan\phi, \text{ and } s(1 + \sec\phi) = 1 - \tan\phi.$$

Hence $\sqrt{rs}\sin^2\phi = \cos\phi\sqrt{\cos 2\phi}(\sec\phi - 1)$.

Also $AF = \sec\phi$ and $FE = 2\cos\phi - \sec\phi = \sec\phi\cos 2\phi$.

Hence $(1/AF + 1/FE) = 2\cos^3\phi/\cos 2\phi$ and $\mathbf{E} = 2\cos^4\phi\,(\sec\phi - 1)/\sin^2\phi\sqrt{\cos 2\phi}$.

Hence $\mathbf{E} \to \infty$ as $\phi \to \pi/4$. Also $\mathbf{E}$ is positive and continuous, and $\mathbf{E} = 1$ when $\phi = 0$. Hence $\mathbf{E}$ takes all positive values ≥ 1 as ϕ varies between 0 and $\pi/4$.

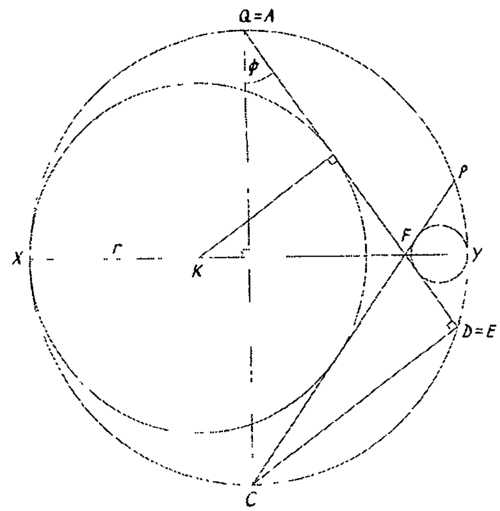

Figure 4

When $\theta = 26°$ as in Figure 3, we find that $\mathbf{E}$ = .954..., and when $\phi = 35°$ as in Figure 4, $\mathbf{E}$ = 1.033... . Hence $\mathbf{E}$ can be close to 1 even in figures that are far from being symmetrical, so it is not surprising that the original conjecture was made, probably as the result of careful drawings.

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SYMMETRY PROPERTIES ON TANGENT CIRCLES

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1 INTRODUCTION

The problems of tangent circles have been discussed in the East and the West. For example, one of them is on the Tangency Problem of Apollonius, and has been studied by Descartes, Gauss, Gergonne, Soddy and Coxeter, etc. (Coxeter, 1968; Dorrie, 1965; Heath, 1921; Pedoe, 1988).

In the East, especially in Japan, solutions to such problems were attempted in *wasan* (Hirayama, 1965; Michiwaki et al., 1985), which was the old Mathematics in Japan (about the 17^{th} ~ 19^{th} cc.).

First, in this paper, we will start from Steiner's Theorem (Coxeter and Greitzer, 1967, Fig. 1).

Next, we will introduce some related theorems in wasan, and show two interesting equations that are symmetrical.

Further, we would like to present new properties on two (non-concentric) circles C, C_o touching four circles C_1, C_2, C_3, C_4 as on Fig. 2.

Lastly, we would like to prove the theorems in wasan by a unified method.

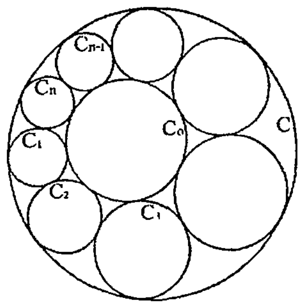

Figure 1

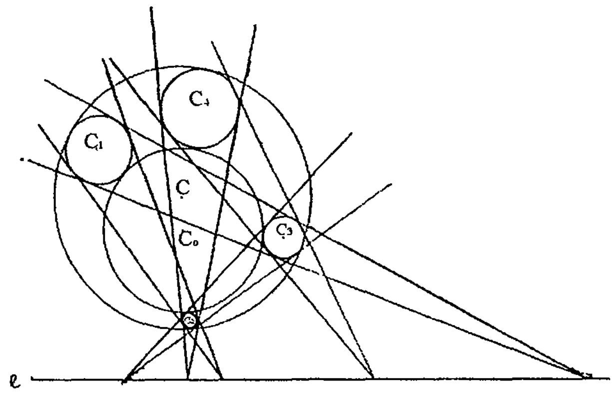

Figure 2

2 WELL-KNOWN TANGENT CIRCLES' THEOREMS

The following theorems are well-known as Steiner's Theorem and Formula (Coxeter and Greitzer, 1967; Pedoe, 1988; Fig. 1).

