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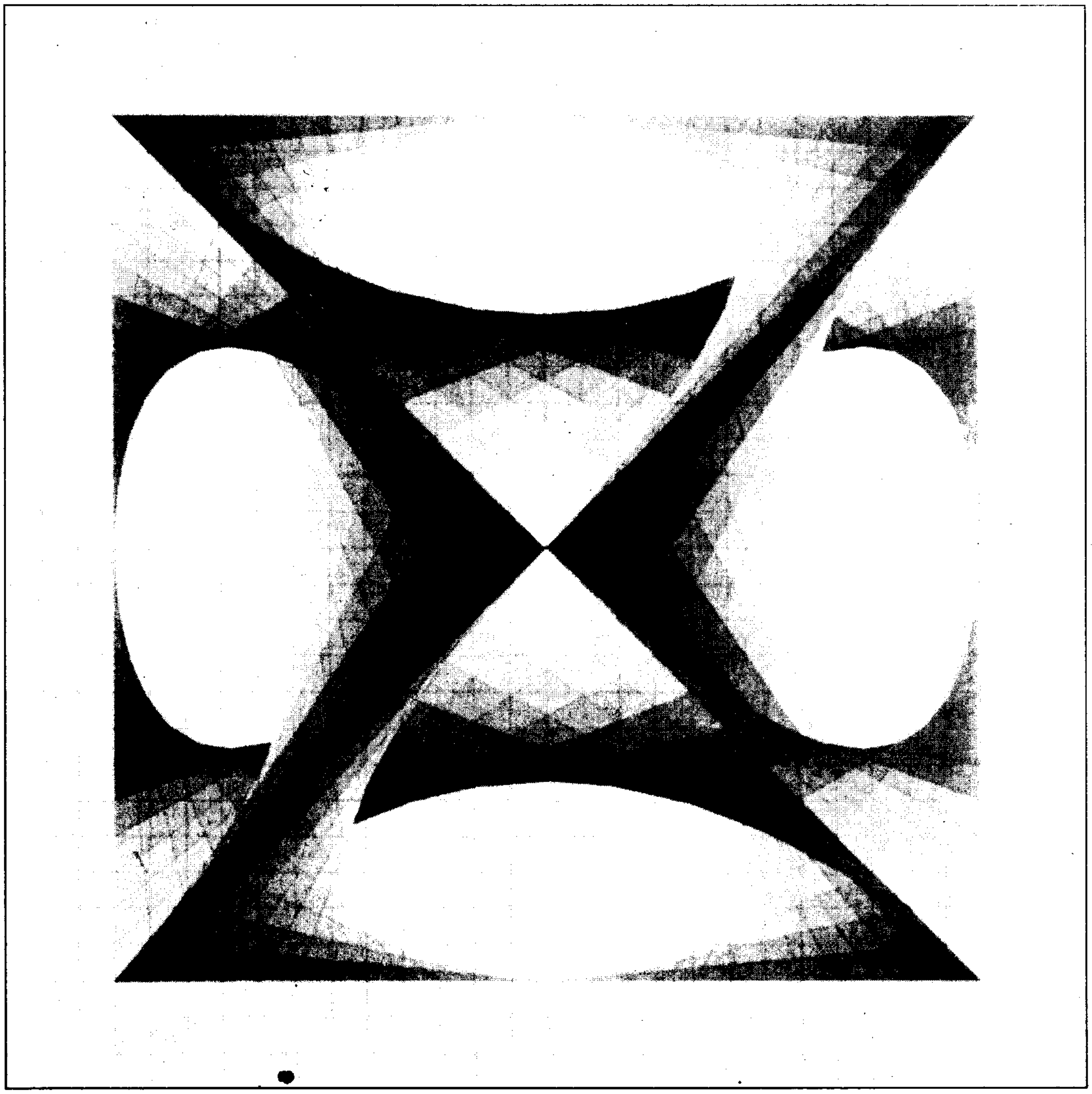

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SYMMETRY: SCIENCE AND ART

CIRCLE PATTERNS ARISING FROM RESULTS IN JAPANESE GEOMETRY

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Abstract: *Japanese mathematics in 17th-19th cc, called* wasan, *left many results. Those results sometimes can be generalized and also be used to construct interesting patterns or configurations. Several such examples will be given.*

1 INTRODUCTION

The mathematics we will present here is one developed in Japan during the *Edo* period (1603-1867). Its root is in Chinese mathematics and it developed rapidly during the *Edo* era. It is called *wasan* (*wa* means "Japan" and *san* means "mathematics", respectively). At the beginning of the *Meiji* period (1868-1911), the new government adopted Western mathematics into its new school system, bringing a sudden end to the short life of wasan. But the wasan tradition was still able to survive.

Results of wasan geometry belong to two- or three-dimensional Euclidean geometry. These results concern elementary figures such as triangles, quadrangles, circles, spheres, etc., where tangent circles are in the majority. The results are stated as problems with their final answers, but in most cases there is no explanation how such answers were achieved.

There were two ways to publish the results in those days: One was in a book and the other was on a wooden board called a *sangaku.* (The characters for *san* and *gaku* mean "mathematics" and "framed board" respectively in Japanese in this case.) When people found interesting problems, they would often write them down on a framed wooden board and dedicated it to a shrine or a temple. Then the board would be hung under the roof on the grounds. Most such problems were of a geometric nature and the figures were beautifully drawn in color. But since there are so many of these wasan problems, many still remain that have yet to be confirmed as correct.

Sometimes we can generalize about many of those problems. Furthermore, patterns or configurations consisting of figures in these problems can often be discovered. In this article we will demonstrate some of these patterns in the plane arising from problems involving tangent circles in the old Japanese geometry.

2 A PATTERN BY FUJITA CONFIGURATIONS

Let $ABCD$ be a parallelogram, E a point on the segment CD, F and G points on the segments CE and DA respectively, and let FG meet BE at H. Suppose the quadrangles $ABHG$, $BCFH$ and $DGHE$ have incircles γ_a, γ_b, γ_d of radii a, b, d respectively. Also we denote the incircle and its radius of the triangle FEH by γ_c and c (see Figure 1). This figure seems to have been first considered by Fujita in his book *Seiyō Sampō* (1781) in the case where $ABCD$ is a square. It seems appropriate to call this figure a Fujita configuration. The problem is as follows:

Problem 1. In a Fujita configuration, where $ABCD$ is a square, $2a+2b+2d+BE+FG+AB$ be given, find AB.

The answer is $AB = 12(2a+2b+2d+BE+FG+AB)/55$. Though several problems involving Fujita configuration where $ABCD$ is a rhombus or a rectangle can be found in some wasan books, they do not demonstrate any interesting properties of the configuration. But there is one remarkable property (Okumura, 1987 and 1989b, see Figures 2 and 3.):

Theorem 1. In a Fujita configuration $a+c = b+d$ holds. If $ABCD$ is a square then c:d:b:a = 1:2:3:4. Conversely, from a triangle with the inradius c and two exradii b and d, we can get two Fujita configurations by setting the radius of the fourth circle equal to $b+d-c$.

