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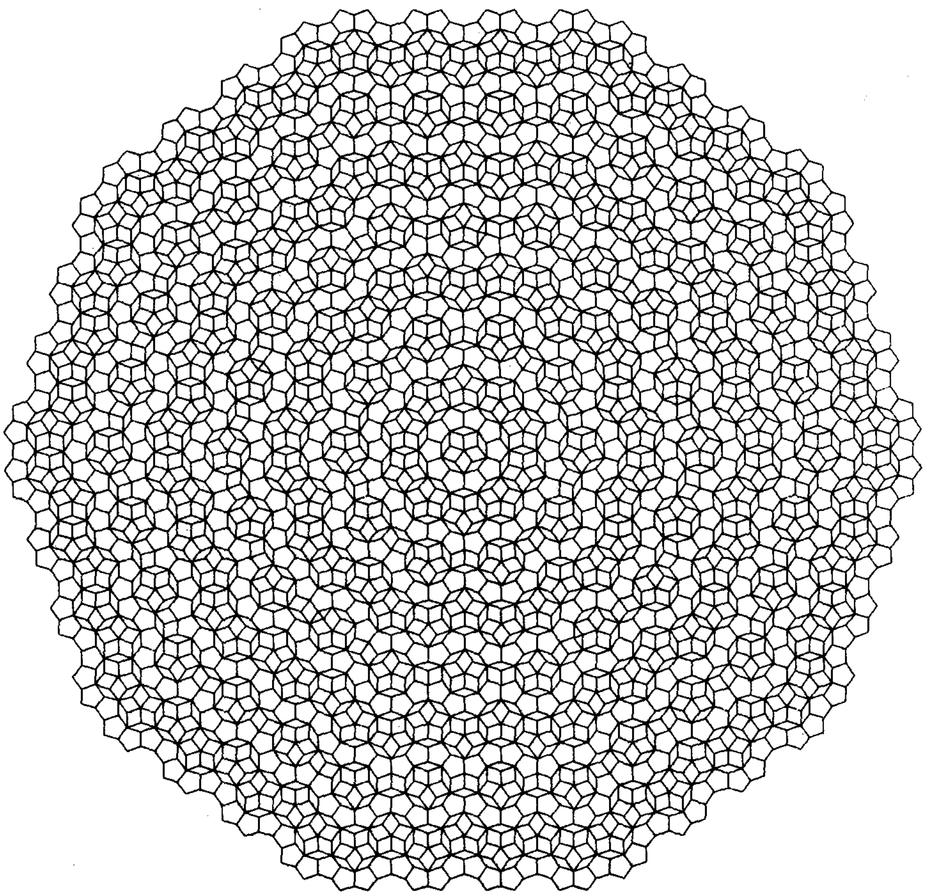

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JUMPING IDENTITIES OF PARTICLES

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Abstract: *The relationship between equality and identity is one of the unsolved problems in science. Leibniz's approach which goes back to Spinoza stresses real space. Suppose there are several identical universes present in an otherwise empty absolute space; then this is equivalent to only a single universe existing. Weyl realized that Leibniz's group-theoretic result in real space remains valid in a more abstract space - configuration space. Suppose mathematically equal particles exist; then configuration space possesses Leibniz's symmetry. The collapse down to a single surviving sub-universe occurs in this space as well. Consequences for the real space in which the particles live follow. The boundaries between adjacent sub-universes in configuration space correspond to well-defined relative positions in real space. Therefore at certain points in real space the particles exchange their identities. If the 2 equal particles live n a ring, the swap occurs under two conditions: coincidence and "anti-coincidence." When the particles pass through opposing positions on the ring, they exchange their identities in a jump. The "leapswap" has implications ranging from chemistry to personal identity.*

1. INTRODUCTION

The idea goes back to Leibniz. In his correspondence with Clarke (1717/1956), he made the following claim: If in Newtonian absolute space, the same universe existed twice, this situation would be "identical" to one in which the universe in question exists only once. This paradoxical conclusion he reached on the basis of his "principle of the identity of the indistinguish-able" (principium identitatis indiscernibilium), which he had learned from Spinoza when visiting him in 1676. Spinoza in turn owed it to the Mutakallimûn (or Mutasilites), early rationalistic Islamic philosophers (Weyl, 1949).

