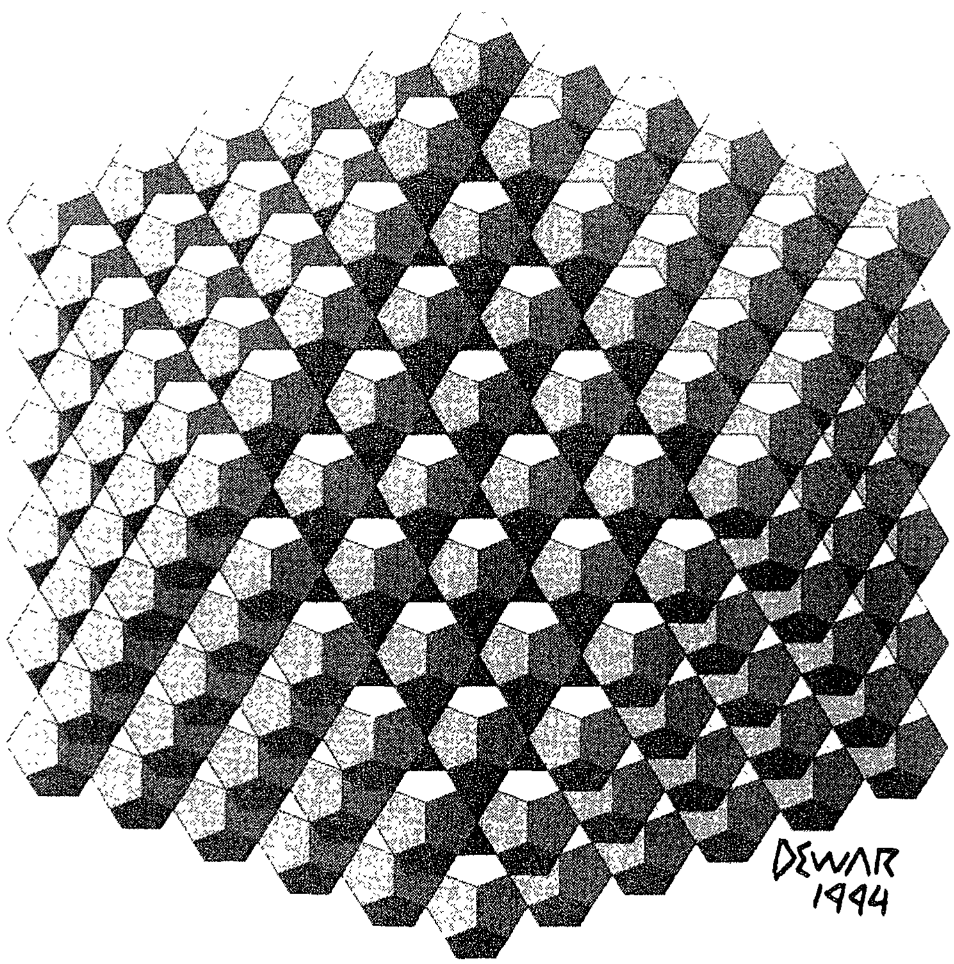
Symmetry: Culture and Science

Symmetry:
Natural and Artificial, 2

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

Volume 6, Number 2, 1995



Third Interdisciplinary Symmetry Congress and Exhibition
Washington, D.C., U.S.A. August 14 - 20, 1995

OPTIMIZATION ON CHAOTIC ATTRACTORS

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Chaotic attractors of dynamical systems are generally found to have a "natural" invariant probability measure which represents the statistical distribution of a typical trajectory which converges to the attractor. Here "typical" means that an initial condition is chosen at random with respect to Lebesgue measure from the basin of the attractor; the "basin" is the set of initial conditions whose trajectories converge to the attractor. Examples of such measures which are known rigorously to exist are ergodic absolutely continuous invariant measures for certain one-dimensional maps and Sinai-Bowen-Ruelle measures for Axiom A systems. Under these circumstances, and given a function defined on the state space of the system, the time average of the function over a typical trajectory is equal to the state space average of the function with respect to the natural invariant measure; in particular the time average has the same value for all typical trajectories.

Nonetheless, there are an infinite number of invariant measures supported on a give chaotic attractor; for instance, every unstable periodic orbit $\{x_1, x_2, \ldots, x_n\}$ embedded in the attractor gives rise to an invariant measure $(\delta_{x_1} + \delta_{x_2} + \cdots + \delta_{x_n})/n$, where δ_x denotes a unit point mass located at x. These measures describe the statistics of atypical trajectories, and as such are often ignored; however, we consider here the question of which invariant measure on a given attractor maximizes (or minimizes) the average of a given function on state space. That is, considering all possible trajectories on the attractor, which ones maximize the time average of the function?

We find, through numerical examples and heuristic analysis, that a time average on a chaotic attractor tends to be maximized by an unstable periodic with low period. The significance of this phenomenon is twofold. First, methods for controlling chaos allow chaotic systems to be kept near unstable periodic orbits with extremely small perturbations. The mean performance level of a chaotic physical system can be modeled as a time average; then our results imply that peak system performance can typically be achieved by controlling the system near an appropriate periodic orbit. Second, our results have implications for "blowout bifurcations", in which a chaotic attractor changes from having an open basin of attraction to having a "riddled" basin as a system parameter changes.

To explain further, we illustrate the mechanism for a blowout bifurcation in terms of a two-dimensional map f with a symmetry about some line L. In particular, f must map L into itself, and in some cases its restriction to L will, viewed as a on-dimensional map, have a chaotic attractor A. Every trajectory in A has an associate "transverse Lyapunov exponent" which represents the mean exponential rate of convergence (negative exponent) or divergence (positive exponent) of nearby trajectories in a direction transverse to the line L. This exponent is the time average of a function which is depends on the partial derivatives of the map f. If the transverse Lyapunov exponents of all trajectories in A are negative, then the trajectories of all initial conditions in a two-dimensional neighborhood of A converge to A; that is, A has an open basin of attraction. A blowout bifurcation occurs when the transverse Lyapunov exponent of some trajectory in A becomes positive as a parameter changes. After the bifurcation, A will typically have a riddled basin—in every neighborhood of A there is a positive measure set of initial conditions whose trajectories are not attracted to A. (But as long as the transverse Lyapunov exponent of the natural invariant measure is still negative, A will continue to attract a positive measure set of initial conditions.) At the bifurcation parameter, the transverse Lyapunov exponents of all trajectories are negative, except for the trajectories whose transverse Lyapunov exponents first reach zero. These trajectories necessarily maximize the transverse Lyapunov exponent, so our results indicate that blowout bifurcations tend to occur when an unstable periodic orbit in A with low period becomes unstable.