Using Figure 2, we can construct a pattern in the entire plane so that the segments BE and FG are portions of some lines (see Figure 4). In this pattern we can superpose similar ones. Two similar patterns which are double and four times the size are drawn on the figure.

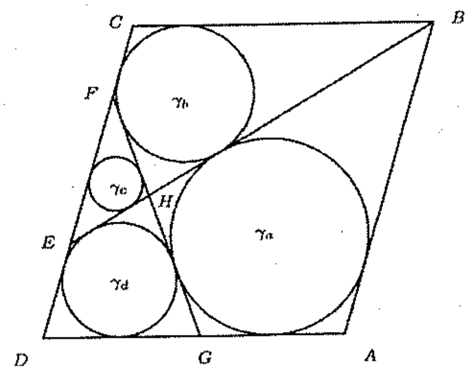

Figure 1

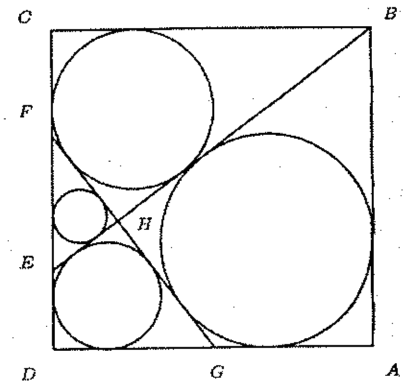

Figure 2

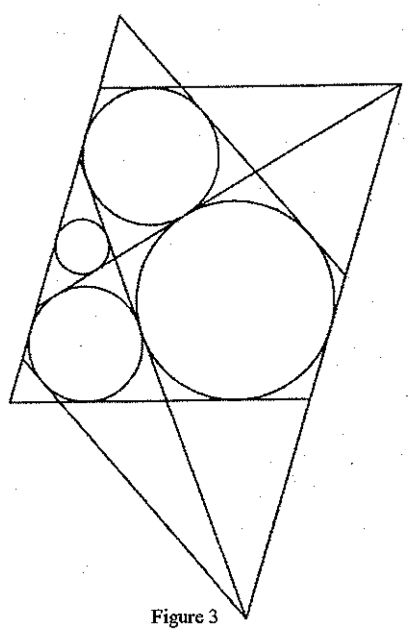

Figure 3

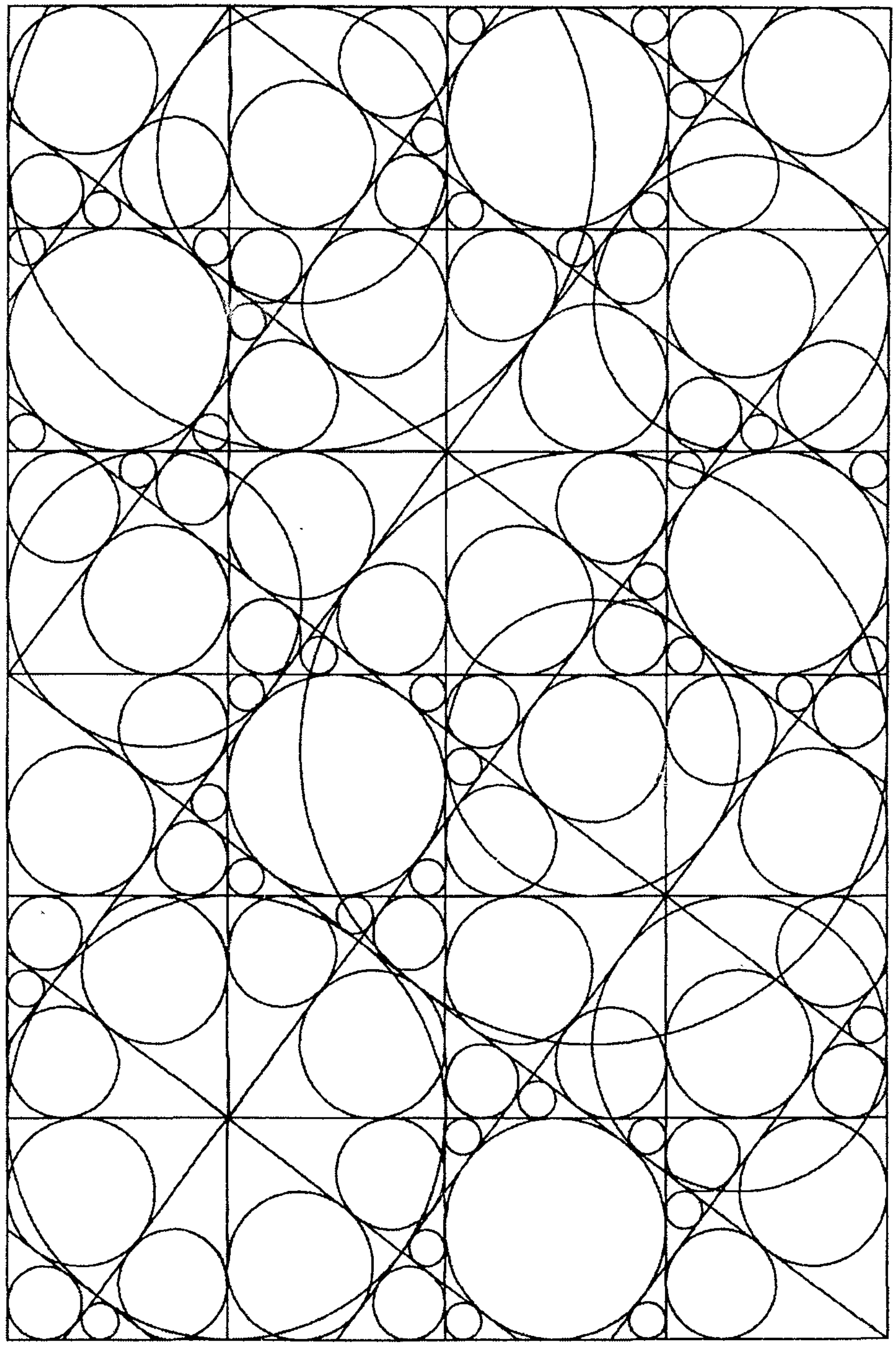

Figure 4

3 PATTERNS OBTAINED BY UNIFYING SEVERAL PROBLEMS

Our patterns in this section are rather trivial. But it is worth-while considering them here because they contain several figures that can be found in several wasan problems. A typical problem is as follows:

Problem 2. Let α_0 be a circle of radius s with center O and a diameter AB, α_1 and α_2 circles with diameters AO and BO respectively. Then let γ_1, γ_2, γ_3 be distinct circles with a radii r contained in α_0 tangent to α_0 and lying on the same side of AB, and β_1, β_2 distinct circles tangent to α_1 at O such that γ_1 touches α_1 externally, β_1 ($\beta_1 \neq \alpha_1$) touches γ_1, γ_2 touches β_1, β_2 touches γ_2, γ_3 touches β_2 and α_2 externally. Show that $s = 5r$ (see Figure 5).

