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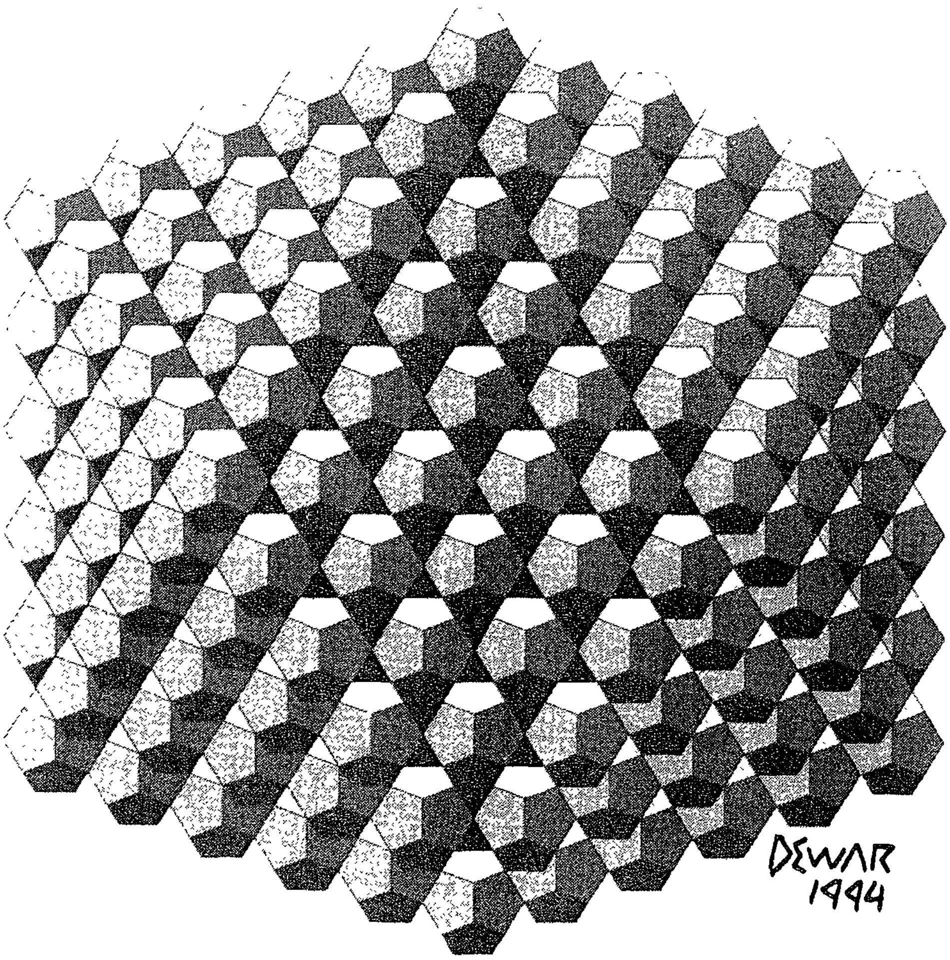

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ON THE STUDY OF POLYHEDRA IN WASAN

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§1. Introduction

A convex polyhedron is said to be regular if its faces are regular and equal, while its vertices are all surrounded alike. If its faces are p-gons and q copies of it are surrounding, the polyhedron is denoted by $\{p,q\}$. $\{p,q\}$ cannot have any other values than

$\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$.[1)]

These five regular polyhedra are commonly called " the Platonic solids ". And semiregular polyhedra, which are called " the Archimedean solids ", occure by cutting each vertice of a regular polyhedron with a plane. The semiregular polyhedron is defined as having regular faces, while its vertex figures are cyclic and equiangular.[2)] A semiregular polyhedron is denoted by $[p_1.p_2.\cdots.p_r]$; r means number of edges of a regular polygon surrounding its vertex. It is said that Archimedean solids are thirteen or may be sixteen.[3)]

In Wasan, traditional Japanese methematics, which developed during the Edo period (1603-1868), we find traces of study concerning polyhedra. But Wasan-ka和算家, traditional Japanese methematicians devoted themselves to calculating volume of them and forming simple equation for that, but they had no interest in the properties of regular polyhedra, such as reciprocity, compound.

This brief paper describes the aspect of study of polyhedra by Wasan-ka.

§2. The Study of Polyhedra by Wasan-ka

In 1639 "Jugai roku" 竪亥録(the Book of Mathematical Formulae for Children) was published by Tomoaki Imamura 今村知商(?-?). In this book T.Imamura gave two solids in the section of "Ho choku shiki"方直式 (Equation for Various Solids), one of which was named "Soba gata"蕎麦形 and the other was named "Kiri ko"切籠 meaning to cut off the corner of a basket(cube) by a plane. Here he showed two figures and equations for calculating volume of "Soba gata" and "Kiri ko"[4)].

