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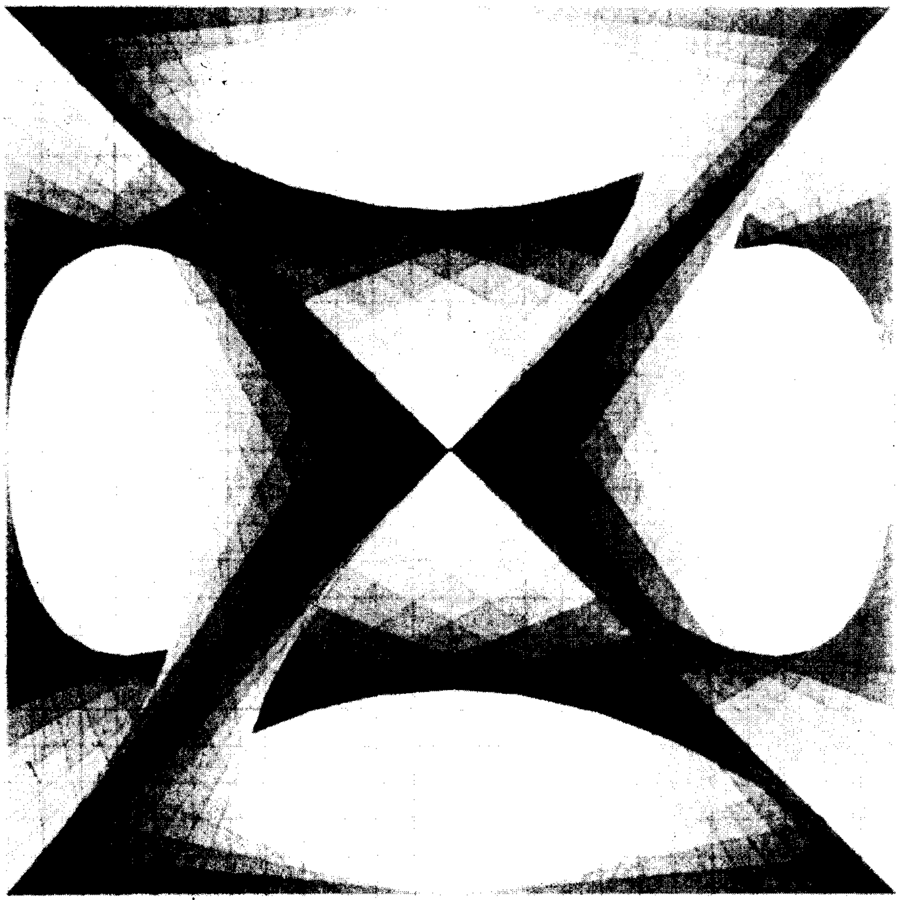

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# SYMMETRY IN TRADITIONAL JAPANESE MATHEMATICS

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***Abstract:*** *The aim of this paper is to show that* symmetry *was a very important concept in traditional Japanese Mathematics (in short, TJM) of the 18$^{th}$ and 19$^{th}$ centuries. Symmetry figures in the plane and space geometry are very useful for students. It is very important for students to work both the right and left sides of the brain, using both instinct and calculation.*

I acknowledge with grateful appreciation my indebtness to my friend the late Dan Pedoe for constant advises and encouragements.

## INTRODUCTION

### Historical overview of the traditional Japanese Mathematics

In the Japan of the 18$^{th}$ and 19$^{th}$ centuries, there was a native traditional mathematics based on Chinese mathematics and independent from the west. In this Edo-period (1603-1867), there were no universities or colleges but many private schools in Japan, some of which were in temples. The ruling family in Japan in those days, the *Tokugawa*s, had a strict policy which excluded Japan from contact with the west. This lasted from 1630 to 1867. And so, some native Japanese cultures developed in this period, including the art "*ukiyoe*". One of them was Traditional Japanese Mathematics, in short, TJM or *wasan.*

Many ordinary people enjoyed the problems of TJM as a creative recreation and lovers of TJM hung the votive wooden tablets (*sangaku*) in near temples or shrines and, on the tablet they wrote the mathematics problems with figures, all colored beautifully. The study of figures flourished in the Japan of the $18^{th}$ and $19^{th}$ centuries.

Japanese Geometry developed to find mainly the metric-relations of tangent circles, the relations of ellipses and circles touched externally or internally, with extensive use of the Pythagoras theorem. There was no development of general theorem with a logical structure, as in Euclidean geometry.

Mathematicians of TJM enjoyed solving the problems of mathematics and calculated intensively to discover beautiful relations.

They left many problems to us in mathematics, geometry, art and mathematics education.

### ***Sangaku.*** **Mathematical wooden tablets of TJM**

In the Edo-period (1603-1867) of Japan, there were neither universities nor colleges, so mathematicians had to make efforts to introduce mathematics to ordinary people their own. All over Japan, there were and are many temples and shrines where people have been hanging votive wooden tablets for worship of gods, or to express their desires, on the tablets many kind of pictures were drawn, for example, houses, flowers and poems. In Japan of nowadays, some students write the hopes for success in passing the entrance examination to high school or Universities on small size (15 by 10 cms) wooden tablets.

Mathematicians of TJM who wanted to display their discoveries to other mathematicians used the precincts of the temples and shrines where they hung wooden tablets of mathematics. The tablets had to be beautiful, interesting and creative for ordinary people to have an interest in the problem. So, in *sangaku* problems, there were many *symmetric* figures. For displaying mathematics problems to other people by means of the wooden tablet *sangaku*, geometry was the best, since the problems on geometry are visual, beautiful and creative.

So many *symmetric* figures are drawn in *sangaku* since *symmetric* figures are beautiful and ordinary people keep having an interest in geometry. This fashion of *sangaku* became popular and there were many tablets all over Japan. Many tablets have been lost.

About 850 tablets survived and even now more tablets are discovered by scholars of TJM. The sizes of *sangaku* are various. One of the small tablets is 50 by 30 cms on which three problems were drawn and is dated 1813. The largest tablet is 620 by 160 cms, on which 22 problems are drawn and dated 1877. Most of the tablets are 90 by 180 cms.

## 1 REPRESENTATIVE PROBLEMS FROM THE TRADITIONAL JAPANESE MATHEMATICS

In this chapter, we see the outline of Traditional Japanese Mathematics, even if the problems are not on symmetry.

### 1.1 Elementary problems from the most famous traditional Japanese mathematics book *Jingouki* (1627)

The problems of this section were for ordinary people to have an interest in arithmetic and mathematics.

Problem 1. Find the volume of your body in the bathtub. Take a bath in a bathtub full of water. See Fig. 1.

Problem 2. Find the weight of an elephant using a boat. Measure the load-water-line on the side of the boat when the elephant is on the boat and out of the boat. See Fig 2.

Problem 3. Find the method to share the 10 $\ell$ (liters) of oil of the pot $A$ into 5 $\ell$ and 5 $\ell$ using the pot $B$ (empty) and pot $C$ (empty), where the pots have no measure and when full, the pots $A$, $B$ and $C$ hold 10 $\ell$, 7 $\ell$ and 3 $\ell$, respectively. For example, at first, using pot $C$, we can share 10 $\ell$ of $A$ into 7 $\ell$ of $A$ and 3 $\ell$ of $B$. We denote this way as $A(10)$-$B(0)$-$C(0)$ $\rightarrow$ A(7)-B(0)-C(3). Then, at second, using the pot $B$, we can share 7 of $A$ and 3 $\ell$ of $C$ into 7 $\ell$ of $A$, 3 $\ell$ of $B$ and 0 $\ell$ of $C$.

$A(7)$-$B(0)$-$C(3)$ $\rightarrow$ (7)-$B(3)$-$C(0)$. Thus, $A(7)$-$B(3)$-$C(0)$ $\rightarrow$ $A(4)$-$B(3)$-$C(3)$ $\rightarrow$ $A(4)$-$B(6)$-$C(0)$ $\rightarrow$ $A(1)$-$B(6)$-$C(3)$ $\rightarrow$ $A(1)$-$B(7)$-$C(2)$ $\rightarrow$ $A(8)$-$B(0)$-$C(2)$ $\rightarrow$ $A(8)$-$B(2)$-$C(0)$ $\rightarrow$ $A(5)$-$B(2)$-$C(3)$ $\rightarrow$ $A(5)$-$B(5)$-$C(0)$.

The desired result is 5 $\ell$ of $A$ and 5 $\ell$ of $B$. We have other methods. See Fig 3.

Figure 1

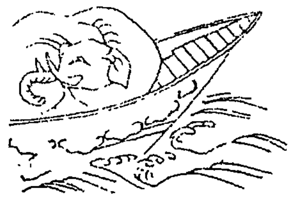


Figure 2

Figure 3

Problem 4. (Mice Arithmetic)

A pair of mice bear six male and six female in each month. On the first of January, a pair of mice bear six couples of mice, so, at the end of January, there are seven couples. On the first of February, each couple bears six couples of mice, so, at the end of February, there are $14 + 7 \times 12 = 2 \times 7^2$ in total.

Figure 4.1

Find the number of mice at end of the December and find the length of a long chain of these all mice, connecting the mouth of one to the tail of the next one, assuming the length of one mouse is 12 cm.