The problem is easily generalized as the following theorem (Okumura 1995a).

Theorem 2. Let α_0, α_1, α_2, γ_1 and β_1 be as in Problem 2. Suppose that two families of distinct circles $\gamma_2, \gamma_3, \ldots, \gamma_i$ with a radii r and $\beta_2, \beta_3, \ldots, \beta_i$ have been drawn such that $\gamma_2, \gamma_3, \ldots, \gamma_i$ are tangent to α_0 and lying in α_0 on the same side of AB, and for any integer j ($2 \leq j \leq i$) γ_j touches β_{j-1} and β_j touches γ_j and α_1 at O. Then define circles γ_{i+1} ($\neq g_i$) with a radius r and β_{i+1} ($\neq \mathrm{b}_i$) such that γ_{i+1} touches β_i and α_0 internally and lies in the same side of AB with γ_1, and β_{i+1} touches γ_{i+1} and α_1 at O. Suppose that $\gamma_1, \gamma_2, \ldots, \gamma_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ have been drawn in this procedure and $\beta_n = \alpha_2$, then we have $s = (n+2)r$.

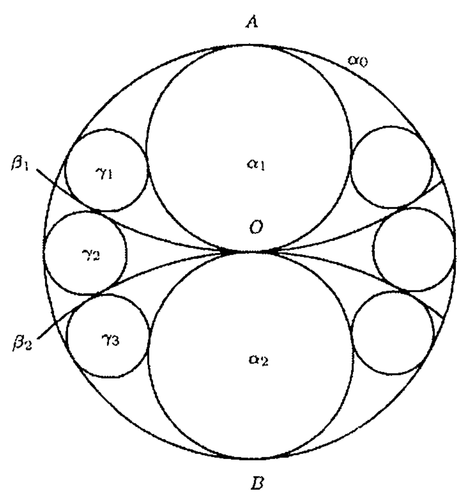

Figure 5

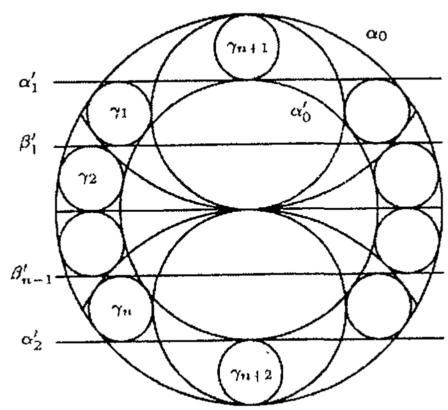

Figure 6

The theorem and our patterns in this section are good examples for demonstrating inversive geometry technique. Now let us prove the theorem (see Figure 6 which shows the case $n = 4$). The following simple proof is drawn from J. F. Rigby (Rigby 1995). Let us add two more circles γ_{n+1} and γ_{n+2} of radii r touching α_0 internally at A and B and an inner circle α'_0 with center O and radius s-$2r$. There is an inversion with center O that interchanges α_0 and α'_0 (the radius of inversion is $\sqrt{s(s-2r)}$). This inversion maps each γ'_i to itself and the circles through O to equally spaced n+1 parallel lines (in the figure α'_i and γ'_i are the images of α_i and γ_i). This proves Theorem 2.

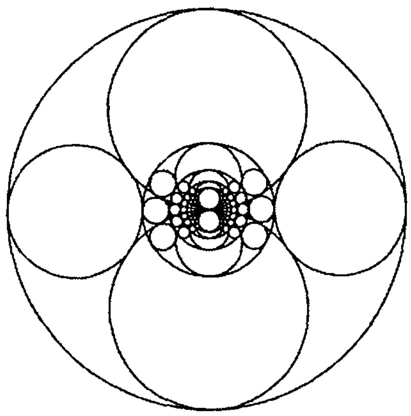

Figure 7a

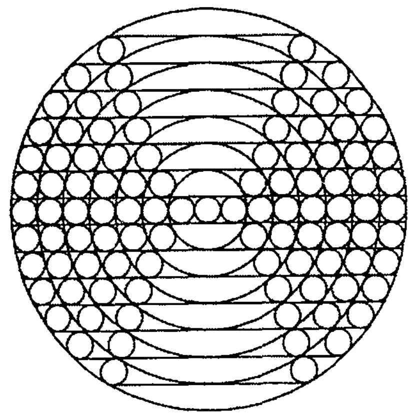

Figure 7b

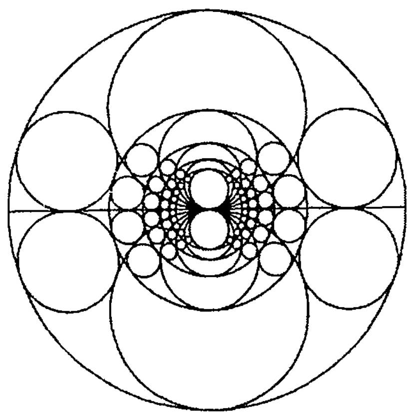

Figure 8a

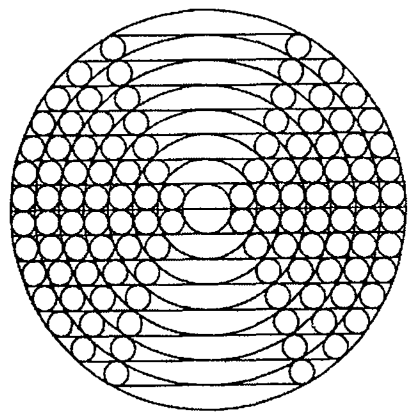

Figure 8b

A problem corresponding to the case $n = 3$ can be found in a sangaku problem dated 1836 (Hirayama and Yamaki, 1967a). The problem was very popular and we could find it in several sangaku books. The case $n = 4$ was published in 1828 (Kimura). Even a problem discussing the case in which n is an arbitrary even number was already proposed in a sangaku problem dated 1839 (Fukushima Wasan Study Association, p. 120).

Using the figures corresponding to the cases $n = 1, 3, 5, \ldots$ of the theorem we can generate a pattern shown in Figure 7a. This pattern can be obtained from a trivial pattern, Figure 7b, which consists of evenly spaced parallel lines, and concentric circles each of which touches two parallel lines, and small congruent circles touching them. Inverting Figure 7b in one of the concentric circles, we can get Figure 7a. Similarly from Figure 8b, we can get Figure 8a, which consists of the figures corresponding to the cases $n = 2, 4, 6, \ldots$ in the theorem.