Hermann Weyl (1949) realized that while the situation of several identical sub-universes in real space is unlikely to gain any practical importance, a realistic case exists in a slightly more abstract space — configuration space. Whenever two mathematically equal particles (or solitons, so we may add today) exist in real space, there applies in configuration space a "two-universes-situation" of the very type envisaged by Leibniz.

What is configuration space? One needs this space to completely describe a configuration in real space of several pointshaped particles. In configuration space, all particle positions occupied simultaneously in real space are represented by a single point. One obtains this space by simply plotting real space (with all its coordinates) against itself so many times as there are particles present — so that indeed a single point suffices to simultaneously characterize all particle positions in real space.

Configuration space possesses a simple structure because it contains only a single moving point. The situation changes, however, when the indistinguishability assumed by Weyl (mathematical equality of particles) is introduced. As an implication of this symmetry assumption, there exist *two* moving points in configuration space, if there are two equal particles present (and *n!* in general when there are *n*). This is because it now makes no difference which one of the *n* copies of real space is associated to which particle. Therefore, indeed several mutually perfectly symmetric sub-universes are generated — in configuration space.

The collapse down to a single sub-universe found by Leibniz therefore indeed exists — in configuration space. The collapse in configuration space, however, has repercussions on real space.

The details of this "reaching back" into real space of the Leibniz collapse are not yet completely understood. In the following, an attempt will be made to capture the "salient point."

2. THE SIMPLEST CASE — TWO EQUAL PARTICLES ON THE INTERVAL

The behavior of two mathematically equal particles in an ordinary 1-dimensional space not closed into a ring is easy to describe. This is because the corresponding configuration space is only 2-dimensional.

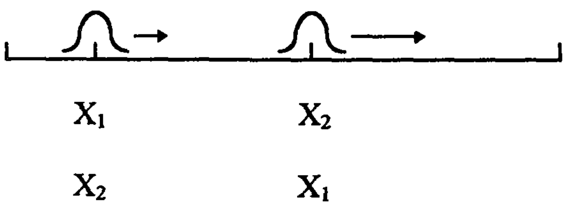

Figure 1: Case of two equal particles or solitonson the interval. X = interval; 1, 2 = possible particle labels.

Figure 1 shows the situation in real space. It is assumed that the two "particles" are each elastically reflected from the two ends of their 1-D box, and that they do not affect each other in any way while travelling. That is, they pass right through each other without interacting as two solitons do.[1]

The positions of the two centers of mass of the two solitons can then be represented by two points on the unit interval: x_1 and x_2 in Figure 1. Each occupies its own position on the interval. If one plots the interval against itself, first with the one position value contained and then with the other, one obtains configuration space. The latter is the unit square: Figure 2. The two positional values — axis sections — jointly determine a unique internal point,

[1] The latter assumption may be dropped in more general situations (like that of Figure 8 below).

the "state point" (plotted bold in Figure 2). The state point moves along a trajectory (path) in configuration space whenever one of the particles (or both) is in motion. A segment of this path — using the two velocities symbolized by different arrow lengths in Figure 1 — is also indicated in Figure 2.

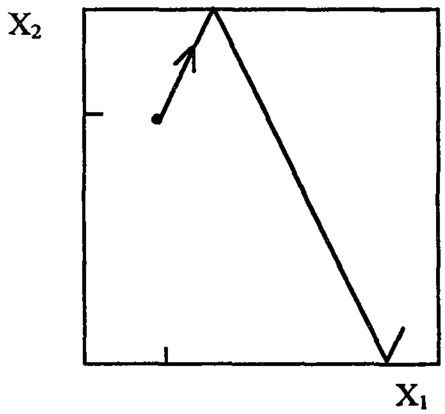

Figure 2: Configuration space for the motions of the two particles in Figure 1. A trajectory is also shown.

The same situation can, however, be represented equally correctly also in the following way: Figure 3. Here the two particle positions are marked, rather than by x_1 and x_2 by x_2 and x_1 (the second alternative indicated in Figure 1). Figure 3 is the symmetry-equivalent case to Figure 2.

