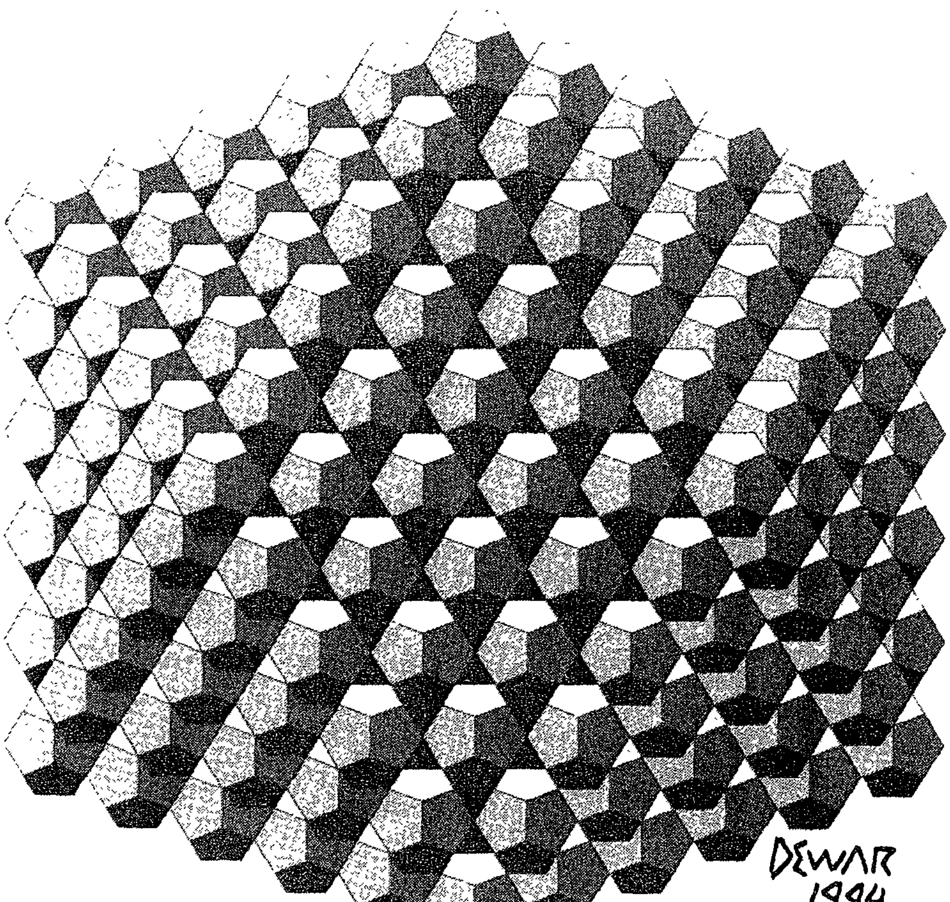
Symmetry: Culture and Science

Symmetry: Natural and Artificial, 3

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

Volume 6, Number 3, 1995



Third Interdisciplinary Symmetry Congress and Exhibition
Washington, D.C., U.S.A. August 14 - 20, 1995

QUASISYMMETRY TYPES FOR ALMOST PERIODIC TILINGS

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A basic idea in the theory of periodic tilings is the classification into *symmetry types* corresponding to isomorphism classes of *symmetry groups.* Here we describe a natural way to generalize the theory of symmetry types to the more general class of *almost periodic tilings* using ideas from dynamical systems theory. A tiling x is said to be *almost periodic* if for every patch q in x there exists a positive number r such that a copy of q occurs within a distance r from an arbitrary location in x. Any periodic tiling is almost periodic, but there are also aperiodic almost periodic tilings. The best known examples of almost periodic tilings are the *Penrose tilings*; many new examples have been found in connection with the theory of quasicrystals.

Almost periodic tilings correspond to the almost periodic points in a type of group action associated with sets of tilings, called a *tiling dynamical system.* A fundamental theorem of W. Gottschalk shows that such actions are *minimal*, a property generalizing the *transitivity* which occurs in the periodic case. In particular, if x is a periodic tiling of $\mathbf{R}^n$, and $\mathbf{Z}_x$ is its translation group, then the quotient $\mathbf{R}^n/\mathbf{Z}_x$ is the n-dimensional torus corresponding to a fundamental domain of x. The translation action by $\mathbf{R}^n$ is transitive. There is also natural action of the point group $\mathbf{H}_x$ of x on $\mathbf{R}^n/\mathbf{Z}_x$. We call the semidirect product $\mathbf{G}_x$ of $\mathbf{R}^n$ and $\mathbf{H}_x$ the *quasisymmetry group* of x, and we show that one can recover the classical symmetry type of x by knowing the appropriate dynamical *isomorphism class* of the natural action of $\mathbf{G}_x$ on $\mathbf{R}^n/\mathbf{Z}_x$.

When we switch to the almost periodic case, we replace $\mathbf{R}^n/\mathbf{Z}_x$ with the *orbit closure*

$\overline{O(x)}$ (in a *tiling topology*) of the tiling x, under the action of $\mathbf{R}^n$ by translation. In this general setting the point group $\mathbf{H}_x$ is defined to be the set of rigid motions leaving $\overline{O(x)}$ invariant. In the aperiodic case, the action of $\mathbf{G}_x$ on $\overline{O(x)}$ is *minimal* rather than transitive. In analogy to the above, the *quasisymmetry type* of an almost periodic tiling is defined to be the dynamical isomorphism class of this action. This provides a classification which generalizes classical symmetry theory while retaining many of its most important features. The new theory includes a version of Bieberbach's second theorem (the rigidity of isomporphism between quasisymmetry groups); on the other hand, there is no crystallographic restriction for almost periodic tilings (hence their place in the theory of quasicrystals), and in several different ways, no generalization of *Bieberbach's third theorem*. There is, however, a close relation between the *discrete spectrum* of the tiling dynamical system corresponding to a tiling x and the x-ray diffraction analysis of almost periodic structures related to x. A few cases, including the Penrose tiling case, have been analized completely.