Steiner's Theorem

If we have two (non-concentric) circles, one inside the other, and other circles are drawn, touching one another successively and all touching the two original circles, as in Fig. 1, it may happen that the sequence of tangent circles closes so as to form a ring of n, the last touching the first.

Let us suppose that r, r_i ($i = 0, 1, 2, \ldots, n$) are the radii of circles C and C_i, respectively, and that d is the distance between C and C_0. In this case, Steiner showed the next formula.

Steiner's Formula

$$(r - r_0)^2 - d^2 = 4rr_0\tan^2(\pi/n)$$

These are treated in geometry textbooks as theorems not only beautiful but also proved easily. That is, it is enough to invert the original circles into concentric circles. So, Fig. 1 is changed to Fig. 3 which is symmetric (Pedoe, 1988).

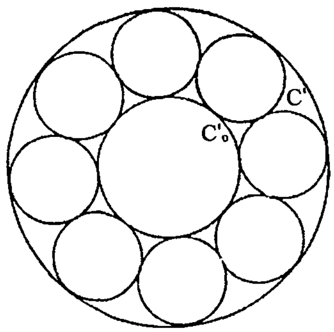

Figure 3

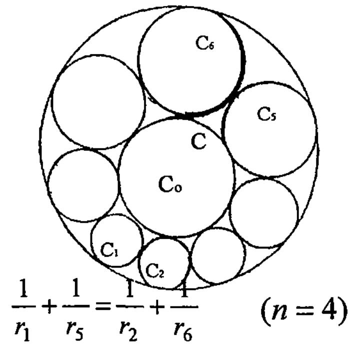

Figure 4

3 TWO BEAUTIFUL THEOREMS IN WASAN

In wasan, the same figure as Fig. 1 was studied, too (Hirayama, 1965; Michiwaki and others, 1985). The main subject is the relation among the radii of circles C_1, C_2, C_3, ... which are touching both circles C and C_o.

Here, we show two theorems, in which the invariants are symmetrical. When i is equal to $2n$ (n = 1, 2, 3, ...), the following extension theorem was given by T. Ikeda (Hirayama, 1965; Michiwaki, Ohyama, and Hamada, 1975; Michiwaki et al., 1985).

Ikeda's Theorem

Circles C_i (i=1, 2, 3, ..., $2n$) are each touching its two neighbors and two given boundary circles C, C_o such that C surrounds C_o (cf. Coolidge, 1971; Coxeter and Greitzer, and see Fig. 4).

Let r, r_i (i = 1, 2, 3, ..., $2n$) be the radii of C and C_i, respectively, then,

$$1/r_i + 1/r_{i+n} = 1/r_j + 1/r_{j+n} \tag{1}$$

This is proved easily by using the invariant in inversion. However, in wasan, it was proved as the result of many calculations by the Pythagorean Theorem.

Further in Ikeda's Theorem, if we can draw two external common tangents as Fig. 5, we get the following theorem. This was proved by S. Kenmochi (1790–1871; Hirayama, 1965; Michiwaki, Ohyama, and Hamada, 1975; Michiwaki and others, 1985) in the case of $n = 4$ and its extension.

Kenmochi's Theorem

Let us suppose that a small circle C_o is in a large circle C. Circles C_i ($i = 1, 2, \ldots, 2n$) touch C_o and C, respectively (Fig. 5). C_1 touches C_2, with C_2 also touching $C_3 \ldots$. And, C_1, C_o and C_{2n} have an external common tangent and so do C_n, C_o and C_{n+1}. Let r_1 be the radius of C_i ($i = 1, 2, \ldots, 2n$), then,

$$(1/\sqrt{r_1})+(1/\sqrt{r_{n+1}})=(1/\sqrt{r_n})+(1/\sqrt{r_{2n}}) \tag{2}$$

Relating (1) with (2), it is interesting to consider why a pair of external common tangents bring the difference.

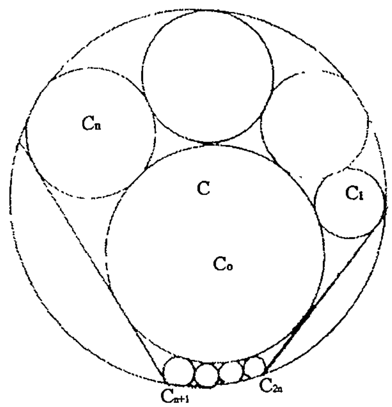

Figure 5

Kenmochi's Theorem seems to be a particular case of Ikeda's Theorem. Among wasan researchers, however, those are known as theorems proved by means of quite different methods (Michiwaki, Ohyama, and Hamada, 1975).