The total number of mice is $2 \times 7^{12}$. The author attempted to help readers to realize that the number is great, and so he added the latter part. The length of the long chain of mice is about 330000 km, that is, 82 times around the earth.

**Figure 4.2**

In old Japan, the mouse was the lucky animal for saving money. In the figure 4.1, we see the god of wealth *daikokusama* and see abacus. At this date, in Japan, arithmetic or operating of abacus was very important for ordinary life. The author aimed for readers to have interest in arithmetic so that he wrote the problem on mice.

Problems 1, 2, 3 and 4 are quoted from the book *Jingouki* (1627) of Mituyoshi Yoshida (1598-1672) which was the most famous mathematics book in Japan.

### 1.2 Cutting some square sheets

The contents of this section are very useful for the education of students nowadays in mathematics.

Problem 5. (Flower)

How do we cut the square paper sheet once by scissors for producing the flower of seven circles (See Fig 5.1)?

The answer is illustrated in Fig 5.2.

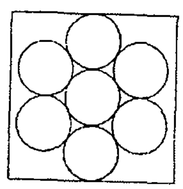


**Figure 5.1**

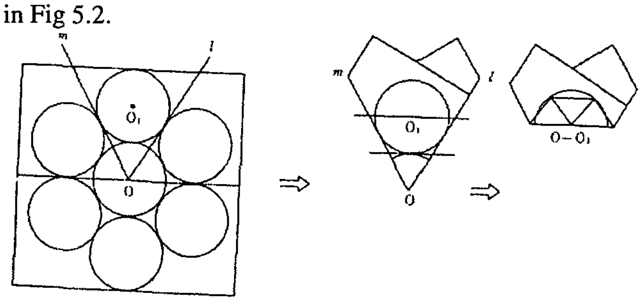



**Figure 5.2**

Problem 6. (Changing the rectangle into a square)

In this problem quoted from the book *Kanjya Otogizoshi* (1743), readers can learn about square roots.

(1) Cut the two unit square sheets for changing into a square of side $\sqrt{2}$ .

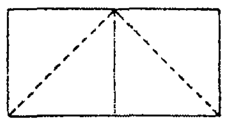


**Figure 6.1.1**

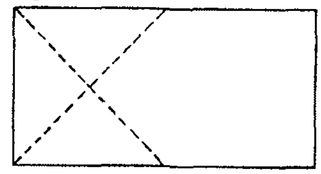


**Figure 6.1.2**

(2) Cut the three unit square sheets for changing into a square of side $\sqrt{3}$ for each case. Answers are illustrated in Fig 6.2.2.

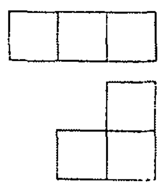


**Figure 6.2.1**

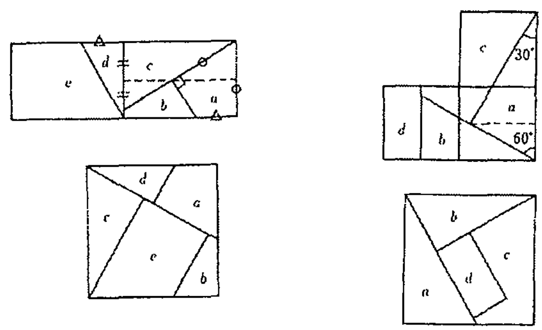
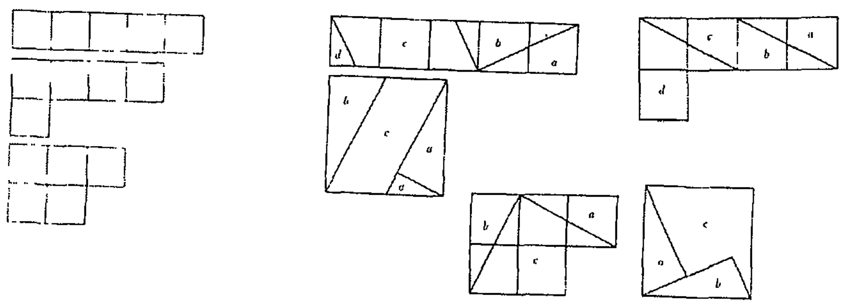



**Figure 6.2.2**

(3) Cut the five unit squares sheets for changing into a square of side $\sqrt{5}$ for three cases. Answers are illustrated in Figures 7.a, 7.b, 8.a and 8.b.



**Figures 7.a, 7.b, 8.a and 8.b**

### 1.3 Arithmetic in TJM

The Japanese mathematicians made enormous efforts to find an approximation of $\pi$ and get some results. They had a great power of operating the Japanese abacus, Soroban, and enormous ability in calculations. This paper is on *symmetry*, so I do not write about this part in detail, but the following are two problems.

Problem 7. (Approximation $\pi$)

The Japanese mathematician Yoshitomo Matunaga (1692-1744) found the following series for $\pi$ and succeeded in finding the right 51 first digits of $\pi$. He obtained the series inductively in 1739.

$$3(a_1 + a_2 + a_3 + a_4 + a_5 + \ldots) = \pi$$

where $a_1 = 1, a_2 = \dfrac{1^2}{4 \bullet 6} a_1, a_3 = \dfrac{3^2}{8 \bullet 10} a_2, a_4 = \dfrac{5^2}{12 \bullet 14} a_3, a_5 = \dfrac{7^2}{16 \bullet 18} a_4, \ldots\ldots$

This expression was very useful for algorithm operations with an abacus or a modern computer.

Problem 8. (Prime decomposition in integers)

Consider the positive integers *f(n)* = 111.... 1, "1" occurs *n* times. Decompose into primes the integers *f(n)* for *n* = 2, 3, 4, 5, 6, ... , 18.

Answer: *f*(2)=11, *f*(3)=111=3x37, *f*(4)=1111=11×101, *f*(5)=11111=41×271, *f*(6)=3x7×11×13x37, *f*(7)=239x4649, *f*(8)=11×73×101, *f*(9)=×37×333667, *f*(10)=11x41x271x9091, *f*(11)=21649x513239, *f*(12)=3x7×11x13x37x101x9901, *f*(13)=53x79x265371653, *f*(14)=11×239x4649x909091, *f*(15)=3x31x37x41x271× 2906161, *f*(16)=11x17x73x101x137×5882353, *f*(17)=2071723x5363222357, *f*(18)=×7× 11x13x19x37x52579x333667.

They found the big number 265371653 in *f*(13) is prime by using only abacus. How excellent their work is! This problem is quoted from Naonobu Ajima's (1732-1798) manuscript *Fujin Issujutu* (recorded in 1869).

### 1.4 Some curves

The Japanese mathematicians studied some curves with the aim of calculating the areas or lengths. The curves which are the trace of a moving point, cycloid or cardioid, were imported from some western book, I guess. But they tried to calculate the areas and lengths using their method only since they did not know the western method. The following are two examples of them.

Problem 9. (Archimedes spiral)

*ABO* is a sector of a circle $O(r)$, with $r = 10$ and $AB = 20$. A point $T$ moves along the arc from $A$ to $B$. A point $P$ on $OT$ moves from $O$ to $T$ in a manner such that $OP = k\,(\angle TOA)$ and $P = B$ when $T = B$. The locus of $P$ is called an Archimedes spiral in western mathematics. Find its length. The length is

$$\frac{5}{2}(2\sqrt{5} + \log(2+\sqrt{5}) = 14.78923....$$

Japanese mathematicians had no concept of logarithm, so they gave the answer as an infinite series.

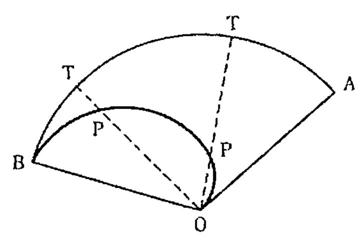



**Figure 9**

Problem 10. (Turtle circle)

A circle $K_1 = O_1(r)$ is in external contact with a circle $K_2 = O_2(r)$, remaining in contact as $K_1$ rotates, say clockwise, around $K_2$. Initially a point $P$ of $K_1$, coincides with a point $A$ of $K_2$. As $K_1$ moves, $P$ rotates clockwise around $O_2$ in such a manner that for each point $T$ at which the circles touch, $AT{=}kTP$ with $k = R/r$ The path $L$ traced by $P$ as it rotates once around $O_1$ is called a turtle circle. Show that the area it encloses is $S = \{(R+r) + 2r^2\}$ which is simple result, and that its length is the same as the circumference of an ellipse with major axis is $2a = 2(R+3r)$ and minor axis $2b = 2(R\text{-}r)$. The former result was found by Yasushi Wada (1787-1840) who suspected plagiarism by the other mathematician, so Wada hung *sangaku* in the precinct of Atagoyama in a hurry, on which his beautiful result was written.