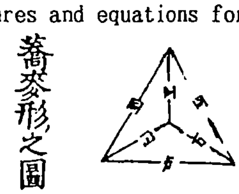

fig.1

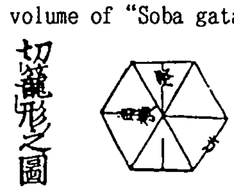

fig.2

$$V = \frac{\sqrt{2}}{12} a^3 \quad \text{---①} \qquad\qquad V = \frac{5\sqrt{2}}{3} a^3 \quad \text{---②}$$

Though his drawn figures are not clear on whether, these are equilateral triangle, plane or solid, two equations for calculating volume, ① and ② ; these are a regular tetrahedron {3,3} and a semiregular polyhedron, cuboctahedron[3,4,3,4]. Since then, in Wasan a regular tetrahedron was called "Soba gata", because it was like a shape of buckwheat. A semiregular polyhedron [3,4,3,4] was called "Kiri ko", because its shape reminded them of Chinese lanterns. It seems that these two solids, except a cube {4,3}, are the first appearance as regular polyhedra and semiregular polyhedra in Wasan.

"Sanpo ketsugi sho"算法闕疑抄 (Mathematical Selection of Lucid Solutions) by Yoshinori Isomura 礒村吉徳 (? -1710) was published in 1661. In Vol.3 of this mathematical book, two regular polyhedra and three semiregular polyhedra appeared under the title of "Soba gata jutsu", "Kiri ko jutsu" and "En kiri ko jutsu"円切籠術. "Jutsu"術 is a technical term with several meanings such as a method, a general solution and a formula. "Soba gata jutsu" is quite the same as "Soba gata" in "Jugai roku". One of these three semiregular polyhedra is a cuboctahedron; the other two solids are a truncated tetrahedron [3,6,6] and a truncated octahedron[3,8,8]. As you see in fig.3 and fig.4, he makes up two semiregular polyhedra by cutting off vertices of a tetrahedron {3,3} with a plane, and a cube respectively. Equations for calculating volume are as follows:[5)]

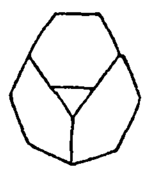
fig.3

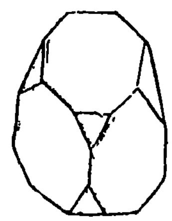
fig.4

$$V = \frac{(14\sqrt{2} + 21)}{3} a^3 \quad \text{---③} \qquad\qquad V = \frac{23\sqrt{2}}{12} a^3 \quad \text{---④}$$

Moreover, he set a problem, which was titled "En kiri ko jutsu", as fig. 5. Here "En" means a circle. His purpose in this problem was to find the remaining volume of sphere when six spherical segments in which the base had the same radius were cut off from a sphere like fig.5.[6)]

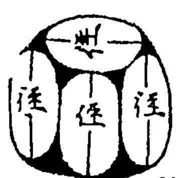
fig.5

We can easily obtain a regular octahedron {3,4} if we make in a sphere equilateral triangle by jointing the vertex of each pair of touching spherical segments. And also we have a semiregular polyhedron [3,4,3,4] if we joint the tangency of each spherical segment. But we do not clearly understand whether Y.Imamura knew this matter or not, because we find no mention on this anywhere in his book. This might be the first appearance of problem of arrangement on a spherical surface in the history of mathematics in Japan.

About the middle of seventeenth century a mathematical manuscript which was titled "Kyuseki kohen"求積後編(Sequel to the Method of Finding Area and Volume) was written by Yoshisuke Matsunaga 松永良弼(? - 1744). Y.Matsunaga, who was a remarkable mathematician in the history of Wasan, belonged to Takakazu Seki's 関孝和 (1642?-1708) academic family. After the death of Y.Matsunaga's teacher, Katahiro Takebe 建部賢弘 (1664-1739), he became a successor of Seki's academic family.[7] In his lifetime he left so many important theses concerning mathematics. "Kyuseki kohen" is one of them. The principal object of this thesis is to find volume of various solids, such as antiprisms and polyhedra. Problems concerning polyhedora are set in the section of "Yo dai"拗台, "To men"等面 and "Konsho dai" 混象台. "Yo dai" means a antiprism turning one of two bases of a prism in its own plane. "To men" means a solid which is constructed by equal faces or regular polygons. The word itself shows regular polyhedra, and similarly, "Kon sho dai", a solid which is construted by different regular p-gons, means a semiregular polyhedron.