It is easily seen that in Figures 7b and 8b we can choose points of intersections of concentric circles and parallel lines so that they lie on a parabola with focus at the common center. Since the image of a parabola by an inversion in a circle with its center as its focus is a cardioid, we can choose sets of points of intersections of two circles so that they lie on a cardioid in Figures 7a and 8a.

4 PATTERNS ARISING FROM A FIVE-CIRCLE PROBLEM

Our patterns in this section do not consist of figures that can be found in wasan problems. But they were found when the author generalized the following problem, which can be found in Yamamoto's 1841 book and in Fukagawa and Pedoe, p. 7. as well. The following is a brief description of the process of finding the patterns.

Problem 3. In the circle γ of radius r let AB be a chord whose midpoint is M. The circle γ_0 of radius r_0 ($r_0 < r/2$) touches AB at M and also touches γ internally. Let P be any point on AB distinct from A, B and M; a circle γ_2 of radius r_0 (equal to the radius of γ_0) touches AB at P on the other side of AB. Distinct circles γ_1 and γ_3 of radii r_1 and r_3 touch AB, and each touches γ internally and γ_2 externally. Show that $r = r_1+r_2+r_3$ (see Figure 9).

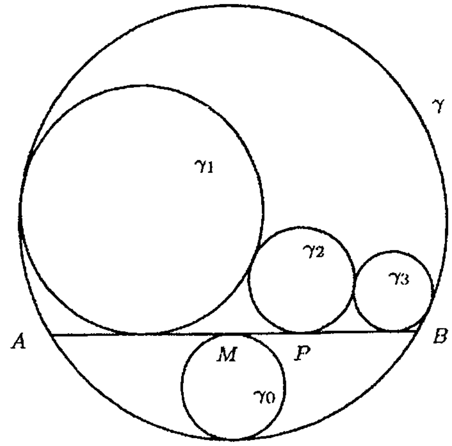

Figure 9

Neither the restriction $r_0 < r/2$ nor $P \neq A, B, M$ are needed for the desired relation among the radii in the problem. Indeed we can generalize the problem as follows (Okumura, 1994, see Figure 10):

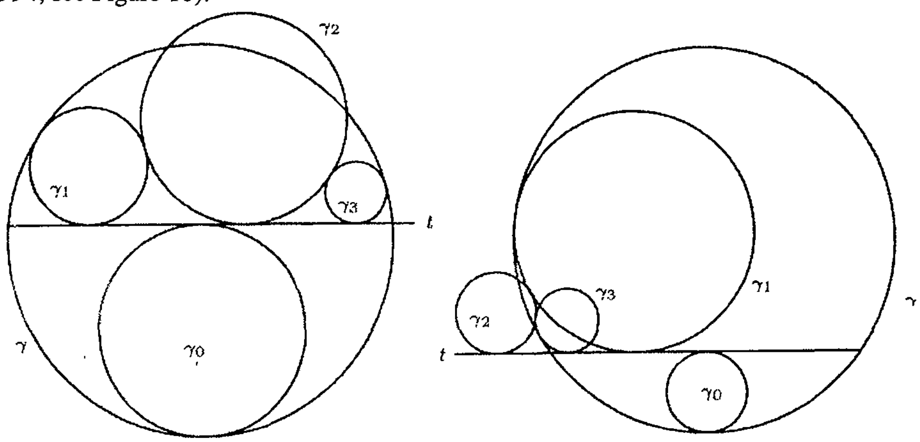

Figure 10

Theorem 3. Let t be a secant of a circle γ of radius r; let $\gamma_1, \gamma_2, \gamma_3$ be circles on one side of t and tangent to it, with γ_1 and γ_3 internally tangent to γ, while γ_2 is externally tangent to γ_1 and γ_3. Let γ_0 be the circle internally tangent to γ on the other side of t, and tangent to t at the midpoint of the segment cut from it by γ. If γ_i has radius r_i and $r_0 = r_2$, then $r = r_1+r_2+r_3$.

If γ_3 degenerates to a point circle in the theorem, γ_2 touches t at an intersection of γ and t. Since $r = r_0+r_1$ in this situation, the centers of γ, γ_0, γ_1 are collinear. This suggests a converse: let γ be a circle and k its chord, γ_0 and γ_1 circles touching γ internally and also touching k at the midpoint on opposite sides. Then the radius of the circle touching k at an end of k on the other side from γ_0 and touching γ_1 externally is equal to the radius of γ_0. But it is possible to prove a more general result (see Figure 11).

Theorem 4. Let γ_0 and γ_1 be externally (resp. internally) touching circles, and γ the circle touching the two at points different from the point of tangency of γ_0 and γ_1 such that the three centers are collinear. For any chord k of γ perpendicular to the line through the three centers, there is a circle of radius equal to the radius of γ_0 touching k at an end of k and touching γ_1 externally (resp. internally).

Proof. Let us assume that γ_0 and γ_1 touch externally. Let r_0 and r_1 be their radii, and O and O_1 the centers of γ and γ_1 respectively (see Figure 12). Then draw a chord k of γ perpendicular to the line through the three centers and a new circle of radius r_0 and center A touching γ_1 externally. If the circle touches k (or its extension) at B and the circle is drawn such that $\overline{O_1O}$ and $\overline{AB}$ have the same orientation, then O_1O and AB are equal and parallel, since $O_1O = r_0$. Hence OO_1AB is a parallelogram and we get $OB = r_0 + r_1$. Therefore B lies on γ. Thus the theorem follows from the uniqueness of the figure. The internal case can be proved in the same manner.

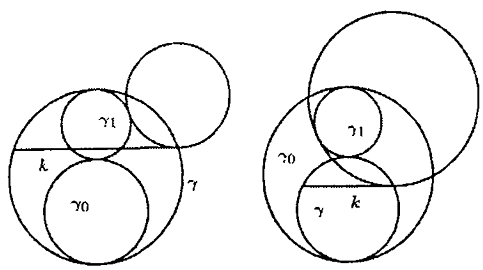

Figure 11

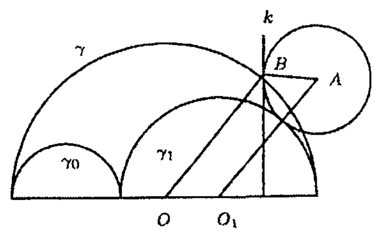

Figure 12

With the aid of this theorem, we can extend the three-circle pattern in the theorem to the entire plane. Let $\ldots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \ldots$ be distinct circles such that all the centers lie on a line, and δ_i and δ_{i+1} touch externally (resp. internally) and the radii of δ_{2i} and δ_{2i+1} are equal to r_0 and r_1, the radii of δ_0 and δ_1 respectively. For each pair of δ_i and δ_{i+1} let us draw another circle $\delta_{i,i+1}$ touching the two at points different from the point of tangency of δ_i and δ_{i+1} such that the three centers are collinear. By Theorem 4, there is a translation T such that d_0^T touches δ_1 externally (resp. internally) and intersects $\delta_{0,1}$ at a point where the tangent of δ_0^T is perpendicular to the line through the centers of δ_0 and δ_1. If we draw the images of the whole figure by the translations T, T^2, ..., and T^{-1}, T^{-2}, ..., we get Figure 13a (resp. Figure 13b).