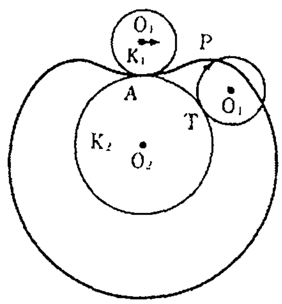



Figure 10

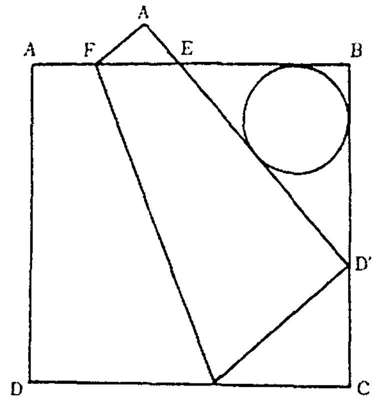



Figure 11

## 1.5 Interesting *sangaku* problems

Problem 11. (Folding a square paper problem)

Fold a square sheet $ABCD$ such as $D$ lies on the side $BC$. Conjecture which one of some segments $AF$, $FE$, $BE$, $BD'$ and $D'C$ is the same as the radius of triangle $EBD'$ by instinct. And then prove the rightness of your injecture.

This interesting problem is written on the *sangaku* of 1893. The old solution runs as follows: The triangle $AF'E$ is a right triangle and $AF' = AF$. Let $r_0$ be the inradius of triangle $A'EF$, and $r$ be the inradius of triangle $EBD'$. Then

$$ED' = A'D' - A'E,\ BE = AB - (A'F + EF),\ 2r_o = A'F + A'E - EF.$$

From $r : BE = r_o : A'E$, $r \cdot A'E = r_o \cdot BE = r_oAB - r_o(A'F + EF)$ and from $r : D'E = r_o : EF$, $r_oEF = r_oD'E = r_o(A'D' - A'E)$. *Hence* $r(EF - AE) = = r_o(A'F + E'F + A'E) = \frac{1}{2}(A'F + A'E - EF)(A'F - A'E + EF) = \frac{1}{2}(A'F^2 - A'E^2 + + 2A'\cdot EF - EF^2) = A'E(EF - A'E)$ *and* $r = A'E$.

Problem 12.

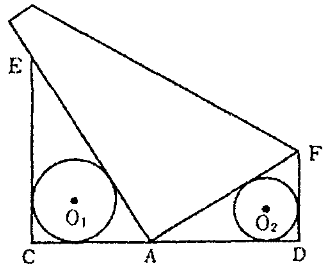



**Figure 12**

A square sheet of paper $ABCD$ is folded so that $A$ falls on a point $A'$ on $CD$, forming the two right triangles $A'CE$ and $A'DF$ shown. Let $O(p)$ and $O(q)$ be the respective incircles of these triangles. Find the maximum possible difference of $p$ and $q$ in terms of the side $a$ of $ABCD$.

Since $A'D = x$, $p = \dfrac{x(a-x)}{a+x}$ and $q = \dfrac{x(a-x)}{a}$.

We can find maximum of the difference $f(x) = p - q = \dfrac{x(a-x)}{a(a+x)}$.

$f'(x) = 0$ implies the maximum of $f(x)$ is $(71 - \dfrac{17\sqrt{17}}{16})a$ when $x = \dfrac{\sqrt{17}-3}{4}a$.

This problem is a *sangaku* problem. The *sangaku* is lost but recorded in the manuscript *Juringi Hougaku Sanpou*, and the *sangaku* was hung about 1865.

Problem 13.

Show that the following ten *sangaku* problems give the result $n$:1 for the $n$-th problem. This *sangaku* of 1865 survived and is 90×120 in size. All problems are in beautiful colors, for example, red circle $R(r)$ implies the circle is colored in red, the center is $R$ and the radius $r$. A worshipper having an interest in mathematics could guess the results, but he may not find the solution. So, he visited the proposer of the *sangaku* problem to ask the solution. Thus, the proposer could gather his fellows as students. The following problems are not so difficult, and the solutions are left for readers.

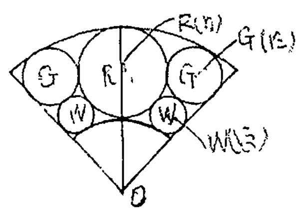



**Figure 13.1**

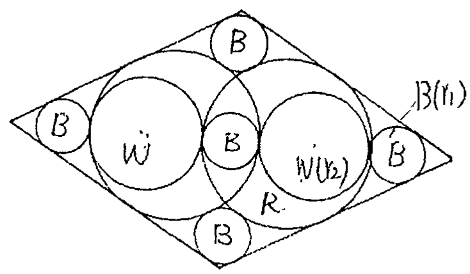



**Figure 13.2**

(1) One red circle $R(r_1)$, two green cirles $G(r_2)$ and two white circles $W(r_3)$ touch, as shown in the figure, in the sector of circle with the center $O$. Show that the ratio $r_1 : r_2 + r_3 = 1 : 1$.

(2) As in the figure, intersecting two red circles, there are five blue circles $B(r_1)$ and two white circles $W(r_2)$ in the rhombus. Show that the ratio $r_2 : r_1 = 2 : 1$.

(3) Yellow circle $Y(r_1)$ is the circle of curvature at the end point $A$ of the major axis of the ellipse, and the line $\ell$ touches the ellipse at $A$. Red circle $R$ touches internally the ellipse at two points, at which points the other two red circles $R$ touch the ellipse externally and the line $\ell$. Blue circle $B(r_2)$ touches the red circle and the ellipse internally. Show that the ratio $r_2 : r_1 = 3 : 1$.

(4) As shown in the figure, there are two blue circles $B(r_1)$, two white circles, two green circles, one red circle $R$ and yellow circle $Y(r_2)$ in the rectangle. Show that the ratio $r_1 : r_2 = 4 : 1$.

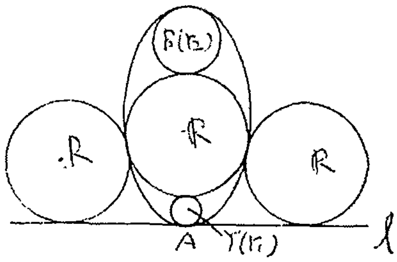



Figure 13.3

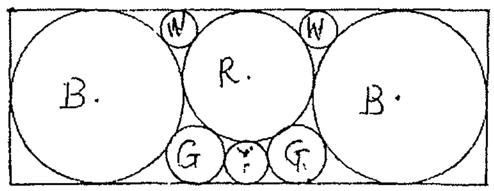



Figure 13.4

(5) In the blue semicircle, two red circles $R(r_1)$, one yellow circle $Y(r_2)$ and two white circles, as shown. Show that the ratio $r_1 : r_2 = 5 : 1$.

(6) White circle $W(r_1)$ passes through the centers of the other two touching white circles. Two green circles having the radius of $r/2$ touch the white circles and the centers of green and white circles lie on a line. Eight red circles $R(r_2)$ touch the green and white circles. Show that the ratio $r_1 : r_2 = 6 : 1$.

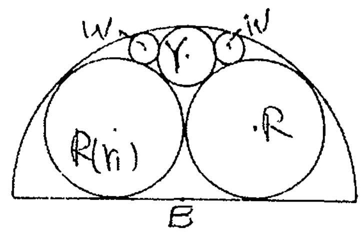



Figure 13.5

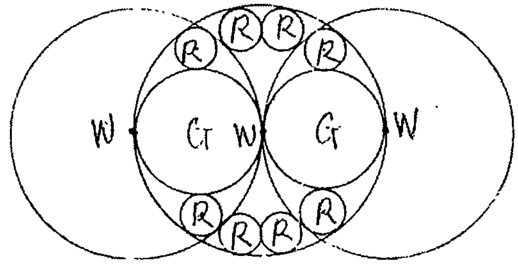



Figure 13.6

(7) $AC$ is the diameter of white circle $W(r)$ and the diagonal of rhombus $ABCD$, both. Three red ellipses with major axis $2a$ and minor axis $2b$ touch in the rhombus with axes on $AC$. Show that the ratio $r : b = 7 : 1$ when $2b$ is maximum.

(8) Twelve circles touch each other externally in a yellow $Y(r)$ circle, as shown in the figure. Show that the ratio $r : t = 8 : 1$ where $t$ is the radius of white circle.

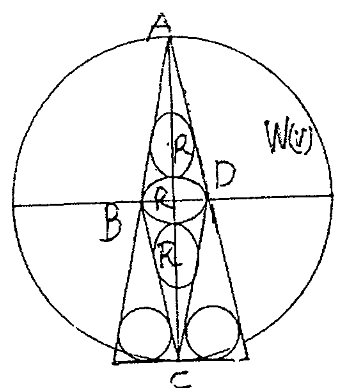



Figure 13.7

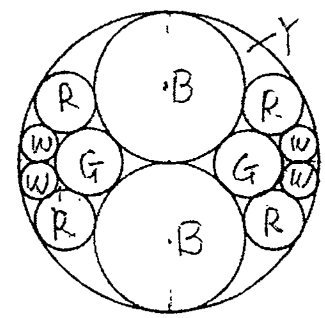



Figure 13.8

(9) Blue circle is the circle of curvature at the end point of the major axis of two congruent white ellipses, both. Draw the red circle with the diameter equal to the major axis. If we can draw four blue circles having the radius $r$ equal to that of the circle of curvature. Show that the ratio $AA' : 2r = 9 : 1$.

(10) In an equilateral triangle *ABC*, each of three green circles touches two sides of the triangle. Seven red circles and six white circles *W(t)* touch externally as shown in the figure. Show that the ratio $t : r = 10{:}1$ where t is the inradius of the triangle.

