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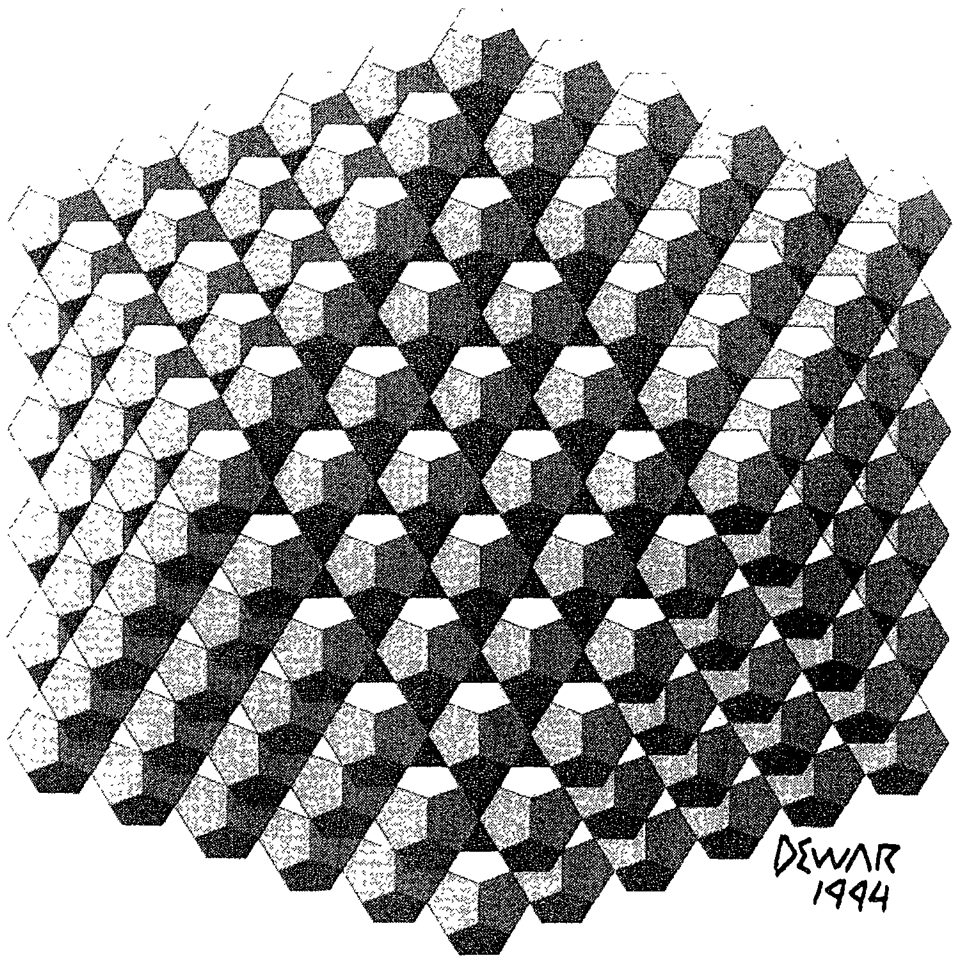

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THE SYMMETRY OF HYPERBOLIC ESCHER PATTERNS

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Introduction

Hyperbolic geometry was discovered by János Bolyai and Nikolai Lobachevsky in the early 1800's. By the end of that century, mathematicians created triangle tessellations (see Figure 1 below), probably the first repeating hyperbolic patterns — which, though attractive, were not created for artistic purposes. However, in the late 1950's, the pattern of Figure 1 (used by H. S. M. Coxeter in a paper) *did* inspire the Dutch artist M. C. Escher to use hyperbolic geometry in an artistic way in his four patterns *Circle Limit I*, *Circle Limit II*, *Circle Limit III*, and *Circle Limit IV* — see Catalog Numbers 429, 432, 434 (and p. 97), and 436 (and p. 98) of [Locher, 1982]. The patterns of interlocking rings near the edge of his last woodcut *Snakes* (Catalog Number 448 of [Locher, 1982]) also exhibit hyperbolic symmetry. It is laborious to create repeating hyperbolic patterns by hand as Escher did. In the late 1970's, the power of computers was applied to the problem of creating such patterns. Since then, much progress has been made in this area which spans mathematics, art, and computer science [Dunham, 1986a], and [Dunham, 1986b].

We will begin with a review of hyperbolic geometry, repeating patterns and tessellations, and symmetries of hyperbolic patterns. Finally, we will discuss a computer-aided hyperbolic pattern-generation process, color symmetry of hyperbolic Escher patterns, and possible further directions of research.

Hyperbolic Geometry

By definition, (plane) hyperbolic geometry satisfies the negation of the Euclidean parallel axiom together with all the other axioms of (plane) Euclidean geometry. Consequently, hyperbolic geometry satisfies the following parallel property: given a line ℓ and a point P not on that line, there is more than one line through P not meeting ℓ. Unlike the Euclidean plane and the sphere, the entire hyperbolic plane cannot be isometrically embedded in 3-dimensional Euclidean space. Therefore, any model of hyperbolic geometry in Euclidean 3-space must distort distance.

