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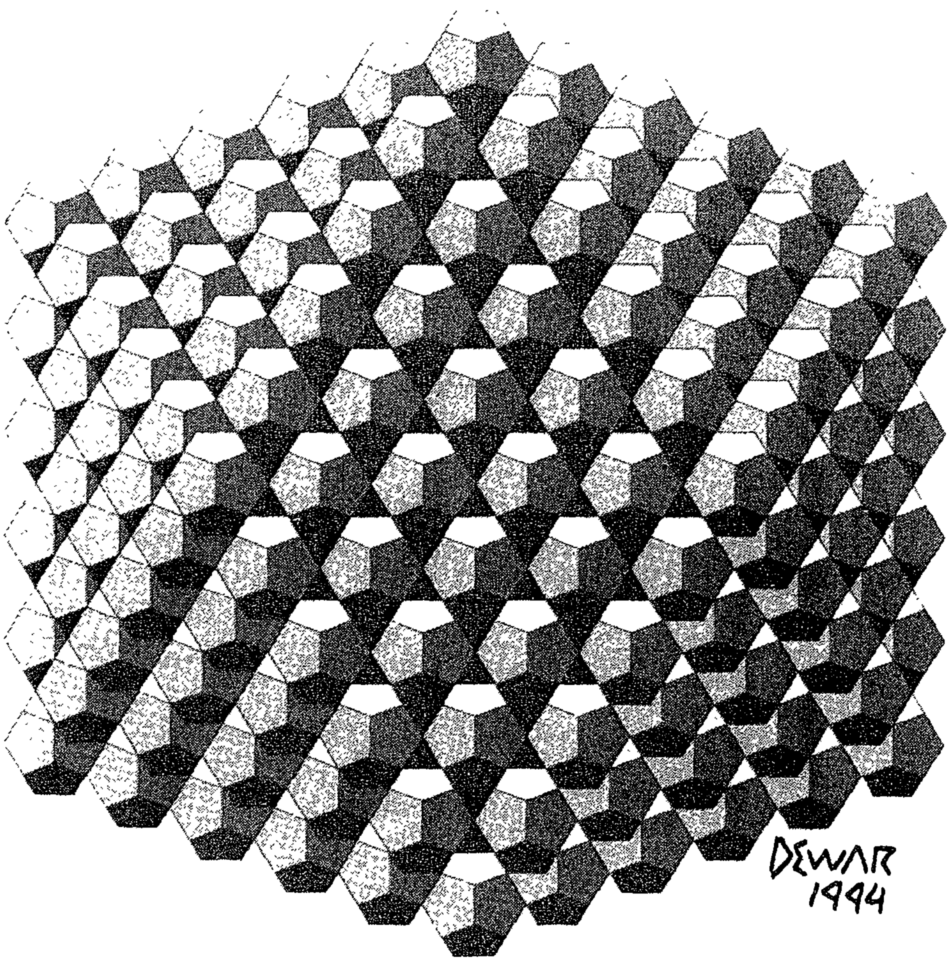

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FOLDED TOROIDAL POLYHEDRA

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A cylinder and a cone are two common examples of surfaces that have zero curvature everywhere (except at the cone point). This makes them, in some sense, flat. Each of these two surfaces can be sliced open and, without stretching, unrolled to lie flat.

The torus is also described as being flat. The meaning of "flat" can be taken in several ways. First, a torus is a quotient space of the flat plane. Second, if we view the torus as $S^1 \times S^1$ (a surface embedded in R^4) then the Gaussian curvature of the surface is zero everywhere. Third, when embedded in R^3 as a surface of revolution the net Gaussian curvature is zero. But it is known that a torus cannot be smoothly embedded in R^3 such that the Gaussian curvature is zero everywhere. In this paper we will show ways of embedding tori in R^3 as polyhedral surfaces such that the concentrated curvature at each of the vertices is zero. And since all curvature on a polyhedral surface is concentrated at the vertices, we will have tori in R^3 that have zero curvature everywhere. These polyhedra are constructed by tiling a piece of the plane with congruent triangles, and then folding the tiling along edges of the tiling. These tori will also have the symmetry properties that all faces in a given polyhedron will be congruent (monohedral), and all vertices will be adjacent to six faces (idemvalent). As abstract polyhedra all of the faces are equivalent, and all of the vertices are equivalent. As geometric polyhedra they have various symmetry groups resulting in some stunning examples. We will show how to calculate these tori by considering the symmetries of the resulting tori and the symmetries of the planer tilings from which they are folded.

A polyhedron is call monohedral if all of its faces are congruent to each other. A polyhedron is called idemvalent if there are the same number of faces incident to each vertex. We will be looking at monohedral idemvalent toroidal polyhedra. Euler's formula tells us that

$$V - E + F = 2 - 2g$$

where V is the number of vertices, E is the number of edges, F is the number of faces, and g is the genus of the polyhedron. In the case of spherical polyhedra $g = 0$ so the formula is $V - E + F = 2$. If each face is a p-gon and there are q faces incident to each vertex, then there are exactly 5 solutions for the pair $\{p, q\}$. They are $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{4,3\}$, and $\{5,3\}$. These correspond to the tetrahedron, octahedron, icosahedron, cube, and dodecahedron respectively; and each of these polyhedra can be realized with regular faces.

