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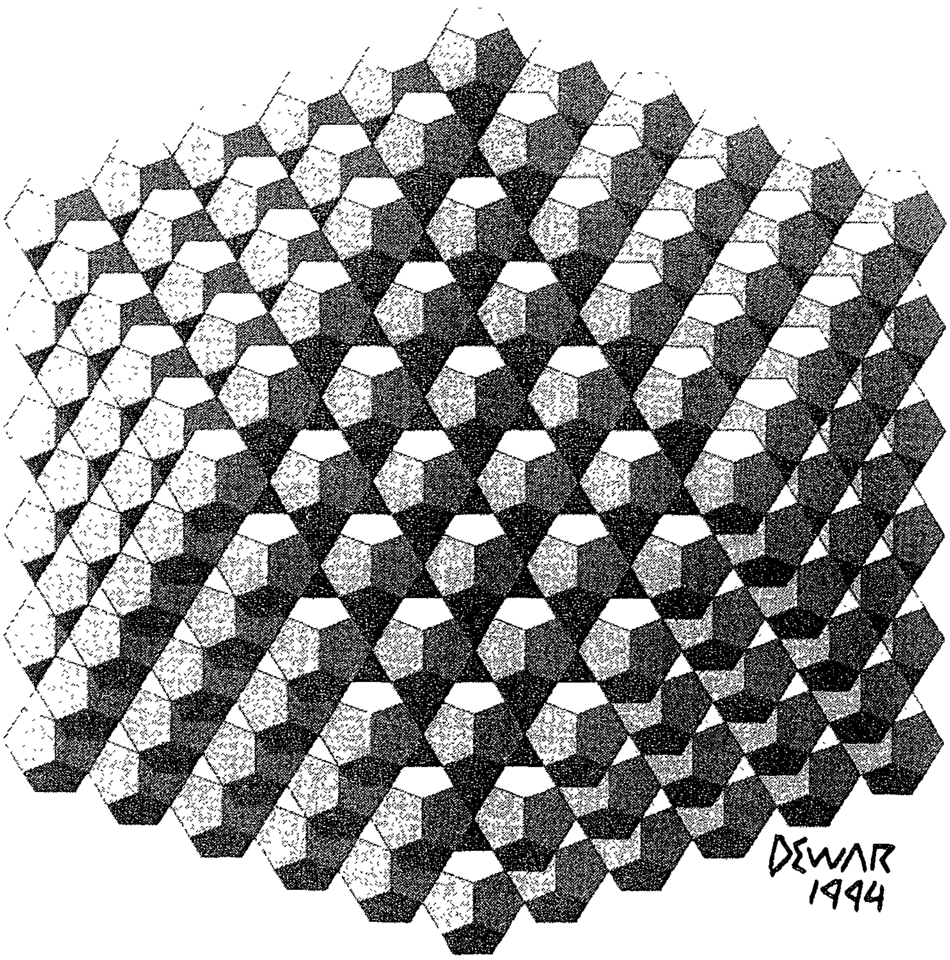

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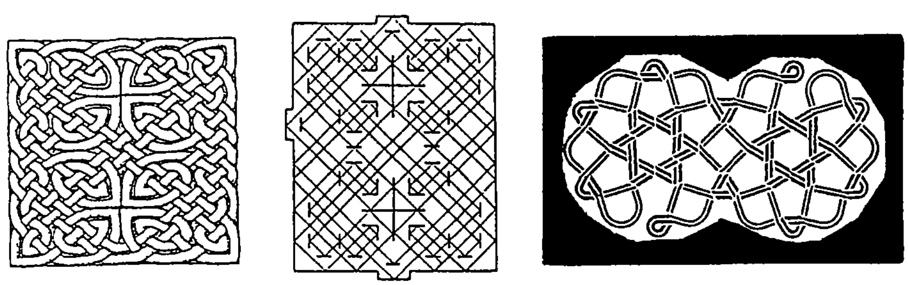
MIRROR GENERATED CURVES

Slavik V. Jablan
The Mathematical Institute,
Belgrade 11001, Yugoslavia

The imitation of the three-dimensional arts of plaiting, weaving and basketry was the origin of interlacing and knotwork interlacing ornaments. Their highlights are the Celtic interlacing knotworks (Bain, 1973; Cromwel, 1993) (Fig.1a). Islamic layered patterns and Moorish floor and wall decorations.