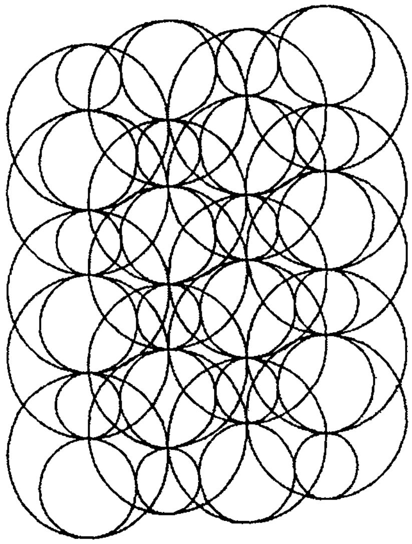

Figure 13a

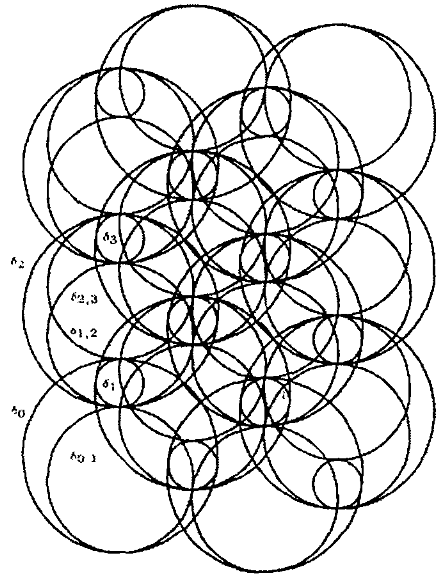

Figure 13b

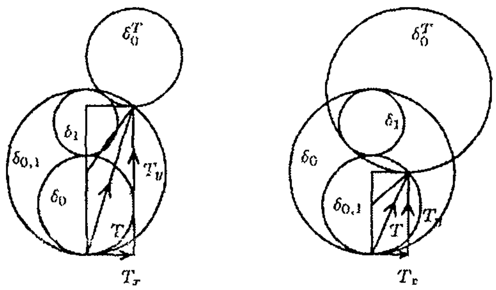

Figure 14

The translation T is equal to the product of two translations T_x ad T_y, where T_x has the direction perpendicular to the line joining the centers of δ_0 and δ_1, and T_y has the direction parallel to the line (see Figure 14). The relation between d_x and d_y is $d_x^2 = r^2 - (d_y - r)^2 = d_y(2r - d_y)$ by the Pythagorean theorem, where r is the radius of $\delta_{0,1}$.

Arranging the ratio of the radii r_0 and r_1 of circles γ_0 and γ_1 and the translation, we can get patterns with further tangency and incidence. Let the length of T_x and T_y be d_x and d_y respectively. Then Figure 15a is obtained by letting $r_0 = 2r_1$ and $d_y = r_0+2r_1$. In this figure, $\delta_{-1,0}{}^T$ passes through the point where δ_1 and δ_2 touch, and $\delta_{3,4}$ and $\delta_{0,1}{}^{T2}$ are tangent. And Figure 15b is obtained by letting $r_0 = (3+\sqrt{5})r_1$ and $d_y = r_0$. In this figure $\delta_{0,1}{}^T$ passes through the point of tangency of δ_0 and δ_1, and δ_4 and $\delta_0{}^{T2}$ are tangent.

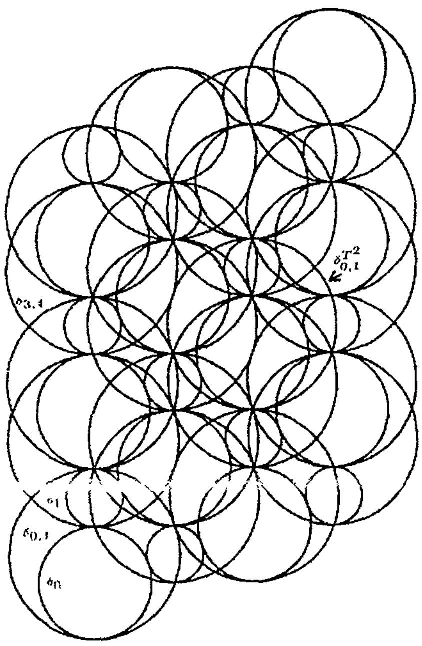

Figure 15a

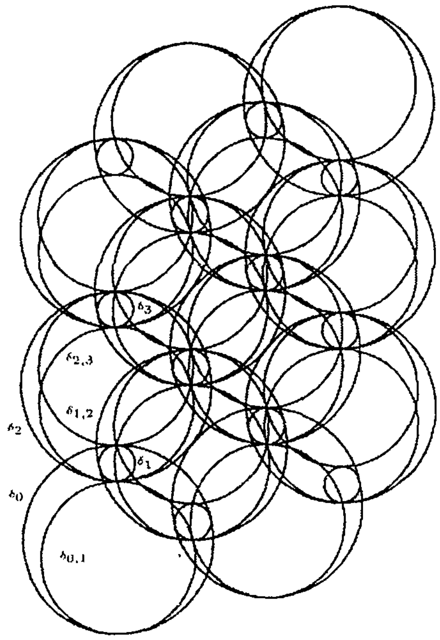

Figure 15b

5 A GENERALIZATION OF THE TILING OF CONGRUENT RHOMBUSES

Before stating the next problem, we briefly introduce oriented circles and oriented lines, which are needed in this section. Circles and lines with orientations are called cycles and rays, and we will describe the orientations by arrows in the figures. Two cycles, or a cycle and a ray touch each other or are tangent to each other if they touch as two circles, or a circle and a line and the orientations at the point of tangency are the same. If the orientations at the point of tangency are the opposite, they are said to anti-touch each other. We consider the sign of the radius of a cycle to be plus if its orientation is counter-clockwise otherwise minus. For a cycle γ and a ray x, $-\gamma$ and $-x$ denote the cycle and the ray along γ and x having the opposite orientations respectively. Two rays are parallel if they are parallel as lines and have the same orientation. For two rays x and y, where neither x, y nor x, $-y$ are parallel, we postulate the existence of a unique cycle which touches all the rays parallel to x or y, and we call it a cycle at infinity. We define the curvatures (reciprocal of radii) of cycles at infinity to be 0.

Our pattern in this section is closely related to two wasan problems. One is the following problem which can be found in Ushijima's 1832 book:

Problem 4. In Figure 16, γ_1, γ_2, γ_3, γ_4 are incircles of the triangles ABD, ADC, $AD'C'$, $AB'D'$ respectively. Given the radii of γ_2, γ_3, γ_4, find the radius of γ_1.