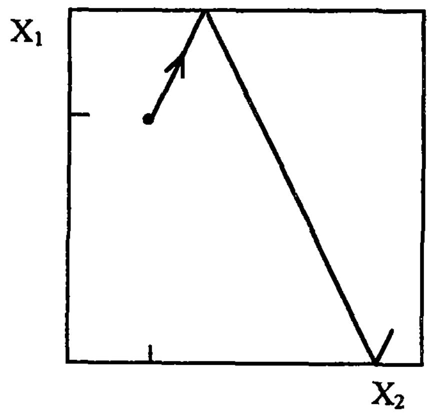

Figure 3: Configuration space for the motions of the two particles of Figure 1, alternative description to Figure 2.

Since neither Figure 2 nor Figure 3 are exhaustive, the question of a "complete description" of the behavior of the two equal particles of Figure 1 in configuration space arises.

A complete description will be reached if both the — correct — description of Figure 2 and the — correct — decription of Figure 3 are admitted on the same footing. Both cases can indeed be combined into a single Figure — provided Figure 3 is first mirror-reflected along the first bisector (which leaves its content unchanged). Both partial Figures (Figure 2 and the mirror-reflected Figure 3) are shown together in Figure 4.

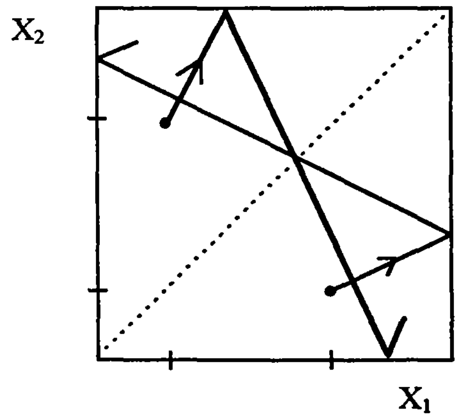

Figure 4: Complete configuration space: Figures 2 and 3, combined. The two partial trajectories are plotted in different degrees of boldness to facilitate understanding (the bold trajectory stems from Figure 2).

One sees from Figure 4 that the exchange symmetry between two particles in real space is represented in configuration space through what may be called a "bi-unique" trajectory (Rössler, 1987a). This fact was first seen by Leinaas and Myrheim (1977).

3. THE LEIBNIZ COLLAPSE

We now have reached the point where Leibniz's argument can be applied. There exists an "absolute space" (the square) containing in its interior two moving "light points" (the two state points) and nothing else. Thus everything that exists in absolute space exists twice. Hence everything exists only once according to Leibniz.

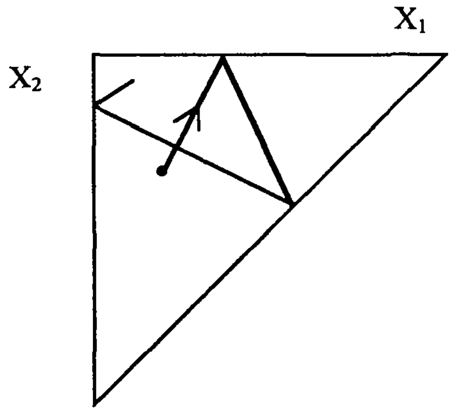

Figure 5: Reduction down to one half of configuration space - by the Leibniz principle.

Specifically, by virtue of the mirror symmetry with respect to the first bisector, there can be no doubt that the sub-universes in question are those two halves of the square that lie symmetric to the first bisector (the identity line). Only one of these "universes" is therefore left: Figure 5.

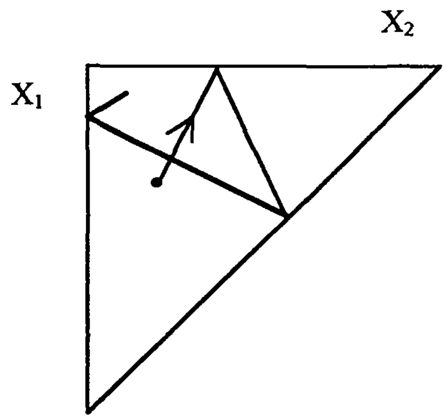

Figure 6: Reduction down to the other half of configuration space. The latter has been folded upwards to facilitate comparison with Figure 5. Note the different labels.

In Figure 5, the "upper" half of Figure 4 has been chosen. Instead, one could as well have elected the "lower" one. The latter is — after reflection along the identity line — shown in Figure 6.

The two cases of Figure 5 and Figure 6 are equivalent. This is because they are both completely represented by Figure 7. In this Figure, x_r means "position of the right particle" and x_l means "position of the left particle." This "neutral" description applies both to Figure 5 and to Figure 6.

