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ON THE VARIOUS PHASES OF CIRCLES - SPHERES IN WASAN

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The main objects of this paper are treated the problems about Descartes Circle Theorem and the related Wasan problems, and more the three dimensional extensions.

1. Short History of the Descartes Circle Theorem.

Descartes Circle Theorem, that is a theorem on curvatures of touched circles, that is: When three circles(curvatures are $e_1, e_2, e_3$) circumscribe each other and the fourth one ($e_4$) contacts the other three, respectively, there is the following relation (Fig. 1, Fig. 2).

$$ (e_1 + e_2 + e_3 + e_4) = 2(e_1^2 + e_2^2 + e_3^2 + e_4^2) $$

Remark. The Descartes Circle Theorem in Rene' Descartes' complete work did not shaped itself into a theorem, because it was his letter to Princes Elizabeth.

Many scholars are extension this Theorem. That is, Steiner solved it on the curvature of the fourth circle:

$$ e_4 = e_1 + e_2 + e_3 + 2(e_1 e_2 + e_2 e_3 + e_3 e_1) $$

Melzak and Stanton started the above formula, and they extended it to n-th circles. H.S.M. Coxeter and Morley moreover Walker are extended the n-th circles. Substituting $\alpha, \beta, \gamma$ and $\delta$ for $e_1, e_2, e_3$ and $e_4$, respectively in Steiner's formula (1), he defined the n-th circle (curvature $\gamma_n$) inductively and he got the following:

$$ \gamma_n = \gamma_0 + 2n \delta + n^2 (\alpha + \beta) $$
where \( \delta = (e_1 e_2 + e_2 e_3 + e_3 e_4) \), that is \( \delta = (\alpha \beta + \beta \gamma + \gamma \alpha) \).

2. The study of it in Japan.

Nushizumi Yamaji 写 (山添 聞, 1724 - 1772) wrote Zeishiki 来日 at 1751. In this, he says about the Descartes Circle Theorem as follows:

When three circles having radii \( r_1, r_2 \), and \( r_3 \) circumscribe each other and fourth one (radius \( x \)) touches the other three, then there is following relation:

\[
(\frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1}) x + (\frac{r_1^2 r_2^2 + r_2^2 r_3^2 + r_3^2 r_1^2}{2 r_1^2 r_2^2 + 2 r_2^2 r_3^2 + 2 r_3^2 r_1^2}) x^2 = 0.
\]

That is, we have rewritten as follows:

\[
x^2 - \frac{s_1}{s_2} = \sqrt{S_3},
\]

where \( s_1 \), \( s_2 \) and \( s_3 \) are expressed the fundamental symmetric expression of \( e_1 \\
\]
\( e_2 \), \( e_3 \).

Theorem Let a point 0 be the center of two concentric circle having curvatures \( f \) and \( 3f \). Let circles \( O', O'' \) (curvature \( 3f \)) be in contact externally with each other and also with the two circles above. We use the same symbols \( O', O'' \) for the centers of these circles.

Let \( P_1(Q_f) \) be the center (curvature) of a circle which is in external contact with the circle \( O', O'' \) and the inner circle 0. Moreover, let \( Q_1(T_f) \) be the center (curvature) of a circle which is in external contact with the circles \( O', O'' \) and is in internal contact with the outer circle 0.

Furthermore, let \( Q_2(T_f) \) and \( P_2(Q_2) \) be the curvature of the center of a circle which is in external contact with the circles \( O', P_1(Q_f) \) and the inner (the outer) circle 0, ... . Repeating this process, are obtain the series \( Q_f, Q_2, \ldots \), \( P_f(T_f, Q_2, \ldots \), \( Q_2, \ldots \), \( P_f(T_f, Q_2, \ldots \), \( Q_2, \ldots \) respectively.

Then the following invariable relation is held (Fig.3):

\[
3 \Gamma^n \cap \phi = 6 \phi^n (n = 1, 2, \ldots , n).
\]

Fig.3 Fig.4
When three circles A, B, C (radius a, b, c) circumscribe each other and more another three circles D, E, F circumscribe each other in that three circles A, B, C's gap. Then the forth circle's radius d is as follows (Fig. 4):

\[ d = \frac{abc}{bc + 2a(b + c) + 2\sqrt{2abc(a + b + c)}}. \]

Remark 1. Let \( e_1, e_2, e_3, e_4 \) be the curvatures \( a^{-1}, b^{-1}, c^{-1}, d^{-1} \), then we have by Wasan experts as following:

\[ e_4 = e_1 + 2(e_2 + e_3) + 2d^2. \]

This is so called 丸（日）切（割）．

Remark 2. This problem is on the mathematical tablets in Soshidō, Bushō Horinou-uchi at Bunkyō 3 (1853) (Fig. 5).

3. The study of it in three dimensions.

The Descartes Circle Theorem easy expand to three dimension. But its omit.

Theorem \( 6 \) A spherical segment has diameter a and height b. Four equal spheres of diameter r are tangential to each other. Two of them are tangent to the spherical segment and the plane base, one is tangent with the spherical segment only and the last is tangential to the plane only (Fig. 6):

Theorem \( 7 \) The large sphere G having curvature \( S \) is packed with equal spheres having curvature \( 3S \). \( A_1, A_2, A_3 \) and \( A_4 \) are four spheres of the above equal ones and they touch each other.

Let \( P_1 \) be a sphere circumscribed with the above four equal ones, and \( P_2 \) be one circumscribed with the spheres \( A_1, A_2, A_3 \) and \( A_4 \). Finally \( P_7 \) is defined as the same process, then \( P_7 \) agrees to \( P_1 \).

And more, let \( Q_1 \) touch the spheres \( A_1, A_2, A_3 \) and large one \( Q_1, Q_2 \) do the spheres \( A_1, A_2, Q_3 \) and large one \( Q_4, \ldots \). Finally \( Q_7 \) is defined as the same process, then \( Q_7 \) agrees to \( Q_1 \).

Furthermore, let \( C_n \) (\( C_n \)) be the curvature of sphere \( P_n \) (\( Q_n \)), then

\[ 3C_n - C_n = 6S \quad (n = 1, 2, \ldots, 6) \]
4. The study of it in Japan (2)

Tansaku Sampō is wrote 1840 by Shokô Kenmochi (1790 - 1871). The seven problems like this are treated by Wasan experts. One of them as follows.

Problem Three spheres A, B and C are circumscribed each other and they put between the flat boards. The fourth sphere D being touched the boards and the spheres A and B. Similarly the fifth sphere E and the sieventh one F in made.

Now, the radiuses of the spheres A, B and C are 6, 3, 4 centimeters, then let seek the radiuses of the spheres D, E and F (Fig. 7).

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