The other is the following problem, which can be found in Aida's 1797 book:

Problem 5. Four lines are tangent to a circle, and a circle γ_i (i = 1, 2, 3, 4) touches each three lines of the four as in Figure 17. Given the three radii of the four circles, find the remaining radius.

The answers of Problems 4 and 5 essentially state that

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4},$$

and

$$r_1 r_3 = r_2 r_4,$$

where r_i is the radius of γ_i.

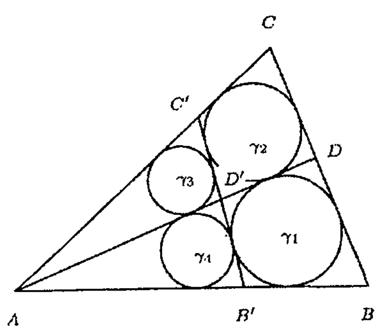

Figure 16

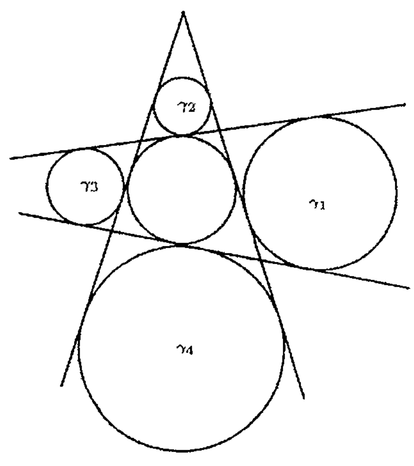

Figure 17

Investigating the symmetry suggested by the relation of the radii of circles in Problem 4, we found that the remaining external common tangent of γ_3 and γ_4 in Figure 16 is concurrent with the lines BC and $B'C'$. Therefore we can inductively draw circles and tangent lines which pass through A or the other point of concurrency (Okumura, 1990, see Figure 18).

Let us use the letter X instead of A and Y to denote the other point of concurrency. Then the configuration consists of two points X, Y and rays x_i and y_j (i and j are integers) passing through X and Y respectively, and cycles touching x_i, $-x_{i+1}$, y_j, $-y_{j+1}$ for any i and j. We denote the cycle touching x_i, $-x_{i+1}$, y_j, $-y_{j+1}$ by (i, j), and call X and Y the vanishing points of the configuration. If x_i and y_j are parallel, then (i, j) is a cycle at infinity.

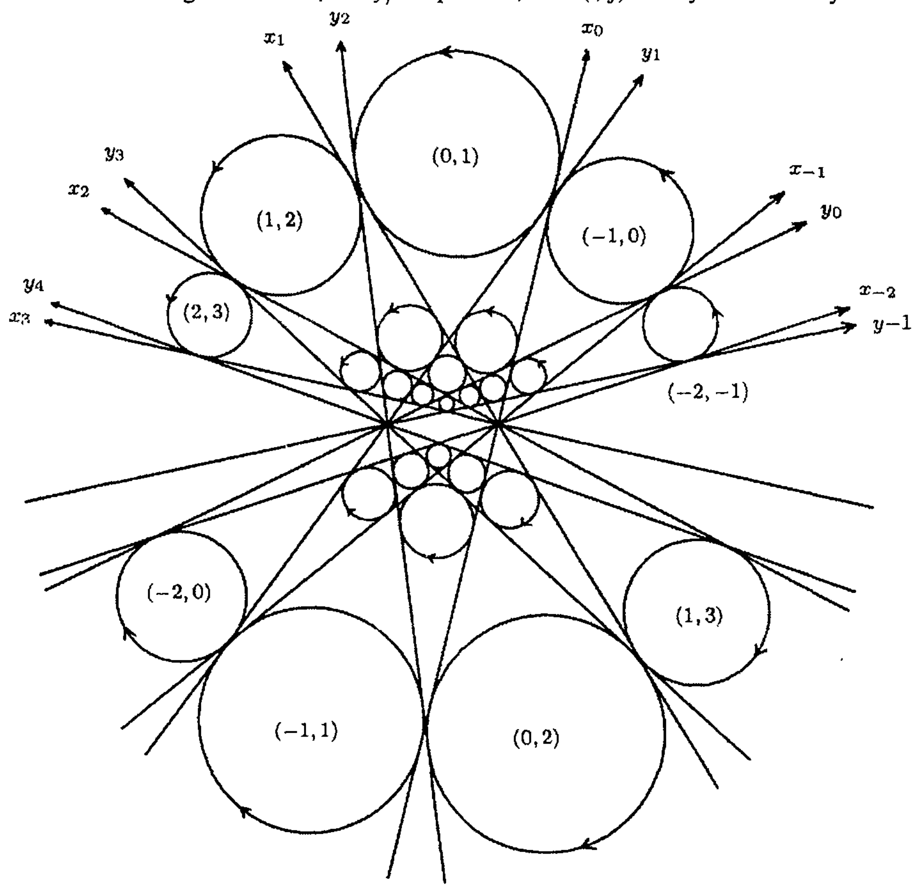

Figure 18

Our pattern can be regarded as a generalization of the tiling consisting of congruent rhombuses where both the vanishing points are points at infinity (see Figure 19). Let us consider the case where one of the cycles, say (0, 0) touches the line XY (as a circle and a line). This can be regarded as a limiting figure of Figure 18, when (0, 0) approaches the line XY and touches it. In this case all the cycles except (0, 0), (1, −1), (1, 0), (0, −1) degenerate to one of the points X and Y (see Figure 20).- Therefore our configuration also can be considered as a generalization of the configuration of a triangle with the incircle and the three excircles.

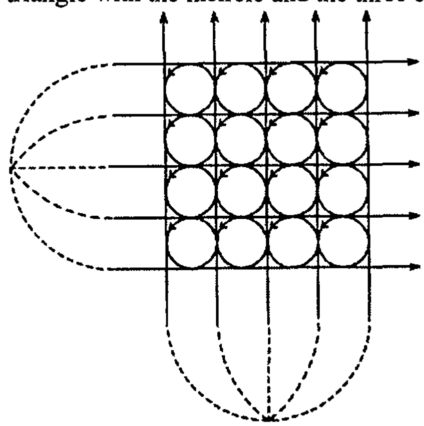

Figure 19

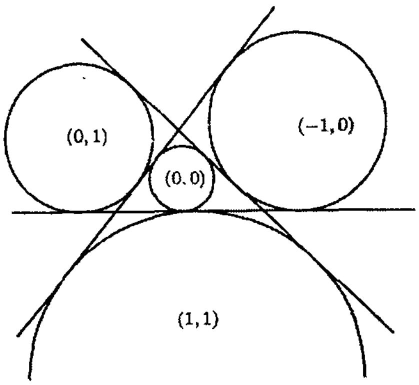

Figure 20

Our configuration has the following properties. (1) For any two cycles, their common tangent rays intersect on the line through the two vanishing points. (2) If we denote the curvature of the cycle (i, j) by $[i, j]$ then

$$[i, j] + [i+m, j+n] = [i+m, j] + [i, j+n],$$

$$[i, j][i + m + n, j - m + n] = [i + m, j - m][i + n, j + n].$$

These formula are generalizations of Problems 4 and 5 respectively. For another generalization of Problem 5, see (Okumura, 1989a). (3) For any integers i, j, k, the four lines x_i, $-x_{i+k}$, y_j, $-y_{i+k}$ have a common tangent cycle. The last fact can be derived from Theorem 4.5 in (Rigby, 1991).