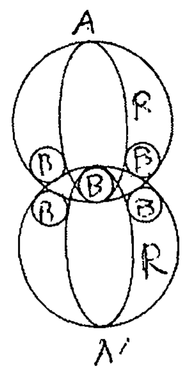



Figure 13.9

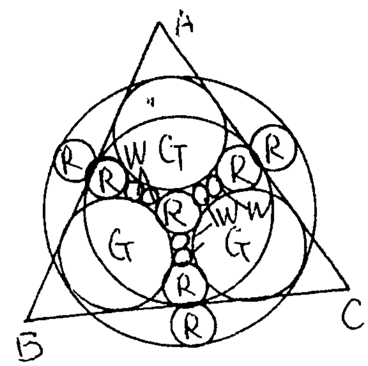



Figure 13.10

## 2. PROBLEMS WITH *SYMMETRIC* FIGURES IN A PLANE

In TJM, there are many problems with *symmetric* figures and with *asymetric* figures, too, in a plane. For the aim of this paper, some problems with *symmetric* figures are introduced in this section.

### 2.1 Some elementary problems with *symmetric* figures quoted from the traditional Japanese mathematics books and *sangaku*.

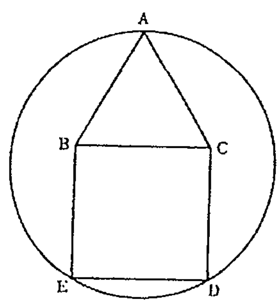



Figure 14

Problems 14, 15, 16, 17 and 18 deal with the inside of a circle.

Problem 14.

In the figure, *ABC* is an equilateral triangle, *BCDE* is a square. If the circle *O(r)* passes through *A, D* and *E,* show that $r = AB$.

Original solution: Construct equilateral triangle $EDA'$ in the square. Then $ACDA'$ is a parallelogram so that we have $r = AA' = A'D = A'E = AB$.

Problem 15.

A loop of six circles $O_1(a)$, $O_2(b)$, $O_3(c)$, $O_4(a)$, $O_5(b)$, $O_6(c)$ touches a circle $O(r)$ internally. Show that $a + b + c = r$.

Solution:
For $\angle O_3OO_2 = \alpha$, $\angle O_2OO_1 = \beta$ and, $\angle O_1OO_6 = \chi$, $\alpha + \beta + \chi = \pi$. Then, $\cos^2\alpha + \cos^2\beta + \cos^2\chi = 1 - 2\cos\alpha \cos\beta \cos\chi$. By Cosine-rule, we have the equation $r^6 - 2(a+b+c)r^5 + (a+b+c)^2r^4 + 16abcr^3 - 16abc(a+b+c)r^2 = 0$, that is, $\{r - (a+b+c)\}\{r^3 - (a+b+c)r^2 + 16abc\} = 0$, and the desired result $a + b + c = r$.

Problem 16. (Lost *sangaku* of 1789)

$O(2r)$ has the diameter $AB$ and $C(r)$ touches $AB$ at $O$ and $O(2r)$ at T. $O_1(r_1)$ is inscribed in the curvilinear $OBT$, and there is a chain of contact circles $O_i(r_i)$ ($i = 2, 3, \ldots$) where $O_2(r_2)$ touches $O_1(r_1)$ and also $C(r)$ and $O(2r)$, $O_3(r_3)$ touches $O_2(r_2)$ and also $C(r)$ and $O(2r)$, and so on. Find $r_n$ in terms of $r$.

We can solve this problem by using "Inversion method", but Japanese mathematicians used the Pythagoras theorem only, very often.

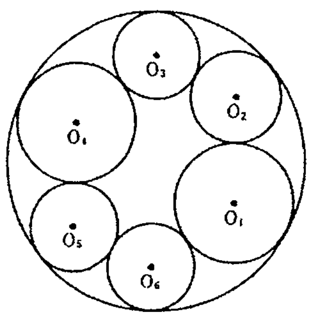



**Figure 15**

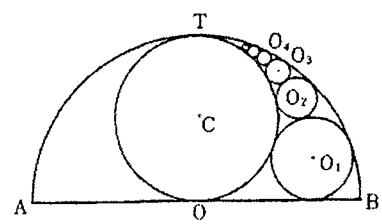



**Figure 16**

Problem 17. (Construction for a regular pentagon)

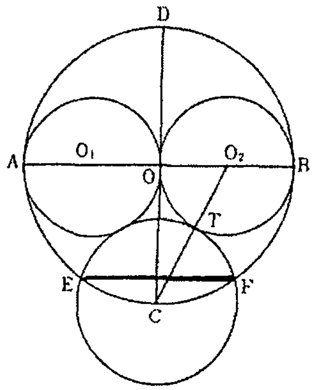


**Figure 17**

We give a simple Euclidian construction for a side of a regular pentagon inscribed in a circle. Let the circle center $O$ have radius 1, and let $AB$ be a diameter. Draw circles $O_1$ and $O_2$ with respective diameters $AO$ and $BO$. Let $DOC$ be a perpendicular diameter, and suppose that $CO_2$ intersects the circle $O_2$ in $T$, where $T$ lies between $C$ and $O_2$.

Finally let the circle center $C$ with radius $CT$ intersect the circle $O$ in $E$ and $F$. Then $EF$ is a side of a rectangular pentagon inscribed in the unit circle. The method was discovered by the Japanese mathematician Yoshifusa Hirano.

Problem 18. (Lost *sangaku* of 1717)

Nine circles of radius $r$ are packed as shown in a circle of radius $R$. Express $r$ in terms of $R$ and show that ten circles of radius $r$ can be packed in the same circle by an appropriate arrangement.

Answer: $r = \dfrac{\sqrt{8}-1}{7}R$. For latter part, see the adjacent figure.

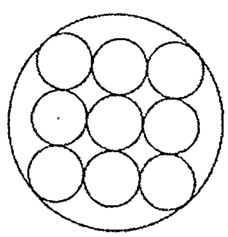


**Figure 18.1**

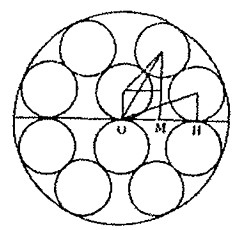


**Figure 18.2**

Problem 19.

Three *sangaku* problems with *symmetric* figures. In each problem, show that blue circle is equal to red circle.

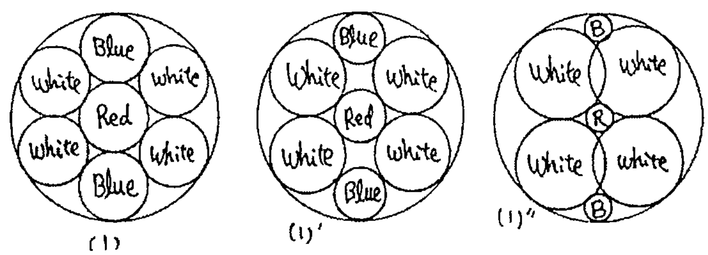



Figure19.1

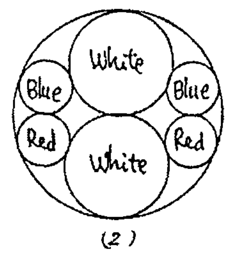



Figure19.2

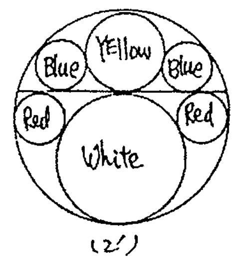



Figure 19.2'

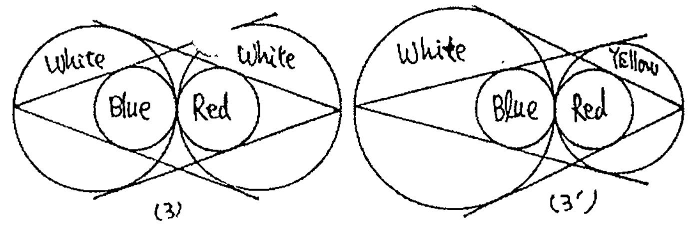



Fig 19.3

Problem 20.

The side $a$ of the square drawn by a thick line is given, then find the radius $r$ of small circles. For each problem, the answer is the same: $r = \frac{\sqrt{2}-1}{2}a$.

How nice and excellent these problems are!

For students, problem figures of geometry should be beautiful and creative. Beautiful and *symmetric* figures evoke an instinct in the brain of students and then the proving becomes a necessary calculation. The study of geometry use makes use of both right and left hemispheres of the brain fully. A group of problems is called *idai doujutu* which means that the problems are different but the result is the same. Problem 20 is an example of *idai doujutu* where 12 problems are introduced. In the book *Youjutu Shindai* of 1878, there are a hundred different problems concerning one square of side a and some small circles of radius r are described but the result is only one $r = a/8$. This is a marvelous work of TJM. See Fukagawa and Dan Pedoe *Japanese Temple Geometry Problems: Sangaku*, Appendix.

