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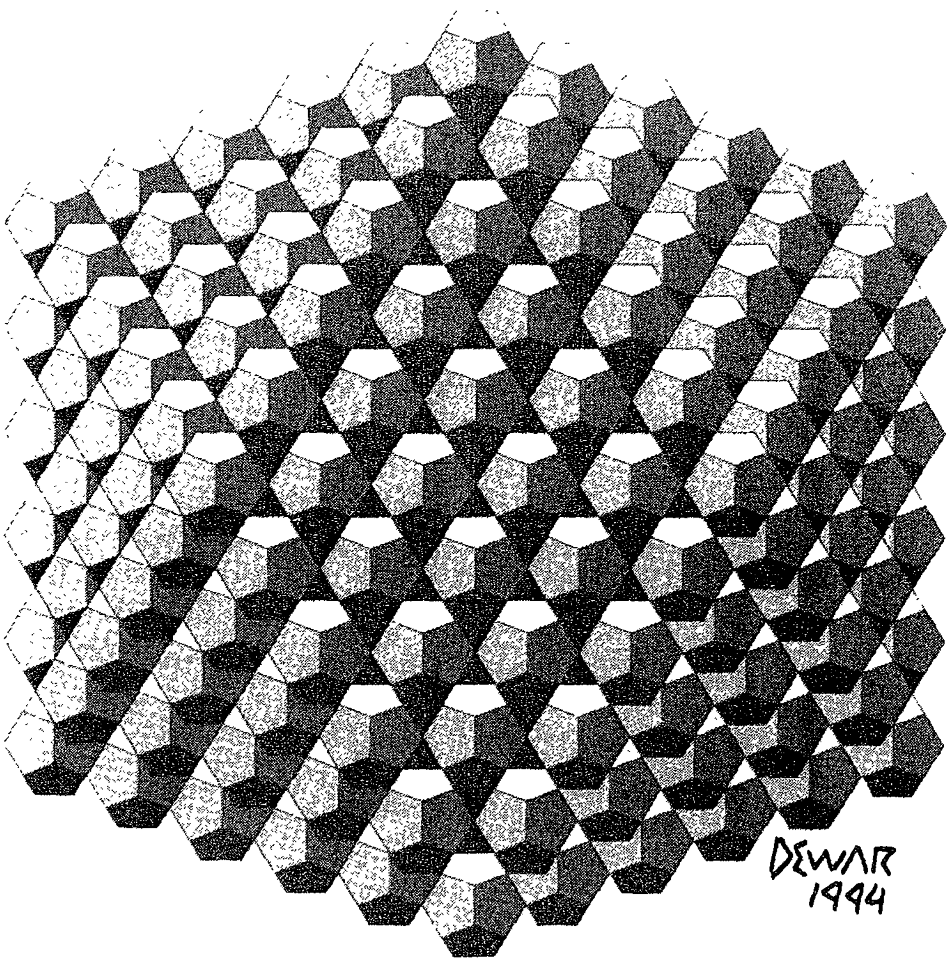

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ON SYMMETRIES AND CONSERVATION LAWS FOR PARTIAL DIFFERENTIAL EQUATIONS

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In the paper we will discuss the problem of the relationship between symmetries and conservation laws for partial differential equations. We consider the approach based on the Noether operator relation which allows the association of conservation laws with symmetries for a large class of differential equations without regard to the existence of the well-defined Lagrangian function. Among the equations of the class are many interesting equations: e.g. Korteweg-de Vries equation, Kadomtsev-Petviashvili equation, Monge-Ampere equation, and Navier-Stokes equations.

As usual by symmetries (Lie-Bäcklund symmetry transformations or generalized symmetries which include classical Lie symmetries) we mean infinitesimal transformations of independent x^i and dependent u^a variables of the form

$$\begin{aligned} x'^i &= x^i + \epsilon\xi^i(x,u,u_j,\ldots) + O(\epsilon^2)\,, \\ u'^a &= u^a + \epsilon\eta^a(x,u,u_j,\ldots) + O(\epsilon^2)\,, \end{aligned} \tag{1}$$

(where ε is an infinitesimal parameter (in general, symmetry groups are multi-parameter) and $u_i^a \equiv \dfrac{\partial u^a}{\partial x^i}$, $u_{ij}^a \equiv \dfrac{\partial^2 u^a}{\partial x^i \partial x^j}$, $i,j = 1,\ldots,n$, $a,b = 1,\ldots,m$) which leave the system of differential equations

$$\omega^a(x^i, u^b, u_i^b, u_{ij}^b, \ldots) = 0 \tag{2}$$

invariant. In other words a symmetry condition is

$$X_\alpha \omega^a \doteq 0\,, \tag{3}$$

where the symbol " $\doteq$ " denotes equality on the solution manifold $\omega^b = 0$, $D_i\omega^b = 0, \ldots$ (see, e.g. [1,2]), and X_α is a canonical symmetry operator

$$X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + (D_i\alpha^a)\frac{\partial}{\partial u_i^a} + \sum_{i\leq j}(D_iD_j\alpha^a)\frac{\partial}{\partial u_{ij}^a} + \dots , \tag{4}$$

(where $\alpha^a = \alpha^a(x, u, u_j, \dots)$ is a symmetry vector, and D_i is the total derivative operator).

By a local conservation law we mean the total divergence of some vector vanishing on the solution manifold.

$$D_i M^i \doteq 0. \tag{5}$$

For Lagrangian systems the correspondence between symmetries (symmetries of the action functional) and local conservation laws is given by the well-known Noether theorem. Moreover it can be shown (see, e.g. [2,4,5]) that all local conservation laws can be associated with Noether symmetries and non-Noether symmetries do not lead to any new local conservation laws.

The central point of our consideration of the correspondence between symmetries and local conservation laws for differential systems is the Noether operator identity [3,5]

$$X_\alpha = \alpha^a E^a + D_i R_{\alpha i} , \tag{6}$$

where

$$E^a = \frac{\partial}{\partial u^a} - D_i \frac{\partial}{\partial u_i^a} + \sum_{i\leq j} D_i D_j \frac{\partial}{\partial u_{ij}^a} - \dots = \sum_S (-1)^l D_S \frac{\partial}{\partial u_S^a} \tag{7}$$

is the Euler-Lagrange operator, l is the parity of the combination S , and $R_{\alpha i}$ is a certain differential operator (see [5] or [3]).

For a class of differential systems (2) defined by the following conditions on ω^a :

$$E^b(\omega^a) \doteq 0 , \tag{8}$$

any symmetry operator X_α $(X_\alpha \omega^a \doteq 0)$ according to the identity (6) leads to conserved currents

$$D_i(R_{\alpha i}\omega^a) \doteq 0 . \tag{9}$$

Among differential systems satisfying Eq.(10) there are equations in the form of a total divergence

$$\omega^a = D_i N_i^a , \qquad a = 1,\dots,m , \quad i = 1,\dots,n . \tag{10}$$

and equations for which

$$E(\omega) = b(u)\,\omega . \tag{11}$$

There are many physically interesting differential equations of the form (10): Korteweg-de Vries (KdV) equation, modified KdV equation, Boussinesq equation, nonlinear heat equation, nonlinear diffusion equation, regularized long-wave equation, Harry-Dym equation, Euler equations, Navier-Stokes equations and others (see [5] for details). For these equations the Noether operator relation (6) provides a natural way to associate conserved quantities with symmetries of the system without regard to the existence of a well-defined Lagrangian function [5,6].

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