The common geometrical construction principle for all such decorations is the use of (two-sided) mirrors incident to the edges of a square, triangular or hexagonal regular plane tiling, or perpendicular to its edges in their midpoints (Fig.1a). In the ideal case, after the series of consecutive reflections, the ray of light reaches its beginning point, defining a single closed curve (Gerdes, 1990). In other cases, the result consists of several such curves.

The construction of such curves was occupied the attention of two most greatest painters-mathematicians: Leonardo and Dürer (Bain, 1973). Some interesting geometrical and aritmethical properties of the curves mentioned are discovered by Paulus Gerdes (1989, 1990, 1993). Let us notice one more beautiful geometrical property: such curves can be obtained using only few different prototiles. For the construction of all the curves with internal mirrors incident to the edges, they are sufficient three prototiles in the case of a regular t· ngular tiling, five in the case of square, and 11 in the case of haxagonal regular tiling. We may also use their combinations occuring in the 11 uniform Archimedean tilings (Grünbaum & Shephard, 1986) (Fig.1b).



(a) (b)

Figure 1

The symmetry of such curves is used for the reconstruction of Tamil designs (Gerdes, 1989), as well as for the classification of the Celtic frieze designs (Bain, 1973). From the ornamental heritage, at first glance it looks that the symmetry is the mathematical basis for their construction and possible classification. But, the existence of such asymmetrical curves suggests the other approach. Trying to discover their common mathematical background, they appear two questions: how to construct such a perfect curve (this means, how to arrange the set of mirrors generating it), and how to classify the curves obtained. Our consideration we will restrict to the curves derived from the square tilings.

In principle, any polyomino (Grünbaum & Shephard, 1986) with mirrors on its border, and two-sided mirrors between cells or perpendicular on the internal edges in their midpoints, can be used for the creation of the corresponding curves. First, we construct all the different curves without use of internal mirrors, starting from different edge midpoints and ending in them, till the polyomino is exausted, i.e. uniformly covered by k curves. After that, we use "curve surgery" in order to obtain a single curve, according to the following rules: (a) any mirror introduced in a crossing point of two distinct curves connects them into one curve; (b) depending on the position of a mirror, a mirror introduced into a self-crossing point of an (oriented) curve makes no change, or breaks it into two closed curves. In every polyomino we may introduce $k-1$, k, $k+1$, ..., $2A-P/2$ internal two-sided *mirrors*, where A is area and P perimeter of the polyomino. Introducing minimal number of mirrors $k-1$ we first obtain a single curve, and in the next steps we try to preserve that result.

There is also a simple way to preserve such single closed curve: to add on the border of a polyomino a cell with three mirror-edges and one empty edge, or delete such a cell. This way, any such polyomino with a single curve can be transformed into a rectangle. Unfortunately, they are rectangular mirror-schemes which cannot be derived that way.

In the case of a rectangle with the sides a,b, the initial number of curves, obtained without use of internal mirrors, is $k=gcd(a,b)$, so in order to obtain a single curve, the possible number of internal two-sided mirrors is $k-1$, $k,\ldots,$ $2ab-a-b$. According to the rules for introduction of internal mirrors, we have the algorithm for the production of designs consisting of a simple closed curve: each from the first internal $k-1$ mirrors must be introduced in crossing points belonging to different curves. After that, when they are conected and transformed into a single line, we may introduce other mirrors, taking care about the number of lines, according to the rules mentioned. The next question is the classification of the curves obtained. First criterion we may use is the geometrical: two curves are equal *iff* there is a similarity transforming one into the other. Instead of considering the curves, we may consider the equal mirror arrangements defined in the same way. Having the algorithm for the construction of such perfect curves and the criterion for their equality, we may try to enumerate them: to find the number of all the different curves (i.e. mirror arrangements) which can be derived from a rectangle with the sides a,b, for a given number of internal mirrors m ($m \in \{k-1, k,\ldots, 2ab-a-b\}$). Unfortunately, we are very far from the general solution of this problem. Reasons for this are: every introduction of an internal mirror changes the whole structure, so it behaves like some kind of "Game of Life" or cellular automata.

Till this time, we have only few combinatorial results, obtained by non-standard use of Pólya enumeration theory (Aigner, 1979; Pólya & Read, 1987). Let be given a rectangle with sides a, b $(a \neq b)$, $k = gcd(a, b)$, and let be introduced the minimal number $k-1$ of two-sided internal mirrors incident to the edges of its square tiling. If $t = (ab - lcm(a, b)) : (k(k-1))$, $x = a : (2k)$, $y = b : (2k)$, we have, for example, for $k=5$, $a=0 (\mathrm{mod}\ 10)$ and $b=5 (\mathrm{mod}\ 10)$, the formula $14720t^4 - 576t^3 + 80t^2 + 32tx - 4xy - x$, giving the number of such curves.