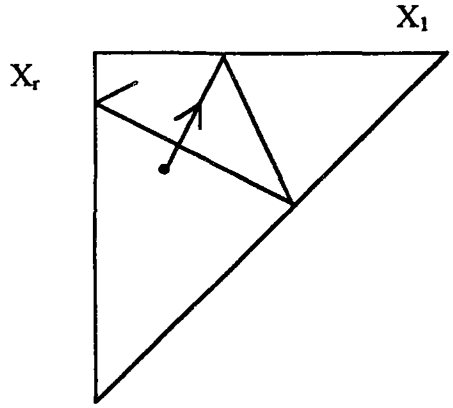

Figure 7: Invariant description of both Figure 5 and Figure 6. (Neutral labels.)

If in Figure 7 (or Figure 6 or Figure 5, respectively), the slanted hypotenuse is touched by the moving light point, no longer an exchange of labels follows suit. That is, everything in existence now is a single universe possessing 3 ordinary reflecting walls and a unique internal light point (Figure 7).

This "alone correct" description of configuration space has consequences for the original (real) space. The newly existing "third wall" — at x_r equal x_l — implies that the two equal particles or solitons exchange their identities at the very moment they eclipse (interpenetrate). For the the left-hand one returns toward the left side from which it came, and the right-hand one also returns to its own side — just as if their two centers of mass had collided with each other in a "hard" interaction. For this is what the configuration space of Figure 7 describes: the behavior of two ordinary impenetrable elastic particles on the unit interval.

This is a surprising result since it seemingly contradicts the assumption made at the outset — mutual penetrability. However, this is not actually the case since the detailed dynamics remains exactly the same as before. Moreover, the result appears not *too* unsettling. On the one hand, one could have guessed it directly — without the roundabout way of using configuration space and its internal multiplicity. On the other, the result seems to change nothing as far as the real world is concerned (for a counterexample, cf. Rössler and Hoffman (1987)). One therefore feels tempted to conjecture that what is at stake here is only a "convention": That is, one may be free to define that whenever two equal particles pass right through each other, an exchange of their identities takes place.

However, it turns out that this "appeasement philosophy" entirely misses the point. This is because there exists a second analogous result which is not covered by it. To that we now turn.

4. THE DECISIVE CASE — TWO EQUAL PARTICLES ON THE RING

The one-dimensional interval of Figure 1 may be closed into a circle as mentioned. The situation thereby obtained is sketched in Figure 8.

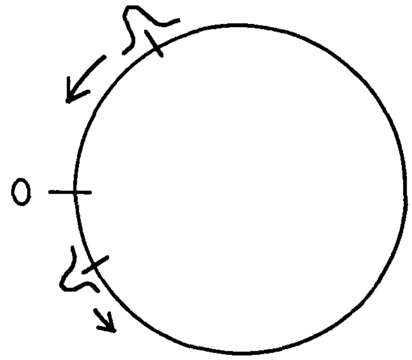

Figure 8: Two equal particles on the ring. O = origin (arbitrary). Compare Figure 1.

The corresponding configuration space — analogous to that of Figure 4 for the interval — is shown in Figure 9. Just as this was the case with Figure 4, again two "wandering light points" are found to lie symmetric to the first bisector. The only major difference to Figure 4 is that the two light points of Figure 9 live, not on a square but on a torus. A torus looks like the inflatable inner rubber hose in the tire of an aldtimer. This hollow-ring-like structure of configuration space is a consequence of the fact that opposing sides of the square are pairwise identical since the original interval has been closed into a circle. The torus is called a "linear torus" because it possesses the same circumference in both directions (around and across) — so taht it cannot be embedded in 3 dimensions but only in 4 if any distortion is to be avoided. This flat structure is preserved automatically if one draws the linear torus as a square and indicates by matching symbols the sides to be identified (as has been done in Figure 9).