Later, we would like to prove the Japanese theorems by a unified method. First, we will show main theorems on them.

4 MAIN THEOREMS

We would like to show our theorems on the above results. First, we introduce Three Circles' Theorem. It is known that each external similarity point of any two circles of the three given circles lies on a line. It is just the radical axis of two circles C, C_o ($C \supset C_o$) which are tangent to the three circles (Dorrie, 1965; Pedoe, 1979).

Second, we present Theorem 1, which is related to Three Circles' Theorem and Gergonne's Theorem (Coolidge, 1971).

Theorem 1

If we have two (non-concentric) circles C, C_o touching four circles C_1, C_2, C_3, C_4 as Figs. 2, 6–8, then an intersection point of external common tangents of any pair of circles C_i, C_j ($i \neq j$, $1 \leq i, j \leq 4$) lies on a line l (Ozone, 1993).

Proof

Let P be the intersection point of external common tangents of arbitrary circles C_i, C_j ($i \neq j$, $1 \leq i, j \leq 4$).

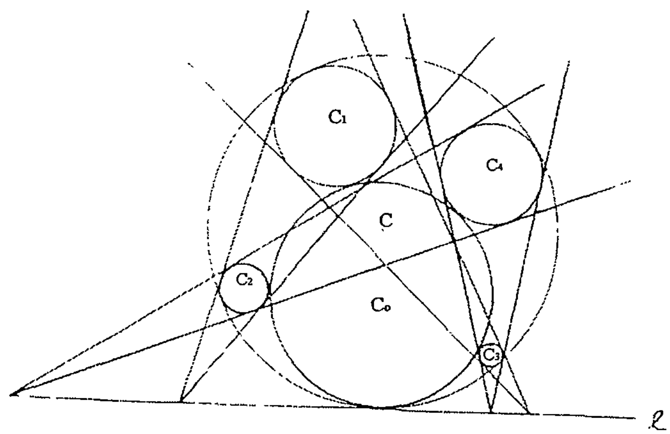

Figure 6

We invert C so as to be invariant with center P of the circle of inversion. So, C_o is invariant, too.

Therefore, P lies on the radical axis of circles C, C_o.

The radical axe is just the line l.

Further, we get the next property in Theorem 1.