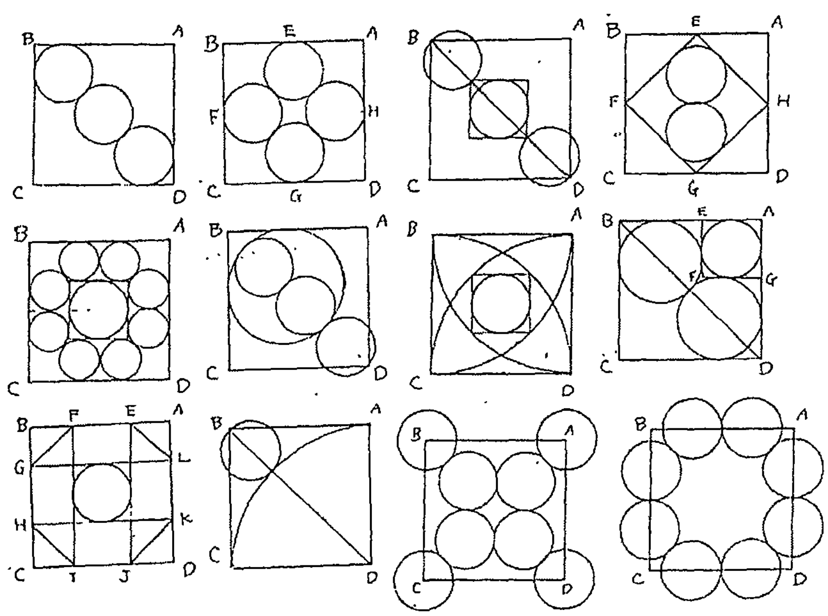
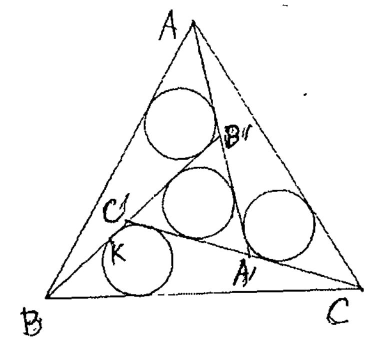


**Figure 20**

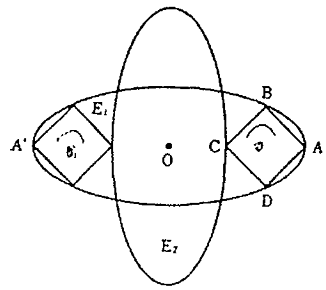
In the past year, a *sangaku* of 1797 has been discovered in a shrine of Aichi prefecture. This *sangaku* is 542 *cm* (width) × 31 *cm* (heigth) and thirty problems are written on the *sangaku* in beautiful colors. We introduce a problem of these below.




Problem 21. (Circle and triangle)

From each vertex of given equilateral triangle $ABC$ of sides $a$, draw three lines $AA'$, $BB'$ and $CC'$, and separate this triangle into four triangles where each triangle has the same inradius $r$. Find $r$ in terms of $a$. Concerning the areas $\Delta ABC = 3\Delta C'BC + \Delta A'B'C'$ or $\frac{\sqrt{3}}{4}a^2 = \frac{3r}{2}(C'B+BC+CC')+3\sqrt{3}r^2$, where $C'B + BC + CC' = 2a + 2C'K = 2a + 2r/3$ for the point $K$ on the side $BC'$. Thus, $\frac{\sqrt{3}}{4}a^2 = \frac{3r}{2}(2a+\frac{2r}{\sqrt{3}})+3\sqrt{3}r^2$ or $16a^2 + 4\sqrt{3}ar - a^2 = 0$. $r = (\frac{\sqrt{7}-\sqrt{3}}{8})a$.

Problem 22. (Squares and ellipses: Lost *sangaku* of 1850)




Children are flying a *kite* in the sky. On the face, two ellipses meet perpendicularly, that is, the major axis of each ellipse $O(a,b)$ are perpendicular lines at the center $O$, and two squares touch the ellipses, the diagonals lie on the one major axis. Show that the side of each square is $b$.

Solution using the equation of the ellipse, which is not the original solution. For convenience, we use the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the point $B((a+b)/2, (a-b)/2)$ lie on the ellipse which implies $\frac{(a+b)^2}{a^2}+\frac{(a-b)^2}{b^2}=4$ or $a^4 - 2a^3b -$ $- 2a^2b^2 2ab^3 + b^4$ thus we obtain $a = (1+\sqrt{2})b$ and the side of square $\frac{\sqrt{2}}{2}(a-b)=b.$

## 2.2 Remark on *sangaku* problems

The Japanese mathematicians tried to make mathematics problems as beautiful as possible.

Problem 23. (*Sangaku* problem that is *asymmetric*)

Show that the distance of two circles $O_1$ and $O_3$ is the distance of two circles $O_2$ and $O_4$, as shown in the figure.

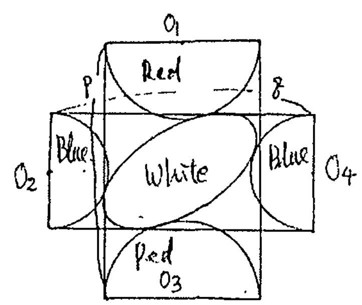



**Figure 23.1:** *Sangaku* problem

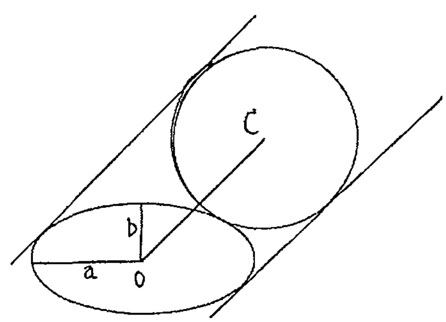



**Figure 23.2:** Mathematics problem

When some mathematicians see a problem, then they may say that there are some needless lines or circles in (1) and the problem is to be (2). The description of the problem is to be the following.

"An ellipse $O(a,b)$ and a circle $C(r)$ touch each other externally, and both touch a pair of parallel lines. Show that $OC = a+b$."

Why did the proposer draw some lines or circles, in addition? Why did the proposer draw the figures in colour?

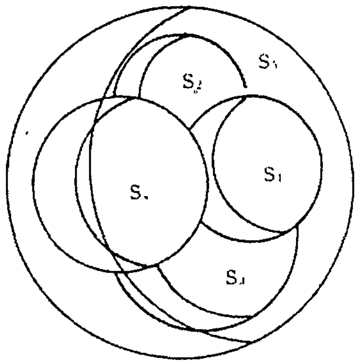
The reason is simple. When ordinary people looked briefly at the *sangaku*, it was hoped that they would show an interest in the problem. Therefore, proposer aimed for ordinary people to have an interest in the problem by drawing the problem in beautiful colour and the figure became more art than mathematics. This point of view of *sangaku* is very important in mathematics education. "Beauty" is important to children, the present author believes.

## 3. PROBLEMS WITH *SYMMETRIC* FIGURES IN SPACE

The Japanese mathematicians studied many figures of space asking the volumes in calculus and relations of contacting spheres. In this chapter, we introduce some theorems on spheres and the volumes of many polyhedra.

### 3.1 Soddy's *Hexlet*

Given three spheres which touch each other, any loop of $n$ contact spheres which touch all three given spheres must be restricted to $n = 6$. The term *Hexlet* was introduced by Frederick Soddy in 1936 which is the following theorem. The same theorem, Problem 24, appeared on a tablet in the Kanagawa prefecture in 1822. The tablet has vanished. Problem 24 is *asymmetric,* but it is very important in solving the problems on contacting spheres.



Problem 24. (Lost *sangaku* problem of 1785)

We have a loop of contact spheres $S_i(r_i)$ ($i$ = 1, 2, 3, 4) and $S_5(r_5)$ is the sphere which touches them all internally or all externally. Show that $3\sum_{i=1}^{5}\frac{1}{r_i^2}=\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2$.

The Japanese mathematicians gave an original solution in seven pages to this problem in a book, in detail, written by wooden block-printing. See Hidetoshi Fukagawa and Dan Pedoe *Japanese Temple Geometry Problems*: *Sangaku*, p. 186. For modern proof, using a symbolic algebra of spheres, see Dan Pedoe *Geometry: A Comprehensive Course.*

Problem 25.(The Soddy's Hexlet. *Sangaku* problem of 1822.)

The spheres $O_1(r_1)$ and $O_2(r_2)$ lie inside $O(r)$, touch each other, and also touch $O(r)$. There is a loop of contact spheres $S_i(t_i)$ ($i$ = 1, 2, 3,..., $n$) which all touch $O_1(r_1)$, $O_2(r_2)$ and $O(r)$. Show that $n = 6$ and $\frac{1}{t_1}+\frac{1}{t_4}=\frac{1}{t_2}+\frac{1}{t_3}$.

Problem 26. (Lost *sangaku* problem of 1839)

Three spheres $O_1(r_1)$, $O_2(r_2)$ and $S_1(t_1)$ lie on a plane $\alpha$ and touch each other. Starting with $S_1(t_1)$, we form a chain of spheres $S_j(t_j)$ ($j$ = which touch each other, lie on the plane $\alpha$, and also touch the spheres $O_1(r_1)$ and $O_2(r_2)$. Show that the construction terminates when $n = 6$, with $S_6(t_6)$ touching $S_1(t_1)$.