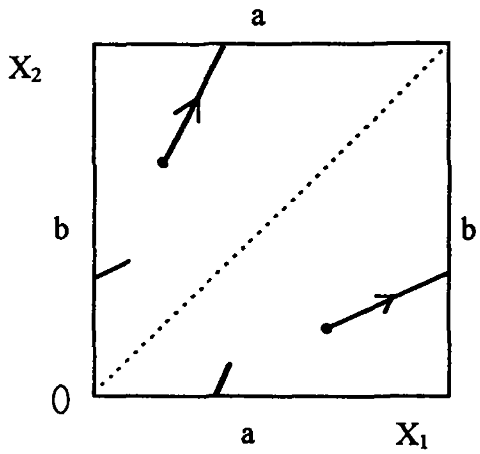

Figure 9: Corresponding complete configuration space (linear torus). a,a and b,b: Pairwise to be identified sides. Compare Figure 4.

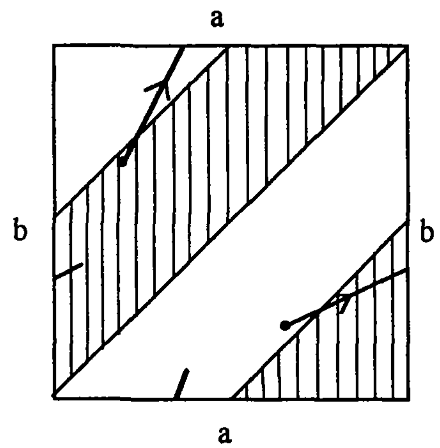

Figure 10: Decomposition of the tours of Figure 9 in two halves that lie symmetric to the first bisector, one cross – hatched, the other not.

We know from the case of the square (Figure 4) that the 2 light points when hitting the first bisector suffer a seeming reflection back into the half-squares from which they came (so that two "half-universes" form). One might therefore expect that the torus of Figure 9 can, too, be dissected in two identical halves by means of a cut taken along the first bisector. However, this is not the case. A single around-going cut can only "open up" a car tire — it becomes perfectly flat but remains in one piece. To slice it up in two pieces, a *second* cutting line is required. The only second cut which generates 2 equal partial spaces possessing parallel boundaries is one that is displaced by 180 degrees (and hence lies diametrically opposed to the first cut on the circular tube). The effect of both cuts taken together is shown in Figure 10.

The constraint just mentioned — parallelism of first and second cutting line — follows from the translation invariance of the doubly populated configuration space relative to the origin. It must make no difference where the "zero point" is located on the circle of Figure 8, and hence on the first bisector of the configuration space of Figure 9. Although other more "wavy" second cutting lines (of the same average orientation) do exist including fractal ones that still generate two identical half universes, this occurs at the expense of the translation symmetry being broken. Hence they are inadmissible. A different way to put the same fact is to quote Leibniz's second favorite principle (his "principle of sufficient reason"). Any other choice of the second cutting line would involve an element of arbitrariness and hence is ruled out for lack of sufficient reason.

The straight second cutting line on the linear torus, shown in Figure 10, is not unfamiliar. It represents what in crystallography is called a "slide reflection" (W. Prandl, personal communication 1992).

5. CONSEQUENCES

How do the two obtained subspaces — the "individual universes" — look like this time?

Figure 11 shows the cross-hatched sub-universe of Figure 10 once more. All pairwise identifications of boundary points implicit in Figure 10 are indicated more explicitly. This includes the identification of the two portions of its upper boundary (the top of the "black belt" and the top of the "black triangle" of Figure 10). The latter identification is called "d" in Figure 11.

The result, when re-drawn, looks like a "cap" (Figure 12). The cap has the unusual property that its upper part has only half the circumference of its lower part. For all pairs of points that face each other diagonally on the top have been identified in a cross-wise manner. The lower part, of course, is open (as befits a cap). This fact is not clearly visible in the Figure since the cap is viewed from above.

The so obtained manifold is known in topology under the name "cross-cap." The cross-cap is topologically equivalent to the "punctured projective plane." At the same time, it is also equivalent to the more well-known "Mobius strip." Compare Hilbert and Cohn-Vossen (1932).

The crosscap (Figure 12) replaces the former triangle (Figure 7). Note that this time, there exist *two* "singular lines" in the sub-universe formed: The "lower boundary" of the cap, which is identical to the former unique singular line (hypotenuse) of the triangle of Figure 7, and the "upper boundary" of the cap which as mentioned consists of the pairwise identified points on the second cutting line marked "d" in Figure 11.