For toroidal polyhedra ($g = 1$) we have $V - E + F = 0$. Counting vertices and edges we see that $V = pF/q$ and $E = pF/2$. Thus $pF/q - pF/2 + F = 0$. Notice that $p \neq 0$ and $F \neq 0$. So dividing by pF we get

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} = 0.$$

In this case the only integer solutions for the pair $\{p, q\}$ are $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$. In the spherical case F was determined by p and q, but in the toroidal case it is not. In fact, there are infinitely many combinatorial solutions for each of the three types. Brückner [2] showed some self-intersecting examples of the type $\{4,4\}$. Alaoglu and Giese [1] showed non-self-intersecting examples of the types $\{4,4\}$ and $\{3,6\}$. In both cases they gave infinite families of examples, but there are many other combinatorial solutions waiting to be realized geometrically. We will here show how some more of the solutions of the type $\{3,6\}$ can be realized geometrically. It would be nice to realize the polyhedra with regular faces, but as we shall see this is asking a lot. So we will have to content ourselves with all the faces congruent.

Alaoglu and Giese [1] showed the existence of non-self-intersecting monohedral idemvalent toroidal polyhedra by building them out of monohedral octahedra. The method we will use here is entirely different. We will tile a patch of the plane with triangles and then fold this along the edges of the tiling to make a torus. The patch of the tiled plane used will be topologically a rectangle. We will play the topologist by identify opposite edges of the rectangle. However unlike a topologist we will not allow any stretching, only folding along edges. Hence the resulting torus will not only have net curvature zero, it will have zero curvature everywhere. A flat torus!

The calculations involved in computing these toroidal polyhedra involve successively solving systems of two or three linear or quadratic equations (depending on the symmetry of the polyhedron). The description is somewhat technical and is given in my thesis [3], and to some extent that forthcoming paper. In this abstract we will content ourselves with showing an example for each of the symmetry types.

The toroidal polyhedra generated by this folding process can have one of five different symmetry types. That is, the polyhedra can have the symmetry of a prism, an antiprism, a skewed antiprism, a dihedral group or a cyclic group. Each of the figures below is shown in stereo images. If the reader can get the left eye looking at the left image and the right eye looking at the right image then the mind should perceive a three dimensional image. Figure 1 shows a torus with prismatic symmetry. It has an axis of 15-fold dihedral symmetry and one central plane of reflective symmetry. Figure 2 shows a torus with antiprismatic symmetry. It has an axis of 15-fold dihedral symmetry and one central plane of roto-reflective symmetry. Figure 3 shows a torus with skew-antiprismatic symmetry. It has an axis of 15-fold rotational symmetry and 15 axes of 2-fold rotational symmetry. It is interesting to note that toroidal polyhedra of this symmetry type have no reflective symmetry and are folded from a tiling of the plane that does not have reflective symmetry either. Figure 4 shows both the top and bottom views of a torus with dihedral symmetry. Unlike the previous three examples the top and bottom of this torus look different. Finally figure 5 shows an example of a torus with only cyclic symmetry. Again top and bottom views are shown since the top and bottom are different. The difference between the polyhedra in figures 2 and 3 is very slight but close inspection will show that they are indeed different both geometrically and combinatorially. The same is true about figures 4 and 5

Each of the figures shows but one example of what can be constructed. The forthcoming paper will show how an infinite number of infinite families of each type can be constructed.

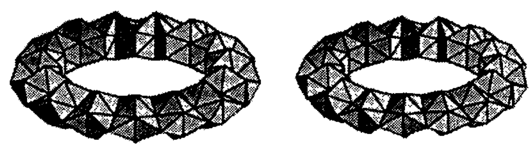

Figure 1: A flat torus with prismatic symmetry.

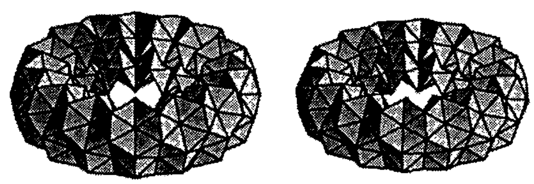

Figure 2: A flat torus with antiprismatic symmetry.

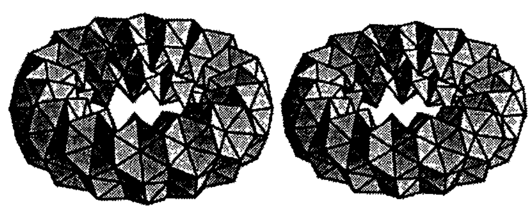

Figure 3: A flat torus with skew-antiprismatic symmetry.

Figure 4: A flat torus with dihedral symmetry.

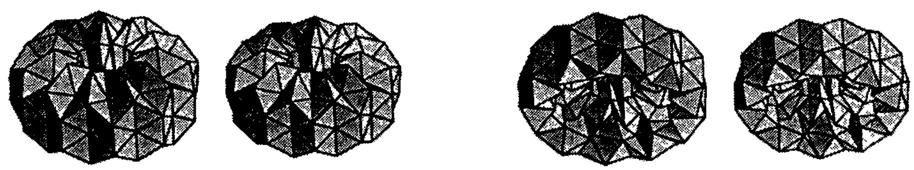

Figure 5: A flat torus with cyclic symmetry.

References

[1] L. Alaoglu and J. H. Giese, *Uniform Isohedral Tori.* Amer. Math. Monthly, 53(1946), 14 - 18.

[2] M. Brückner, *Vielecke und Vielflache.* Teubner, Leipzig 1900.

[3] W. T. Webber, *Monohedral Idemvalent Toroidal Polyhedra.* Ph.D. Thesis, University of Washington, 1994.