The other point of view on the classification of such perfect curves is that of the knot theory. As it is mentioned before, every such curve can be simply transformed into an interlacing knotwork design, this means, a projection of some alternating knot. In the history of ornamental art, such curves occured most frequently as knotworks, then as plane curves. Even the name *Brahma-mudi* (Brahma's knot) (Gerdes, 1989) denoting such Tamil curves refers us to the knot theory (Burde & Zieschang, 1985; Kauffman, 1987; Kohno, 1989). In order to classify them, we will first transform every such knot projection into a proper (reduced) knot projection (Kohno, 1989)- a knot projection without loops, by deleting cells with loops.

This way, we will obtain proper knot projections with the minimal number of crossings. Two such projections or knot diagrams are equal *iff* they are isotopic in projection plane as graphs, where the isotopy is required to respect overcrossing respectectively undercrossing (Burde & Zieschang, 1985). For the classification of knots they are used different kinds of knot invariants: Alexander polynomials (Burde & Zieschang, 1985; Kauffman, 1987; Kohno, 1989), Jones polynomials (Kohno, 1989), Conway polinomials (Kauffman, 1987), etc. In order to classify the knot projections (Dowker & Thistlethwaite, 1983) we will define a new invariant of knot (or link) projections. Let be given a proper oriented knot diagram D with generators $g_1, \ldots, g_n$. If the meeting point of generators g_i, g_j, g_k is "right", then $a_{ii}=t$, $a_{ij}=1$, $a_{ik}=-1$; if it is "left", then $a_{ii}=-t$, $a_{ij}=1$, $a_{ik}=-1$; in all the other cases $a_{ij}=0$. The determinant $d(t)=|a_{ij}|$ is the polynomial invariant of D.

The writhe of D, denoted by $w(D)$, is the sum of signs of all the crossing points in D, where the sign is +1 if the crossing point is "right", and -1 if it is "left" (Kohno, 1989). There is the most simple visible property of every knot projection: $|w(D)|$ is the type of the knot projection. By the use of a computer program, based on algorithm developed by Dowker & Thistlethwaite (1983), it is derived the complete list of non-isomorphic alternating knot projections for $3 \leq n \leq 12$.

There are some important properties of the integer polynomial invariant $d(t) = c_n t^n + \ldots + c_1 t$: (a) for every alternating knot projection, the degree of $d(t)$ is n and $|c_n|=1$; (b) for every knot projection $|c_1|$ is equal to the type of the knot projection (i.e. $|c_1| = |w(D)|$); (c) $d(t)$ and $d(-t)$ correspond to the obverse (enantiomorphic, mirror symmetrical) knot diagrams; (d) for $n=0 (\mathrm{mod}\ 2)$, a change of the orientation of an alternating knot projection results in the change of $d(t)$ to $d(-t)$; (e) for $n=1 (\mathrm{mod}\ 2)$ a change of orientation of an alternating knot projection results in the change of $d(t)$ to $-d(-t)$. According to (c),(d) and (e), in the set of all the knot invariants $d(t)$ we may distinguish even functions ($d(t)=d(-t)$), containing only even degrees of t, corresponding to amphichiral knot projections, and odd functions ($d(t)=-d(-t)$), containing only odd degrees of t, which are invariant to a change of orientation of the knot projection. Let us also notice that invariant introduced makes distinction between non-isomorphic knot projections of composite knots (i.e. direct products of prime knots).

This invariant may be simply transferred to the alternating link projections. In this case, the result is the polynomial invariant of the form: $d(t)=c_n t^n+...+c_k t^k$, where n is the number of crossing points, and k is the number of the link components. For every link, $|c_n|=1$. If a_i are the link components, $a_{ii}=w(a_i)$, and if $a_{ij}=lk(a_i,a_j)$ denotes the linking number of the components a_i,a_j, then $|c_k|=|\det(a_{ij})|$.

The problem exposed shows how the same (old) structures-perfect *pavitram* curves (Gerdes, 1989, 1990), may be regarded from the three different points of view: that of the theory of symmetry, combinatorial geometry and topology, taking us to a trip through mathematics, and introducing a new class of mirror-structures: mirror-generated curves.

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