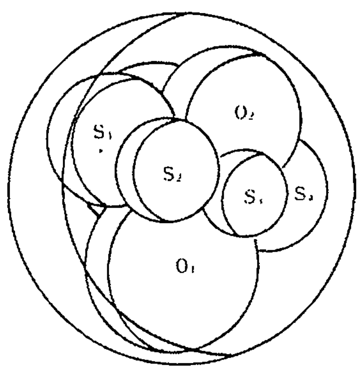


**Figure 25**

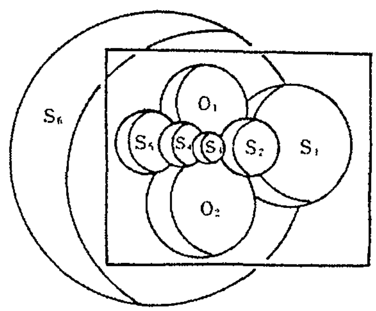


**Figure 26**

## 3.2 Find the number of small spheres in the bounded region

The contents of this section is peculiar to TJM, according to the present author.

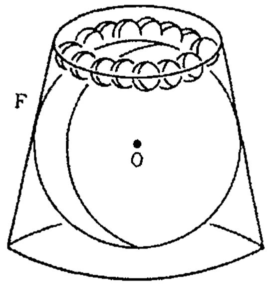


Figure 27

Problem 27.(Lost sangaku problem of 1814)

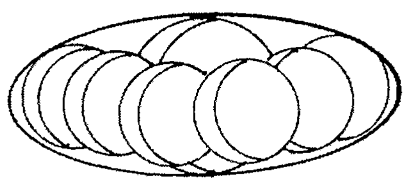
A sphere $O(r)$ is the inscribed in a frustum of a cone of height $h$ and base radii $a$ and $b$ internally, $a<b$. Allowing $a$ and $b$ to vary, find the maximum number of equal contact spheres touching $O(r)$ externally and $F$ internally as shown.

A traditional Japanese mathematician presented the above mentioned problem and gave the result $n$ = 16. They used an inequality that Fukagawa described definitely, using modern terms;

$$n \le \frac{\pi}{\sin\dfrac{\pi}{n}} < n+1 \quad \text{or} \quad \left[\frac{\pi}{\sin\dfrac{\pi}{n}}\right] = n.$$

Thus, the result is $n$ = 16. The symbol [ ] implies to omit the figure below the place of decimals, that is, if $n \le x <$ n+1, then $[x] = n$.

Problem 28 (Lost *sangaku* problem of unknown date)



An oblate ellipsoid is obtained by the revolution of the ellipses $O\ (a,\ b),\ a>b$, about its minor axis. Sphere $O(b)$ of radius $b$ is inscribed in the oblate ellipsoid. Consider the loop of small spheres of same radius touching $O(b)$ externally and each of them touching the ellipsoid at two points.

Find the maximum number of small spheres. For convenience, we write the equation of the ellipse as $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The equation of small circle is $(x\text{-}p)^2 + y^2 = r^2$ and $p = b+r$.

Double contact of the ellipse and circle implies $p^2b^2 = (a^2 - b^2)(b^2 - r^2)$ and $r=\dfrac{b(a^2-2b^2)}{a^2}$ which implies $p = b + r = 2b(a^2 - b^2)/a^2$. Fukagawa's inequality shows that the number of spheres is $n=\left[\frac{p}{r}\pi\right]=\left[2\pi\frac{a^2-b^2}{a^2-2b^2}\right]=\left[2\pi\frac{1-k^2}{1-2k^2}\right]$ $(0<k<1)$.

If $n = 6$, then six equal spheres touch one sphere, so these six spheres touch two parallel planes which is wrong. $n \geq 7$. The radius of circle of curvature at end of major axis is $r = b^2/a$ which implies $a = 2b$ and $n \leq 9$. $n = 7, 8, 9$.

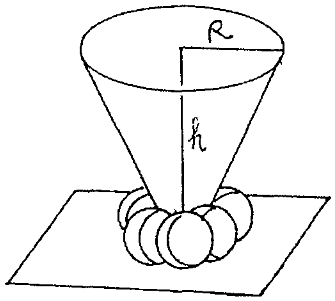



**Figure 29**

Problem 29. (Lost *sangaku* problem of 1799)

The cone of height $h$ and circle of radius $R$ as base touches on the plane perpendicularlly, as in the figure. Construct the chain of equal spheres of radius $r$ touched on the cone externally and the plane. Find the number $n$ of spheres in terms of $R$ and $h$. The result is $n=\left\lfloor\sqrt{1+\left(\frac{R}{h}\right)^2}+\frac{R}{h}\pi\right\rfloor$ obtained by using Fukagawa's inequality.

### 3.3 Polyhedron in TJM

The Japanese mathematicians studied many different kinds of polyhedra. The idea of TJM in this part was to find the volume only in terms of the side $a$, not to find the relations theoretically since they had a powerful ability of calculation. Polyhedra of this section are quoted from the manuscripts *Syukisanpou kaigi* (1801) and *Surimujinzou* (1830).

Problem 30.

Find the volume of the following regular polyhedra in terms of the sides $a$.

(1) Tetrahedron. (2) Cube. (3) Octahedron.(4) Dodecahedron.(5) Icosahedron. See Fig 30.1.

Answer: the original solution of (1), as shown in the Fig. 30.2, is very useful in mathematics education. Remove four right pyramids with the equilateral triangle of side $a$ as the base from the cube of side $a/\sqrt{2}$.

$$V=\left(\frac{a}{\sqrt{2}}\right)^3-4\left[\frac{1}{3}\frac{a}{\sqrt{2}}\left(\frac{a}{\sqrt{2}}\right)^2\right]=\frac{\sqrt{2}}{12}a^3.$$

(2) $a^3$ is clear. (3) $\frac{\sqrt{2}}{3}a^3$. (4) $\frac{15+7\sqrt{5}}{4}a^3$. (5) $\frac{15+5\sqrt{5}}{12}a^3$.

For obtaining the results, they used Pythagoras theorem only, very often. Copy of original solution to (5) is introduced below.

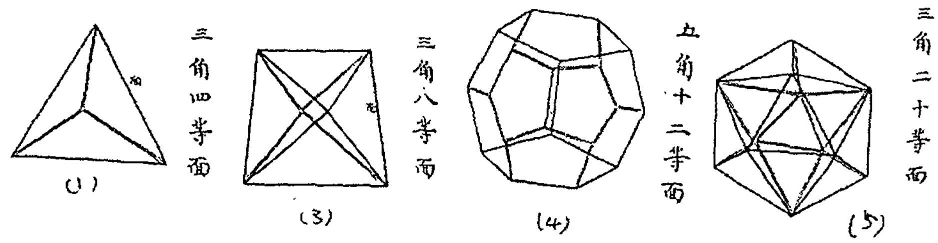



Figure 30.1

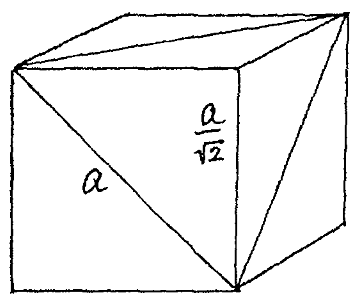



Figure 30.2

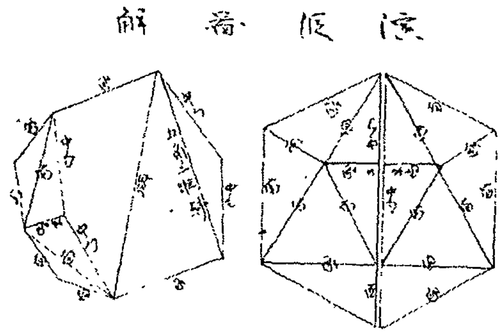
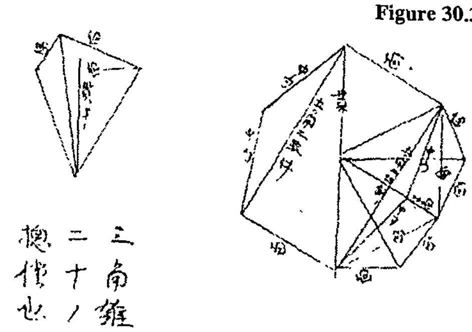
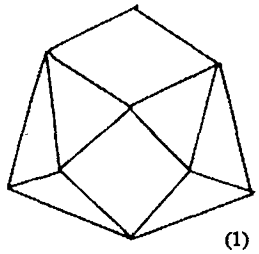
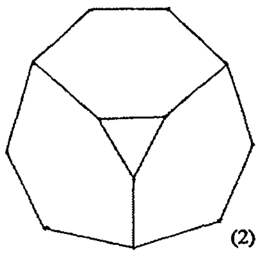


Figure 30.3

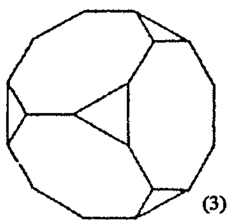




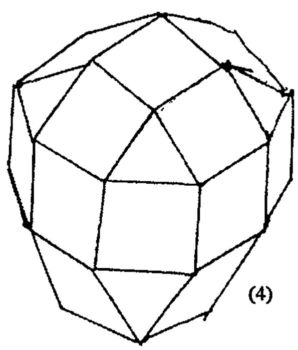
(1)



(2)



(3)



(4)

Problem 31.

Find the volume V of the following semi-regular polyhedrons in terms of the side $a$.

