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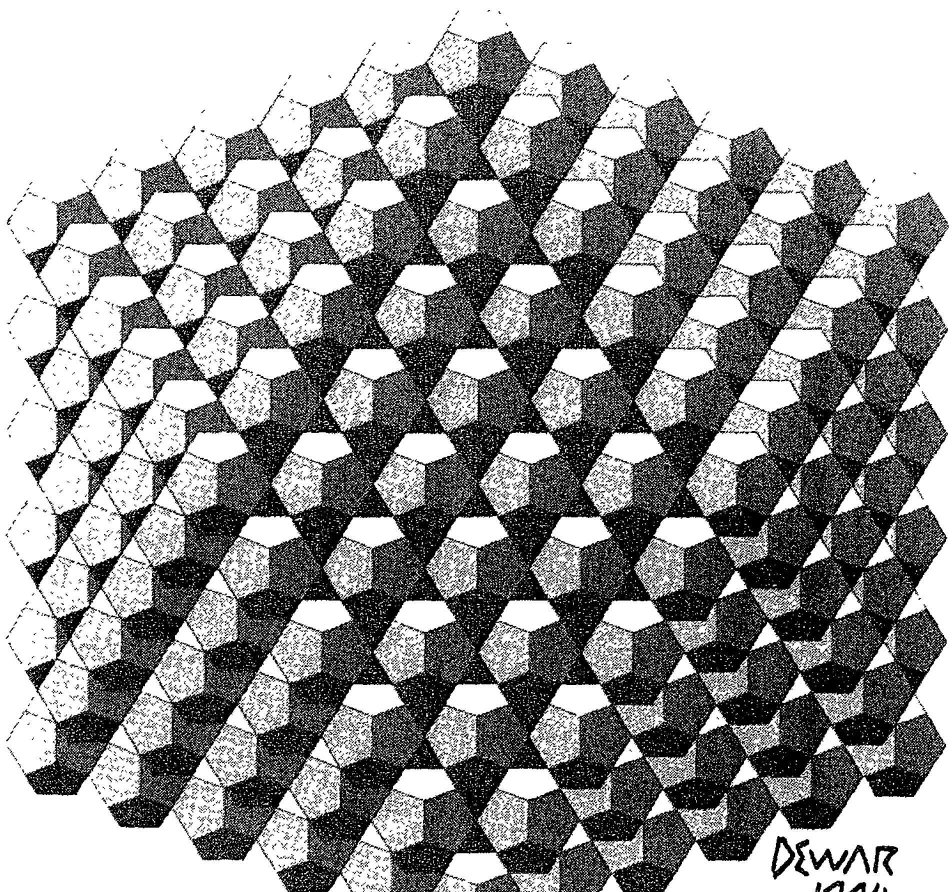

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INDEX OF HARMONY AND DELAUNAY TRIANGULATION

Oleg R. Musin
department of Cartography and Geoinformatics
Moscow State University, Moscow, 119899, Russia
e-mail: musin@gislab.geogr.msu.su

One of the most important problem of the modern science is to find a quantitative and qualitative characteristics needed for estimation of the harmonic, symmetrical and optimal properties of the natural and artificial geometric objects. In this paper we will consider the index of harmony of polygons, polytops and tessellations. This index is very simple, pure number and prove its name in practice. There are at least two reasons why the name "harmony" takes place here: the first of all this index came from so-called "harmonic function" in Mathematics and the second one is that polygons, polytops and partitions giving the minimum of index of harmony are more appropriate and symmetrical i.e. harmonic.

Index of Harmony for Polygons

For polygon P the ***index of harmony*** is equal the sum of squares of length of the sides P divided by area of P i.e. if we denote this index as hrm(P) then

$$hrm(P) = \sum a_i^2 / S(P)$$

where $a_1,\dots,a_n$ are the length of sides P and S(P) its area. This index is the same for similar polygons.

Let us call n - gon P as ***harmonic*** if and only if the hrm(P) achieves its minimum for P. It is clear that harmonic triangle is equiangular. The same result hold for arbitrary n. For harmonic n-gon P we can prove the following statements:

- P is convex;
- P is circumscribed polygon;
- P is equilateral.

From this statements it follows that harmonic polygon is rectilinear. It is easy to calculate the hrm for harmonic n-gon P,. $\mathrm{hrm}(P) = 4\mathrm{tg}(\pi/n)$.

We have proposed the generalization of the index of harmony for polygons. Let us denote for the polygon P:

$$\mathrm{hrm}(P,k) = \sum a_i^{2k} / S(P)^k ,$$

where k is the parameter. For this functional the minimal polygon is rectilinear too. In the case k = ½ this result gives classic isoperimetric unequality for polygon [1].

The Delaunay Triangulation

The Voronoi diagram is a natural and intuitively appealing structure, repeatedly reinvented by researches in several fields. While computer scientists general name it for the mathematician Voronoi, mathematicians called it Dirichlet (or Voronoi - Dirichlet) tessellation, meteorologists, geographers and geologists associate two dimensional version with the name Thiessen and physicist honor Wigner and Seitz for the 3D version. It has been used by geologists, foresters, agriculturalist, medical researches, geographers, crystallographers, astronomers, etc.

The ***Voronoi diagram*** on a set of points in the plane, called sites, is a subdivision of the plane into regions, each site corresponding to a single region consisting of all points in the plane that are closer to that site than to any others. The Delaunay tessellation is the straight-line dual graph of the Voronoi diagram. When no degeneracies are present, the Delaunay tessellation is a triangulation of the sites, called the ***Delaunay triangulation***. Lawson (1972), Sibson (1978), Guibas and Stolfi (1985), Rippa (1991) and Musin (1993) showed that Delaunay triangulation is the best from the surface's interpolation point of view.

A triangulation is regarded as 'good' for the different purposes if its triangles are nearly equiangular. When the data sites are placed almost as part of a regular triangular lattice, there is a little doubt about which triangulation to choose and no difficulty over constructing it. But in practice this even spacing mail fail to hold. With arbitrary placed points a close approach to equiangularity is seldom possible, and a criteria is needed for assessing the acceptability of a triangulation.

The harmonic functional gives its minimum for triangle if and only if this triangle is equiangular. For triangulation t of the set of sites S let denote hrm(t) the sum of hrm its triangles. It is clear, if hrm(t) gives its minimum, that t is the best among all triangulations of S. Our main result is following:

Theorem. hrm(t) achieves its minimum if and only if t is the Delaunay triangulation.

3D and Higher Dimensions

Let us consider a index of harmony in 3 and higher dimensions. It is clear, that right extension of hrm for polytop P is following

$$\mathrm{hrm}(P) = \sum F_i^d / V^{d-1},$$

where F_i are the volumes of faces of P, V is its volume and d is dimension.

As in 2-dimensional case, a harmonic tetrahedron (simplex) is rectilinear. It seems to us that all Platonic solids are harmonic. For general case it need to study properties of harmonic polytops more carefully.

For d > 2 it is not true that Delaunay triangulation gives minimum of hrm. It is not surprise, the same effect we can see that Delaunay triangulation is not the best from the surface's interpolation point of view [4]. It is open problem to find "harmonic" triangulation for $d > 2$. This and other problems can possibly be solved with further study of the index of harmony for polytops and tesselations.

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