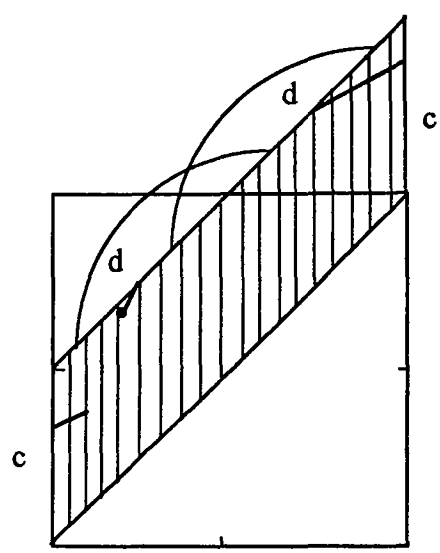

Figure 11: One of the two subspaces on the torus of Figure 10. (Note that the lower "black triangle" has been moved up for clarity.) c,c and d,d: Pairwise to be identified boundaries.

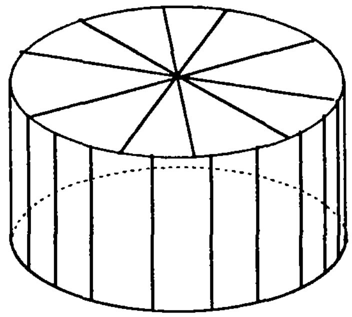

Figure 12: Equivalent description of the cross-hatched sub-universe of Figure 11 to show how it looks like in space: "Cross-cap."

In all those cases in which the *lower* boundary is approached and reached by a light point, there clearly is no difference between Figure 12 and Figure 7: simple reflection. Hence the behavior of two equal particles on the interval and on the ring, when they meet and interpenetrate, is the same: identity exchange.

What happerns, however, when the *upper* boundary of the cross-cap is reached? At such points, again a switch-over to the other trajectory occurs. The path continues on the *other* side of the cap in a seemingly reflected fashion (when the cap is assumed ironed-flat and transparent). To understand this, it is best to focus on a point inside the "black belt" of Figure 10. As the upper boundary of the belt is hit from below, the original trajectory continues into the upper white triangle. Simultaneously, however, the second trajectory (in the lower-right half picture) enters the "black triangle" from above. Since the point of departure, out of the black belt, and the point of entry, into the black triangle, are identified (top of cap), indeed a switch of trajectories occurs at the very same moment. This switch-over from the one trajectory to the other is once more accompanied by a change of labels of the axes — and hence a swapping of particle identities.

In consequence, we have reached the result that there exists, apart from the familiar identity exchange "at a place," also an identity exchange "at a distance." This swap-in-a-jump occurs whenever the two equal particles on the ring pass through a configuration of maximum possible distance. For this is what characterizes the second cutting line which runs parallel to the first bisector in Figure 10: It describes those pairs of positions on the ring which are in mutual "opposition." This line thus — unlike the first bisector which corresponds to the points of "coincidence" — corresponds to the points of "anti-coincidence."

6. DISCUSSION

The described "identity exchange at a distance" is a rather surprising result. How seriously should it be taken?

The new finding comes not completely unexpected. It belongs into classical phase-space theory since configuration space is a part of phase space. Gibbs' (1902) first found that a reduction of phase-space volume is implicit if one assumes classical indistinguishability (the presence of mathematically equal particles). His famous factor 1/n! by which phase space volume is reduced plays a role in the calculation of equilibrium entropy, cf. Rössler (1987b). However, Gibbs did not focus on the more detailed topological mechanisms which underlie the scalar volume reduction — that there is a cellular structure of phase space present with but one cell ("sub-universe") surviving.

This fact was first glimpsed by Weyl (1949). However, Weyl (who was not aware of Gibbs's earlier finding) also did not arrive yet at the description of a well-defined sub-universe. He confined himself to introducing the term "Leibniz-Pauli-principle." In a footnote (on page 237 of the second German edition of his book (Weyl, 1949)), he emphasizes: "So written in the year 1926!" Weyl apparently believed that Pauli had already reaped the most important results in that same year.

However, Pauli actually had not arrived at the Weyl cell. One of the reasons for that oversight is that Pauli's theory is quantum-mechanical — and there are no trajectories in existence in quantum mechanics. A second, deeper, reason is that the "spin," postulated by Pauli to explain the fact that cells in atoms almost always contain two equal particles rather than a single one, is unexplained classically up to this day. Although Finkelstein's famous "rubberband lemma" for non-rotation-symmetric solitons (Finkelstein and Rubinstein, 1968) is a first step, it fails to be applicable to rotation-symmetric (point-shaped) classical particles like electrons; cf. Rössler (1996) for a preliminary new proposal.