(1) Cuboctahedron with 14 faces of 8 equilateral triangles and 6 squares. $V=\dfrac{5\sqrt{2}}{3}a^3$.

(2) Truncated tetrahedron with 8 faces of 4 equilateral triangles and 4 regular hexagons. $V=\dfrac{23\sqrt{2}}{12}a^3$.

(3) Truncated cube with 14 faces of 8 equi-lateral triangles and 6 regular octagons. $V=\dfrac{14\sqrt{2}+21}{3}a^3$.

(4) Truncated rhombic dodecahedron with 26 faces of 8 equilateral triangles and 18 squares. $V=\dfrac{10\sqrt{2}+12}{3}a^3$.

(5) Truncated octahedron with 14 faces of 6 squares and 8 hexagons. $V=8\sqrt{2}a^3$.

(6) Icosidodecahedron with 32 faces of 12 regular pentagons and 20 equilateral triangles. $V=\dfrac{17\sqrt{5}+45}{6}a^3$.

(7) Truncated icosahedron with 32 faces of 12 regular pentagons and 20 hexagons. $V=\dfrac{43\sqrt{5}+85}{4}a^3$.

(8)Truncated dodecahedron with 32 faces of 20 equilatral triangles and 12 regular decagons. $V=\dfrac{235\sqrt{5}+495}{12}a^3$.

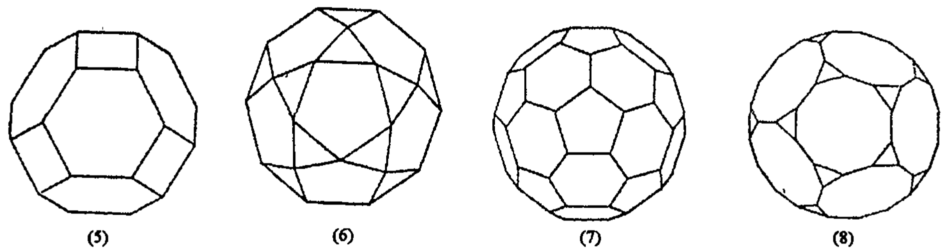



Figure 31 (1-8)

Problem 32. Twisted polygonal-prisms.

Find the volumes of the following twisted polygonal-prisms which are named by the present author.

(1) Equilateral $A'B'C'$ lie on the equilaterals $ABC$ of the same sides $a$. We twist triangle $A'B'C'$ on the triangle $ABC$ such that six vertices (A, A', B, B', C, C') construct a regular hexagon and pass an elastic string through each vertex. Lift triangle $A'B'C'$ a distance $h$ from the triangle $ABC$, then we have twisted equilateral prism with height $h$ and equilateral of side $a$ as base. Find the volume in terms of $a$ and $h$.

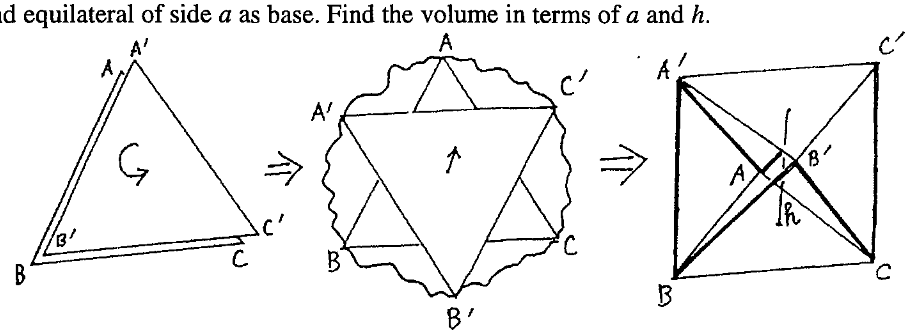



Figure 32.1

Original solution.

Consider hexagon prism, $AEBFCD : D'A'E'B'F'C'$. The base $AEBFCD$ is a hexagon of side $a/\sqrt{3}$ and height $h$. The volume of the hexagon prism is $\frac{\sqrt{3}}{2}a^2h$, from which we subtract the volume of six triangle pyramids, one of them is $\frac{h}{3}\Delta BFC = \frac{\sqrt{3}}{36}a^2h$. The desired volume is $\left(\dfrac{\sqrt{3}}{2}-\dfrac{\sqrt{3}}{6}\right)a^2h=\dfrac{\sqrt{3}}{3}a^2h$.

(2)Twisted square prism with height $h$ and square of side $a$ as base is defined as (1). Find the volume in terms of $a$ and $h$.

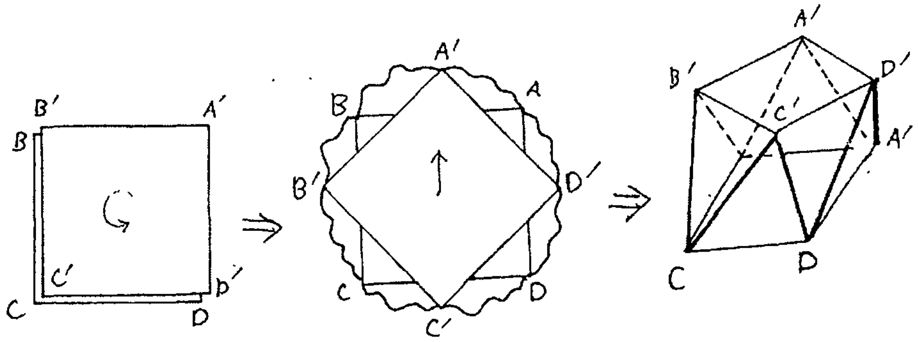



Figure 32.2

Original solution. In the same way as (1), we have the volume $\dfrac{2+\sqrt{2}}{3}a^2h$.

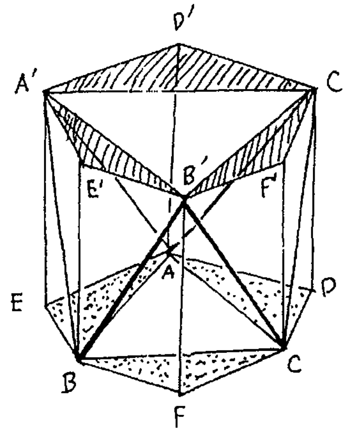



Figure 32.3

(3) Twisted regular $n$-polygon prism with height $h$ and square of side $a$ as base which is defined as (1). Find the volume in terms of $a$ and $h$.

(4) Original solution. In the same way as (1), we have the volume $\left\{2\cot\dfrac{\pi}{n}+1/\sin\dfrac{\pi}{n}\right\}na^2h/12$.

Problem 33.

Find the volume of the following polyhedra considered by traditional Japanese mathematicians only, the present author guesses.

(1) Truncated twisted equilateral pyramid, the above is equilateral of side $a$, the bottom is equilateral of side $b$ and the height $h$, which is constructed analogous to Problem 27(1). Find the volume $V$ in terms of $a$, $b$ and $h$.

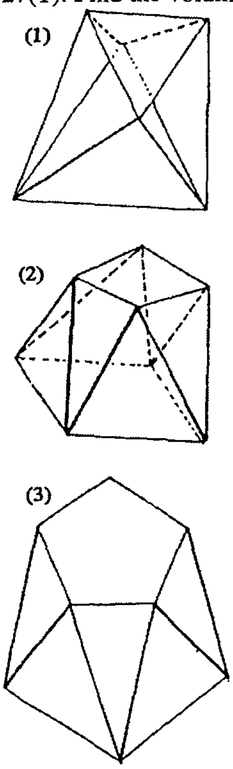


Figure 33

The answer is $V = \frac{3h}{6}\left(\frac{a^2+b^2}{2\sqrt{3}} + \frac{ab}{\sqrt{3}}\right)$

(2) Truncated twisted square pyramid, the above is square of side $a$, the bottom is square of side $b$ and the height $h$ which is constructed analogous to Problem 27(2).

Find the volume $V$ in terms of $a$, $b$ and $h$. The answer is $V = \frac{4h}{6}\left(\frac{a^2+b^2}{2} + \frac{ab}{6}\right)$

(3) Truncated twisted regular $n$-polygon pyramid, the above is regular $n$-polygon of side $a$, the bottom is regular $n$-polygon of side $b$ and the height $h$. Find the volume $V$ in terms of $a$, $b$ and $h$.

$$V = \frac{nh}{6}\left(\frac{(a^2+b^2)cot(\pi/n)}{2} + \frac{ab}{2\sin(\pi/2}\right)$$

We obtain the original solutions to the above Problem (1), (2) and (3) in the same way as problem 27(1).

Problem 34. (Lost *sangaku* problem of 1796)

30 equal small spheres of radius $r$ are touching on the large sphere of radius $R$ so that each small sphere touches the other four small spheres and large sphere. Find $r$ in terms of $R$.

Original solution runs as follows: Join the centers of small spheres then we obtain *icosidodecahedron* (semi-regular polyhedron) and can find that some sides lie on a plane passing through the center of large sphere and the one circle of radius $R+r$ which implies $\frac{r}{R+r} = \sin\frac{\pi}{10}$ and $r = \frac{R}{\sqrt{5}}$.