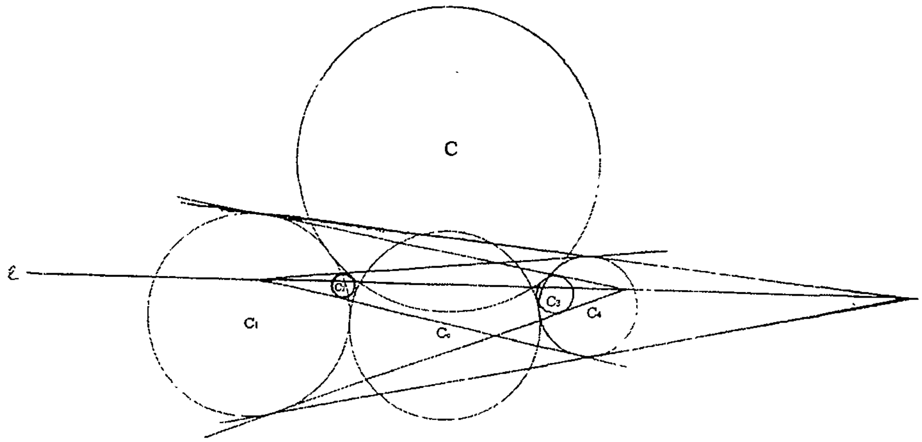

Figure 7

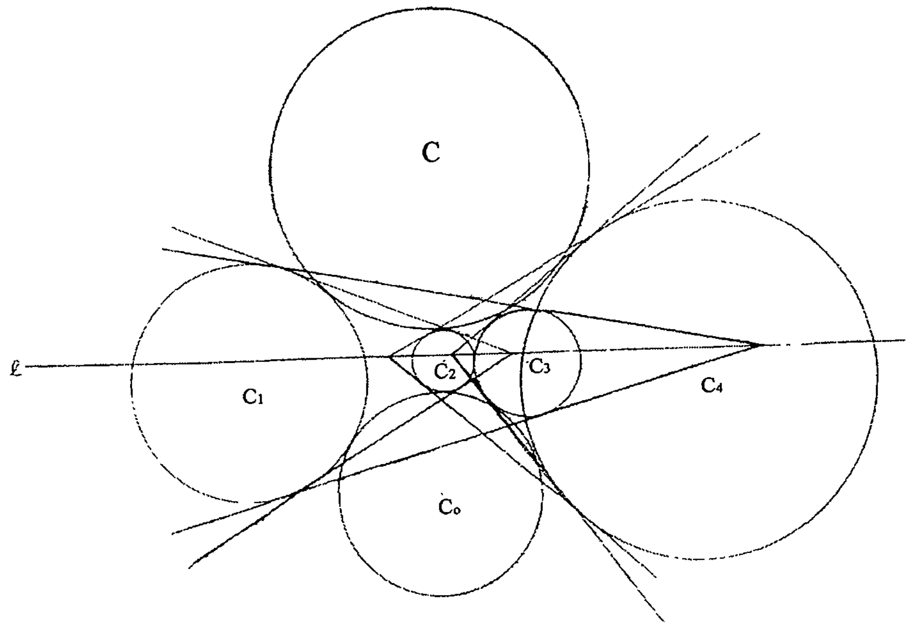

Figure 8

Theorem 2

If we draw a pair of external tangents from any point P on the line l to an arbitrary circle C_i which touch C and C_o, we can find a circle C touching C, C_o and having the same external tangents, too.

Proof

If we invert C so as to be invariant with center P of the circle of inversion, we can find a circle C_j which is the inverse of C_i. And we see that C_i, C_j have the same external tangents.

We can get Theorem 3 by Theorem 1 easily.

Theorem 3

In Steiner's Chain (Dorrie, 1965; Michiwaki, Ohyama, and Hamada, 1975), an intersection point of external common tangents of arbitrary circles C_i, C_j lies on the radical axis of circles C, C_0 (Ozone, 1993; Fig. 9).

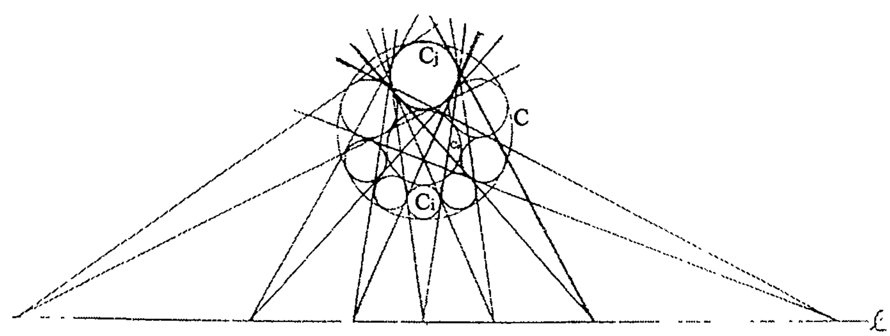

Figure 9

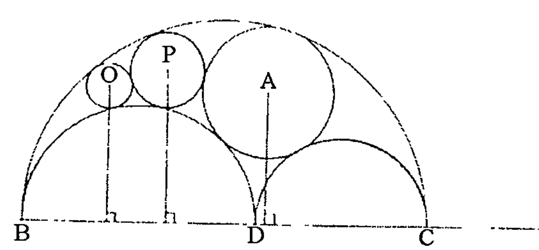

Figure 10

Here, we introduce the Theorem of Shoemaker's Knife by Pappus of Alexandria as follows (Heath, 1921; Fig. 10).

Theorem of Shoemaker's Knife

Let successive circles be inscribed in the Shoemaker's Knife touching the semicircles and one another as shown in Fig. 10, their centers being A, P, O, ... Then, if p_1, p_2, p_3, ... are the perpendiculars from the centers A, P, O, ... on BC and d_1, d_2, d_3, ... the diameters of the corresponding circles,

$$p_1 = d_1,\quad p_2 = 2d_2,\quad p_3 = 3d_3,\ \dots .$$

Next, we can show a property related to the above theorem, as l is the radical axis of circles C, C_0.

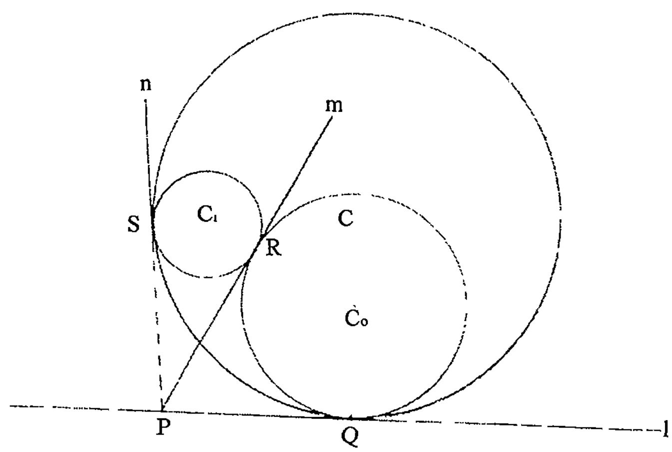

Figure 11

Theorem 4

Let a point Q (on a line l) be tangent to circles C, C_0, and C_t be an arbitrary circle touching C, C_0. If we draw two lines m, n such that m is a line tangent to C_t, C_0 and n is one to C_i, C, then an intersection point P of m and n lies on l.