6 PATTERNS ARISING FROM A SEVEN-CIRCLE PROBLEM

Our last patterns were discovered when the author was solving the following problem, which can be found in Sakuma's 1877 book (Okumura, 1995b).

Problem 6. Let γ be a circle of radius r, and γ_a, γ'_a circles of radii a touching γ internally at the end of a diameter of γ, and γ_b, γ'_b circles of radii b touching γ internally at the end of a diameter of γ and touching γ_a and γ'_a externally respectively, and γ_c, γ'_c circles of radii c touching γ internally at the end of a diameter of γ and touching γ_b, γ'_a and γ'_b, γ_a externally respectively. Suppose that the centres of γ_a, γ_b, γ_c, γ'_a, γ'_b, γ'_c form vertices of a convex hexagon. Given a, b and c, find r (see Figure 21).

The next problem is also related to our pattern, which can be found in Matsuzaki (1997, p. 42) and Hirayama and Yamaki (1967b), and essentially the same fact stated in Yamamoto's book which is cited in Fukagawa and Pedoe (1989, p. 32) with no solution:

Problem 7. Three circles touch each other externally, and another circle contains and touches them, and the centers of the three small circles form the vertices of a right triangle. Given the sum of two sides of the triangle, find the radius of the largest circles (see Figure 22).

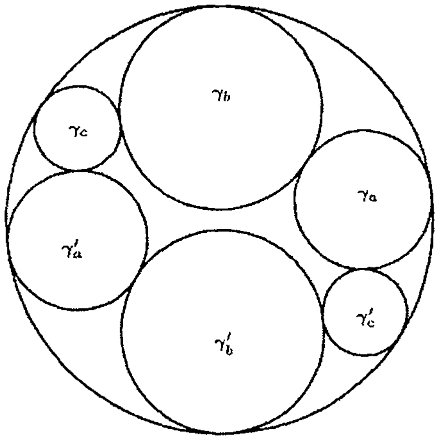

Figure 21

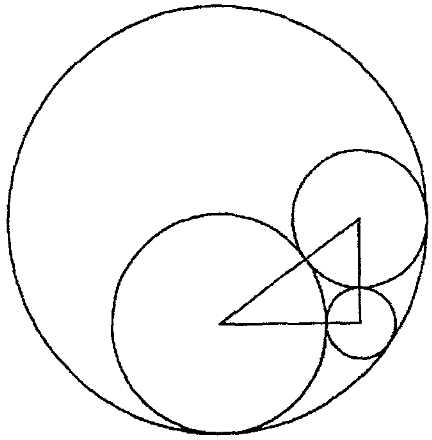

Figure 22.

The answer of Problems 6 is $r = a+b+c$, and the answer of Problem 7 is that the sum of the three sides equals a diameter of the largest circle. Hence they stated essentially the same fact. Later on we will show that Figure 22 is a special case of Figure 21 in a sense.

Now let us consider Problem 6. It is not appropriate to reproduce the long proof in Sakuma's book here, which is based on Pythagorean theorem. A proof using trigonometric functions can be found in (Fukagawa and Sokolowsky, p. 25), which derives a polynomial of six degrees. Let us assume that three circles of radii a, b, c touch each other externally. Next, tessellate the plane without gaps or overlaps by copies of the triangle formed by the three centers and consider a vertex D of the tessellation (see

Figure 23). There are six segments joining the neighboring vertices to D, which have length $a+b$, $b+c$, $c+a$ in pairs. We denote these vertices by C, C'; A, A'; B, B' and assume that A, B, C, A', B', C' lie around D in this order. Then draw six circles γ_A, γ_B, γ_C, $\gamma_{A'}$, $\gamma_{B'}$ and $\gamma_{C'}$ of radii a, b, c, a, b, c with centers A, B, C, A', B', C' respectively. It is obvious that each of the six circles touches its two neighbors externally. Since the distance between D and one of the intersections of the line DA and γ_A is $a+b+c$, γ_A touches the circle of radius $a+b+c$ with center D internally. Similarly the remaining five circles are tangent internally to the seventh circle with center D. Therefore a solution of Problem 6 follows from the uniqueness of the figure.

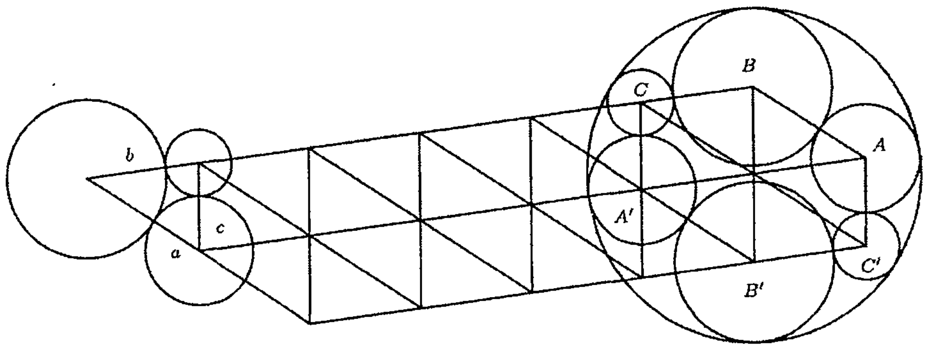

Figure 23

Now we refer to Problem 7. Let us again tessellate the plane as in the solution of Problem 6, using the triangle formed by the centers of the three small circles. Then we can see that there is a circle touching the small three, and that its radius is equal to the sum of the radii of the three. Therefore the uniqueness of the figure gives an immediate solution of Problem 7.

Let us denote the seventh circle in Figure 23 by γ_D. The tessellation suggests that we can construct a circle pattern in the plane consisting of copies of γ_A, γ_B, γ_C and γ_D. Let S and T be the translations mapping A into A' and B into B' respectively. If we draw the images of γ_A, γ_B, γ_C and γ_D by S^mT^n for all the integers m and n, we get a pattern as in Figure 24.