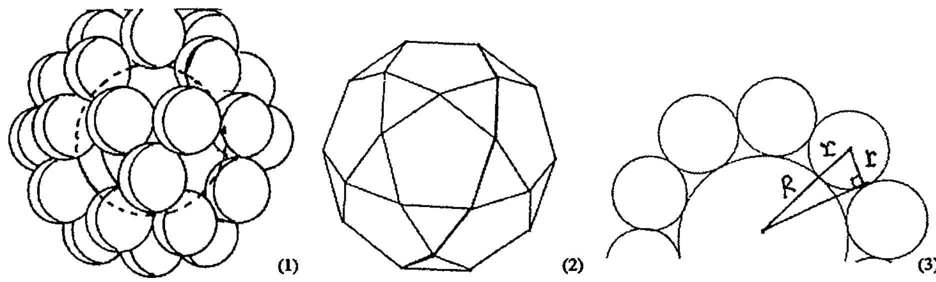



**Figure 34**

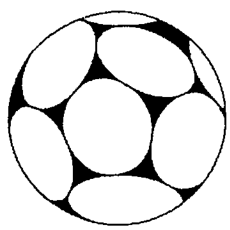


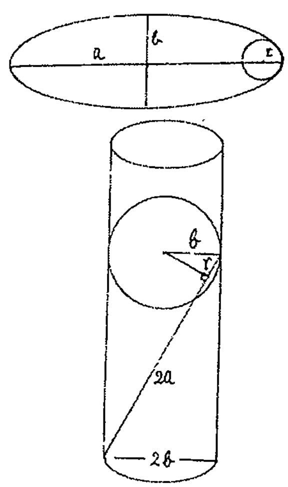
**Figure 35**

Problem 35.

A set of $n$ disjoint circles $O(r)$ ($i = 1, 2, \ldots, n$) packs the surface of a sphere $S$ so that each region of surface exterior to the circles is bounded by arcs of three of the circles. Find the possible values of the number $n$ of circles. If we join vertices of the group, then we have regular polyhedrons. The number of distinct ways is five.

This problem is quoted from the book, *Sanpou Kaiun* (1849).

### 3.4 Ellipse gained as section of right cylinder



In TJM, the concept of the ellipse is obtained by a section of cylinder by the plane.

Problem 36.

Find the radius $r$ of curvature at the end point $A$ of the major axis $AA'$ of the ellipse with the major axis $2a$ and minor axis $2b$. By analytic geometry, we have $r = \dfrac{b^2}{a}$.

Original solution: Cut the right cylinder of radius $b$ by the plane passing through the point $P$ at which a sphere of radius $b$ is inscribed. From the figure, we can derive the ratio $b : r = 2a : 2b$ which implies $r = \dfrac{b^2}{a}$.

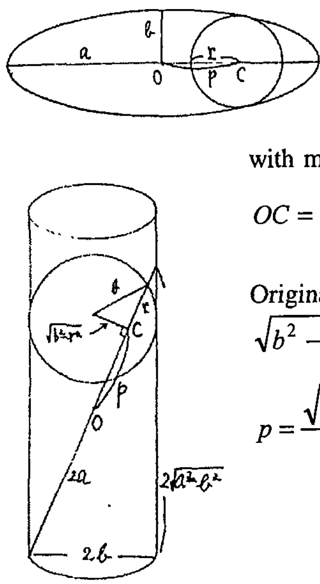



Figure 37

Problem 37.

Let $C(r)$ be a circle inscribed in the ellipse $O(a,\ b)$ with major axis $2a$ and minor axis $2b$, and the center $O$. Then

$$OC = p = \frac{a^2 - b^2}{b^2 - r^2} b.$$

Original solution: From the figure, we have the ratio $\sqrt{b^2 - r^2} : p = 2b : 2\sqrt{a^2 - b^2}$ which implies

$$p = \frac{\sqrt{(a - b^2)(b^2 - r^2)}}{b}.$$

### 3.5 Some curves gained as the section of doughnuts and conoid

Concerning the problems of this section, see, Hidetosi Fukagawa, Algebraic Curves in Japan during the Edo Period, *Historia Mathematica* 14 (1987), 235-242. pp., Academic Press, USA.

Problem 38.

A torus is obtained by rotating a circle of radius $r$ about an axis in the plane of the circle at a distance $d \geq r$ from its center. Consider the section of a torus by a plane parallel to the axis at a distance $t$. Show that the section is a *Cassinian* oval if $t = r$.

Problem 39.

In Problem 38, show that the section is a leminiscate if $d = 2r$ and $t = r$.

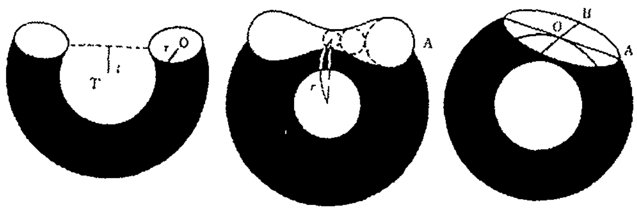


Figure 38 (1-3)

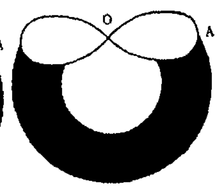



Figure 39

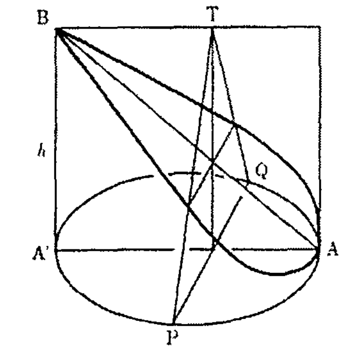



Figure 40

Problem 40.

$PQ$ is any chord perpendicular to the diameter $AA'$ of circle $O(r)$. Isosceles $PTQ$, the sides $PT = TQ$, is perpendicular to the base circle $O$, the height $h$ is constant.

If $P$ and $Q$ move on the circumference, the body constructed by the isosceles $PQT$ is called a *Conoid.*

As shown in the figure, the section of the body by the plane parallel to $PQ$ through $A$ and $B$, $A'B = h$ is *Wedge circle* or *senen.* Find the area of the *Wedge circle* in terms of $h$ and $r$. Let $AB$ be $x$-axis, the midpoint of $AB$ be the origin and perpendicular to $AB$ at the origin be $y$-axis. Then, the equation of *senen* is $y = \pm r(a+x)\sqrt{\dfrac{a^2-x^2}{a^2}}$. We have the desired result that the area is $\dfrac{ar\pi}{2}$, $a = \sqrt{r^2+h^2}$.

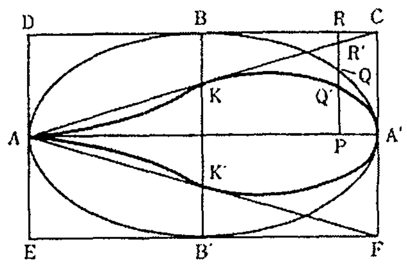



Figure 41

Problem 41.

An ellipse with major axis $AA' = 2a$ and minor axis $BB' = 2b$ is inscribed in a rectangle $CDEF$ with $AA' \parallel CD$, $BB' \parallel CF$. Let $P$ be an arbitrary point on $AA'$ and let the perpendicular to $AA'$ at $P$ meet the ellipse at $Q$, $CD$ at R, and $AC$ at $R'$ as shown.

Let $Q'$ be the point on $PR$ such that $PR' : PQ' = PR : PQ$. The locus of $Q'$ is called *seitoen* or *flame circle.* Show that the *flame circle* is congruent to a wedge circle as in

Problem 39 and that its area is $ab/2$. The equation of *seitoen* is $y = \frac{\pm b(a+x)\sqrt{a^2-b^2}}{2a^2}$ where $AA'$ is $x$-axis and $BB'$ is $y$-axis. If $r = 2b$, then *seitoen* is *senen*. The result is clear.

In Japan of the 18th and 19th centuries, mathematicians hung tablets under the roof of shrines or temples. On these many original problems were written as challenges to other mathematicians. Lovers of TJM hung the tablets under the roof of shrines and temples near their neighborhoods, on which learned problems were written. Sometimes they were not original but copied. Almost all of the problems were on geometry and finding the volumes or length of curves of some figures. On one tablet, five or six problems were written with the last result only, without the solutions. The size was not constant. The most popular size was 180 by 90 cms. Some gifted people who had not so much mathematics knowledge and wanted to study mathematics were excited to see *sangaku* in the precincts of some shrines or temples shown in a glamorous way. And they developed an interest in mathematics. So, there were many attractive problems with *symmetric* figures on the tablets. The present author believes that *sangaku* problems are very attractive to school students nowadays.

## REFERENCES

Fukagawa, H., Pedoe, D. (1989) *Japanese Temple Geometry Problems*, Winnipeg, Canada: C.B.R.C.

Fukagawa, D., Sokolowsky, D., *Traditional Japanese Mathematics Problems of 18th and 19th centuries*, U.S.A.: Math Pro Press.

Fukagawa, H. (1987) Algebraic curves in Edo-period, *Historia Mathematica*, 14-3, U.S.A.

Pedoe, D. (1988) *Geometry: A comprehensive course*, Dover publications.

Fukagawa, H., Rigby, J.F., (2001) *Traditional Japanese Mathematics Problems of 18th and 19th centuries*, Singapore: SCT-publishing.