The basic idea of regular solids and semiregular solids was partly formed by some Wasan-ka before Y.Matsunaga, as we mentioned above, so he discussed how to construct "To men" and "Konsho dai" and also how to find the equations for them. The following equations of $\{p,q\}$ and $[p_1, p_2, \cdots, p_r]$ for calculating volume of these are found by himself.

$\{p,q\}$ is shown below (① and ② are mentioned above.):

$\{3,5\}$(icosahedron)

$$V=\frac{(5\sqrt{5}+15)}{12}a^3$$

$\{5,3\}$ (dodecahedron)

$$V=\frac{(7\sqrt{5}+15)}{4}a^3$$

$[p_1, p_2, \cdots, p_r]$ is shown below (③ and ④ are already known.) :

$[4,6,6]$(truncated octahedron)

$$V=8\sqrt{2}a^3$$

$[3,5,3,5]$(icosidodecahedron)

$$V=\frac{(17\sqrt{5}+45)}{6}a^3$$

$[5,6,6]$(truncated icosahedron)

$$V=\frac{(43\sqrt{5}+85)}{4}a^3$$

$[3,10,10]$[8] (truncated dodecahedron)

$$V=\frac{(235\sqrt{5}+495)}{12}a^3$$

About this time, he discovered all the five Platonic solids and also seven of the thirteen Archimedean solids including the past results. His basic idea to consutruct polyhedra was to cut the vertex of solid by a plane. However, in the cases of $\{3,5\}$ (icosahedron) and $\{5,3\}$ (dodecahedron), it was very difficult to do so. Hence he explaned in this thesis as follows :

1) 是三角拗台之十二楞等者也。此形四角錐之面楞相等者、両個接合也。

2)此形執面梁斉等五角錐両個、而使面与面相対、中間容面梁斉等五角拗台

They are as follows:

1) By placing two equal regular pyramids base to base (the common base is a {p}), and by adjusting the altitude of the pyramids we can make the triangles equiletaral. If p=4, we obtain a octahedron {3,4}. And by adjusting the altitude of an antiprism, we can make its 2p lateral triangles that could be equilateral. If p=3, we have the octahedron. 2) If p=5, we can place regular pyramids on the two bases, and then we have the icosahedron {3,5}.

§3. Conclusion

We are not sure whether Y.Matsunaga's idea to make the octahedron and icosahedron, by using the antiprism, was purely drawn from the idea of Wasan. We suppose that he knew it from the mathematical and astronomical books which were brought over to Japan from Holland or China[9]. Here we are not to give a detail on this matter; we will discuss this on another occasion.

Notes

1) H.S.M.Coxeter, Regular Polytopes (New York : Dover Publications,1973),Third edition, p.5.
2) H.S.M.Coxeter,Regular Polytopes,p.18.
3) Shin Hitotsumatsu,Solution of Regular Polyhedron (Tokai University Press., 1983), pp.96-123.
4) Compiled by Kan Yosano, Atsuo Masamune and Akiko Yosano,Collected edition of Japanese classics (Nippon Koten Zensho Kankokai:Tokyo,1927),pp.110-102, see Ken-ichi Sato,Jugai roku kana sho (Kenseisha: Tokyo,1988),pp.200-204 or Kazuo Shimodaira, A history of Wasan (Fuji College Publisher:Tokyo,1970),Vol.1, p. 42.
5) ibid.,pp.31-38. L.Fejes Toth call this figure "spherical ball". See his book, Lagerungen in Ebene auf der Kugel und im Raum, Translated by Isao Higuchi and Masami Tanemura, Haichi no monndai-- Heimen·Kyumen·Kukan ni okeru--(Misuzu Shobo:Tokyo,1983),pp.116-117.
6) ibid.,p.39.
7) Edited by Akira Hirayama and Jun Nito, Selection of Yoshisuke Matsunaga (Tokyo Horei Publisher:tokyo,1987), biography,pp.1-2.
8) See Selection of Yoshisuke Matsunaga, pp.328-336.
9) Concerning the antiprism see H.S.M.Coxeter,Regular Polytopes,pp.5-6. And on the origin of antiprism H.S.M.Coxeter in his book, Regular Polytopes,p.14, says that antiprism do not seem to have been recognized before Kepler(A.D 1571-1630). On the other hand, S.Hitotsumatsu, in his book, Solution of Regular Polyhedron, p.15, says that he could see it in Eculed's Vol.13.