The reader may consider that the largest circles (the copies of γ_D) play a special role among the others. But we will show that each of the circles plays exactly the same role if we ignore the relation "one circle contains another" or "one circle is contained in another". To see this fact we use cycles as in the previous section. Let us assign counter-clockwise orientations to the circles γ_A, γ_B and γ_C and their copies, and clockwise orientations to γ_D and its copies so that each pair of touching circles anti-touch as cycles as in Figure 24. Then for each cycle, there are six cycles anti-touching it. The six cycles fall into three pairs such that each of the pairs consists of two congruent cycles anti-touching the first at the end of a diameter. Since the orientations of the largest cycles are opposite to the other, their radii have different signs from the others. Therefore the sum of the four different radii of the cycles in the pattern is equal to zero.

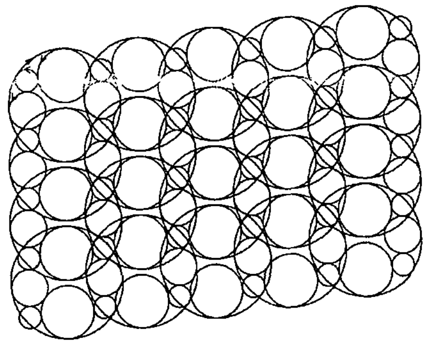

Figure 24

The centers of the circles form a triangular lattice in our pattern. Let us consider the case where $A'BC$ is a right triangle with the right angle at C. Then AA' and BB' are perpendicular, and the figure consisting of γ_A, γ_B, $\gamma_{A'}$, $\gamma_{B'}$ and γ_D is symmetric in the line AA'. This implies that the images of γ_C and $\gamma_{C'}$ by the reflection in the line AA' coincide with $\gamma_C{}^T$ and $\gamma_C{}^{T\text{-}1}$ respectively. Therefore $\gamma_C{}^T$ and $\gamma_{C'}{}^{T\text{-}1}$ touch γ_D internally. Similarly symmetry in the line BB' implies that γ_A and $\gamma_{B'}$ (and γ_B and $\gamma_{A'}$) also touch. Hence if we draw our pattern in this situation, it is symmetric in the lines AA' and BB', and each of the circles has eight tangent others (see Figure 25). In this sense, Problem 7 can be regarded as a special case of Problem 6.

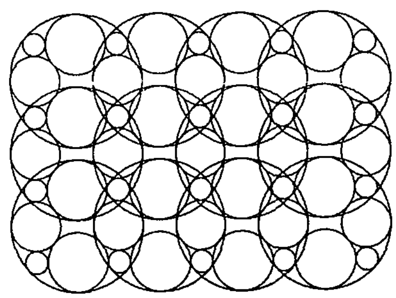

Figure 25

Arranging the shape of the triangles of the triangular lattice, we get patterns with various tangencies. Every circle is always tangent to six others in our pattern. When every circle is tangent to exactly n other circles in our pattern, we may say that the pattern is of the n-type. Then our pattern is of 6-type in general. Since our pattern is symmetric in the center of any circle, n is always even. As we have just seen, if the triangles of the lattice are right triangles, we can get a pattern of 8-type.

The fact stated in Problem 7 can easily be generalized, that is, the sum of the three sides of a triangle of the triangular lattice in Figure 23 is equal to a diameter of a largest circle. Indeed, if we construct our pattern with a triangle with the sides x, y and z, then the four different radii of the circles are $(x+y+z)/2$, $(-x+y+z)/2$, $(x-y+z)/2$, $(x+y-z)/2$.

If we construct a pattern with a triangle with the sides 1, $\sqrt{2}$, $\sqrt{5}$ (see Figure 26a), we get Figure 27, which is of 10-type. If we complete the pattern so that it has symmetry **p4m**, we get a pattern in which every circle touches 16 others (see Figure 28). Similarly we can take another triangle with the sides 1, $\sqrt{5}$, $\sqrt{17}$ (see Figure 26b). Then we can get another circle pattern with further tangencies. But unfortunately the resulting pattern is so messy that it is hardly worth describing it here.

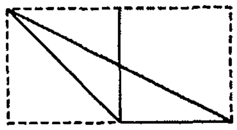

Figure 26.a.

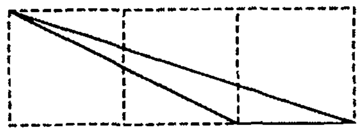

Figure 26.b.

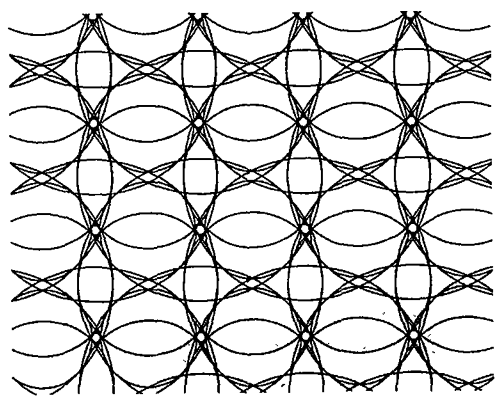

Figure 27

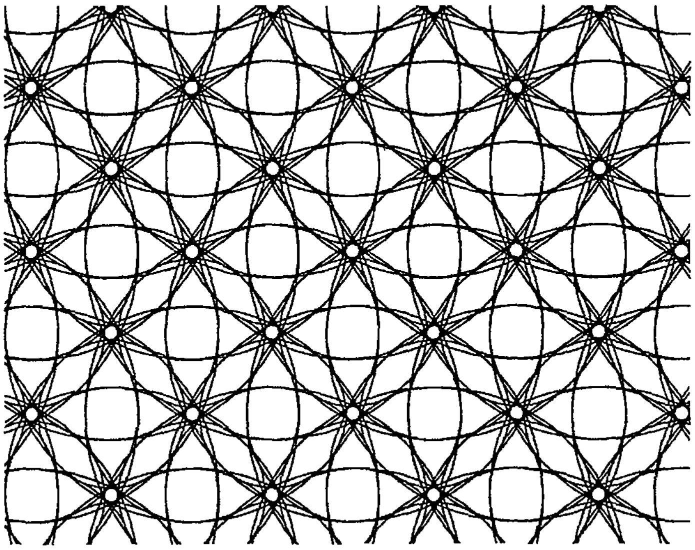

Figure 28

7 CONCLUSION

Almost all of the figures in geometric wasan problems consist of several elementary figures such as triangles, circles, rectangles and so forth. Therefore to generalize such problems is not to consider properties of a single elementary figure such as a triangle and a circle, but to try to find some relationship between such figures, and this can sometimes result in finding some configurations or patterns. Since wasan people liked to consider a certain inner area of an elementary figure, it is useful to try to extend such figures to some outer area. Also it is useful to try to embed such figures into more symmetric figures. By constructing several patterns arising from wasan problems, we were able to show that they are good sources for such experiments.

Since we confined our configurations to ones involving tangent circles, we do not refer to patterns consisting of polygons. But we can also construct such new patterns or configurations arising from geometric wasan problems. We hope to discuss them in a later paper.

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There is a pair of homonyms among the key words. In the title of Ushijima's book, *"Sangaku"* does not mean

"mathematical table", but "mathematics" itself.