Further, let R be a point tangent to C_t, C_0, and S to C_t, C, then $PQ = PR = PS$ (Fig. 11).

5 PROOF OF THE THEOREMS IN WASAN

Lastly, we would like to try to give a simple proof to the theorems in wasan by the use of main ones.

First, we prove Ikeda's Theorem.

Proof of Ikeda's Theorem

We draw a pair of common external tangents to arbitrary circles C_i, C_{i+1} $(1 \leq i \leq 2n-1)$.

Let P be the intersection point of the tangents. From Theorem 1, P lies on the radical axis of circles C, C_o. Then we see that a circle C_{i+n+1} is the inverse of a circle C_{i+n} and P is equal to the intersection point of the external common tangents of C_i, C_{i+1}.

Let $2\Theta_i$ be the angle between the external common tangents of C_i, C_{i+1}, and $2\Theta_{i+n}$ be that of circles C_{i+n}, C_{i+n+1} (Fig. 12).

If we regard P as the center of the circle of inversion, we get,

$$r_i\, r_{i+1} / \tan^2\Theta_i = r_{i+n}\, r_{i+n+1} / \tan^2\Theta_{i+n} \tag{3}$$

And we easily have,

$$\tan\Theta_i = r_i - r_{i+1} / 2\sqrt{r_i\, r_{i+1}} \tag{4}$$

$$\tan\Theta_{i+n} = r_{i+n+1} - r_{i+n} / 2\sqrt{r_{i+n+1} r_{i+n}} \tag{5}$$

From (3) – (5), we get,

$$1/r_i + 1/r_{i+n} = 1/r_{i+1} + 1/r_{i+n+1}$$

Therefore, we can gain similarly,

$$1/r_i + 1/r_{i+n} = 1/r_j + 1/r_{j+n} \tag{1}$$

Proof of Kenmochi's Theorem

We can get this relation by the use of main theorems (Fig. 13). This case is different from the above in regards to the next point.

$$r_1 r_{2n} / \tan^2\Theta_i = r_n r_{n+1} / \tan^2\Theta_n$$

$$\tan\Theta_i = \sqrt{r_1} - \sqrt{r_{2n}} / 2\sqrt{r_o}$$

$$\tan\Theta_n = \sqrt{r_n} - \sqrt{r_{n+1}} / 2\sqrt{r_o}$$

From these connections and (3), we can get,

$$1/\sqrt{r_1} + 1/\sqrt{r_{n+1}} = 1/\sqrt{r_n} + 1/\sqrt{r_{2n}} \tag{2}$$

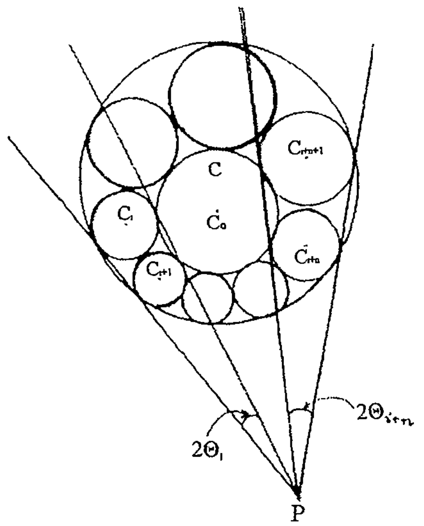

Figure 12

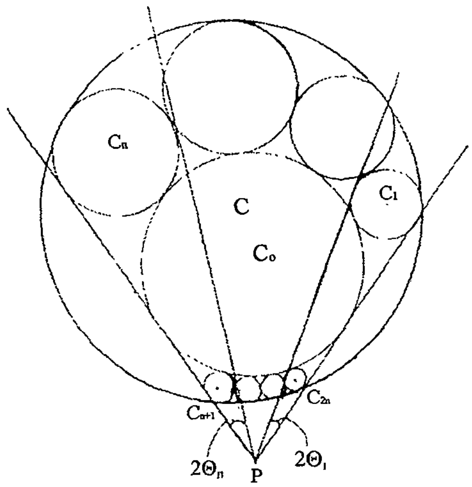

Figure 13

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