Two examples of Weyl cells are: (1) The two halves of the straight line of Figure 1, one occupied on average by the right and the other by the left particle since they do not penetrate each other. (2) The two regions on the half ring (assuming appropriate corotating coordinates) occupied on average by each particle. Despite its "too great simplicity" as far as quantum mechanics is concerned, the Weyl cell possesses an important qualitative feature. It for the first time permits the prediction of a "classical chemistry." Up till now, any foundation of chemical identity has always been "non-classical" in the sense that it was derived with the aid of new axioms, cf. Primas and Müller-Herold (1984).

Ortho-Helium is the first atom that can be understood classically in the sense that its cellular structure can be predicted. Here, 2 impenetrable equal particles ("classical electrons") live near a center of infinite-mass in an otherwise empty space. In the simplest (2-dimensional) case, the configuration space possesses not twice as many dimensions (four) as one would expect, but only three since rotation-equivalence allows one to eliminate one dimension of configuration space. A three-dimensional configuration space, however, can still be inspected. It is (essentially) a product of a crosscap and a circle. The "identity swap" occurs whenever 2 particles pass through the same radial distance from the center. This result remains valid in 3 dimensions ("fully 3-D classical ortho-Helium" (Rössler, Meier and Hoffmann, 1990)). This cellular structure — two concentric cells — would if confirmed provide the most sophisticated example of Weyl cells as yet — no. (3).

The main point to be discussed, however, is not generalization but existence. The new symmetry effect of "jumping identities" violates common sense to an almost intolerable extent. Is a swap of identities in a leap *really* implicit in classical exchange symmetry? Any simultaneously "absurd" and "hard" result is bound to have unfathomable consequences. This is why Leibniz therefore placed great hopes in his and Spinoza's principle. However, such results are scarce and rare between as is well known.

A single unacceptable feature of the swap would be sufficient to eradicate it. One obvious such feature is its lack of relativistic invariance. However, it turns out that the same verdict applies to virtually all of statistical mechanics — so that it indeed is not decisive. Classical molecular dynamics with long-range (Coulomb-type) interactions was recently used with success to generate a deterministic Newtonian simulation of an oscillatory chemical reaction (Diebner and Rössler, 1990). The many forces that impinge on each particle, simultaneously determining its direction of motion, cannot be described relativistically as is well known ("noninteraction theorem" of Goldstein and Kerner, cf. Rössler (1994)).

The "swap" therefore has yet to be disproved. A more general question is therefore to be addressed finally. Does the the Weyl-cell possess any relevance for the "identity question" in the sense of one's own identity and its origins (Weibel and Steinle, 1992)?

The candle flame comes to mind as an illustration. The flame remains constant as a light-giving source even though it is continually passed through by new particles in what is called a "steady flux" (Bertalanffy, 1953). This is not surprising — there exist many other only apparent self-identities — like that of a wave on a wheat field, for example. However, the flame may be profoundly different. Particle indistinguishability — with its implied identity swaps — opens up the prospect that, despite the material flow, a permanent identity applies to most internal molecular constituents of the flame at roughly the same place. Pauli and Fermi made a similar claim concerning the movement of an electric current through a metal wire — with all electrons staying in place. However, this time the absurdity is part of a classical prediction.

The analogy "flame-brain" then suggests that, should subjective identity possess an identifiable material correlate, the above-described jumping mechanism may contribute to its formation.

To conclude, an idea of Gibbs and Weyl has been taken up. After the "triangle," the "cross-cap" appears to be the simplest prototype cell in configuration space under a condition of classical exchange symmetry. An equivalent result ("Möbius strip") has already been obtained by Leinaas and Myrheim (Leinaas and Myrheim, 1977) on the basis of a more indirect description (center-of-mass-coordinates). The implication described above — identity swap at a distance — appears to be new. Since it is at odds with common sense, speculations as to the significance of the "jump" are, perhaps, premature. Pythagoras'idea of "metempsychosis" (soul-jumping) would lose some of its strangeness if particle identities did the same thing.

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