The *Poincaré circle model* of hyperbolic geometry has two properties that are useful for artistic purposes: it is conformal (i.e. the hyperbolic measure of an angle is equal to its Euclidean measure), and it lies within a bounded region of the Euclidean plane — allowing an entire hyperbolic pattern to be displayed. The "points" of this model are the interior points of a *bounding circle* in the Euclidean plane. The (hyperbolic) "lines" are interior circular arcs perpendicular to the bounding circle, including diameters. The edges of the curved triangles in Figure 1 and the backbones of the fish in Escher's *Circle Limit I* pattern (Figure 2) represent *hyperbolic lines.*

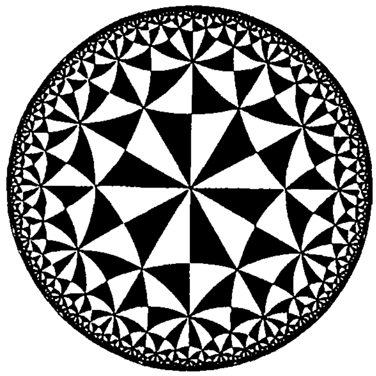

Figure 1. A pattern with symmetry group $[6,4]^+$.

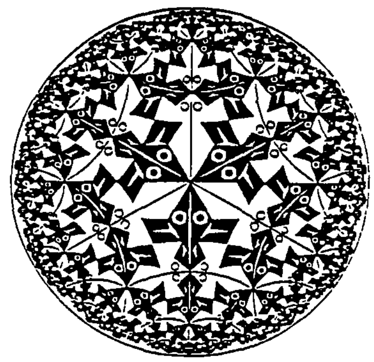

Figure 2. A computer generated rendition of M. C. Escher's *Circle Limit I* pattern.

Repeating Patterns, Tessellations, Symmetries

A *repeating pattern* of the hyperbolic plane is a pattern made up of hyperbolically congruent *copies of a basic subpattern or motif.* For instance, any adjacent black-white pair of triangles of Figure 1 forms a motif. Similarly, a black half-fish plus an adjacent white half-fish make up a motif for Figure 2.

An important kind of repeating pattern is the *regular tessellation*, $\{p,q\}$, of the hyperbolic plane by regular p-sided polygons, or *p-gons*, meeting q at a vertex. It is necessary that $(p-2)(q-2) > 4$ to obtain a hyperbolic tessellation. Figure 3 shows the tessellation $\{6,4\}$ (solid lines) and its dual tessellation $\{4,6\}$ (dotted lines).

A *symmetry operation* or simply a *symmetry* of a repeating pattern is an isometry (hyperbolic distance-preserving transformation) of the hyperbolic plane which transforms the pattern onto itself. For example, reflections across the backbones in Figure 2 and across any of the lines of Figure 3 are symmetries of those patterns (reflections across hyperbolic lines of the Poincaré circle model are inversions in the circular arcs representing those lines — or ordinary Euclidean reflections across diameters). Other symmetries of Figure 2 include rotations by 180 degrees about the points where the trailing edges of fin-tips meet, and translations by four fish-lengths along backbone lines (in hyperbolic geometry, as in Euclidean geometry, a translation is the product of successive reflections across two lines having a common perpendicular; the product of reflections across two intersecting lines produces a rotation about the intersection point by twice the angle of intersection).

The *symmetry group* of a pattern is the set of all symmetries of the pattern. The symmetry group of the tessellation $\{p,q\}$, denoted $[p,q]$, can be generated by reflections across the sides of a right triangle with acute angles of $180/p$, and $180/q$ degrees; i.e. all symmetries in the group $[p,q]$ may be obtained by successively applying a finite number of those three reflections. Thus, $[6,4]$ is the symmetry group of the tessellation $\{6,4\}$ formed by the solid lines of Figure 3 — in fact $[6,4]$ is the symmetry group of the entire pattern of Figure 3. The orientation-preserving subgroup of $[p,q]$ consisting of symmetries made up of an even number of reflections is denoted $[p,q]^+$. The symmetry groups of Figures 1 and 4 are $[6,4]^+$ (it is just $[6,4]$ if the color of the triangles is ignored), and $[5,5]^+$ respectively. We took our inspiration for Figure 4 above from Escher's notebook pattern number 20 (page 131 of [Schattschneider, 1990]). For more about the groups $[p,q]$, see Sections 4.3 and 4.4 of [Coxeter and Moser, 1980].

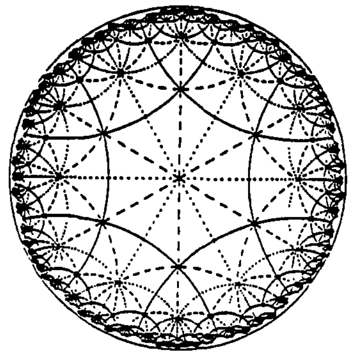

Figure 3. The tessellations $\{6,4\}$ (solid lines) and $\{6,4\}$ (dotted lines), and *other lines* (dashed) of reflective symmetry of the pattern.

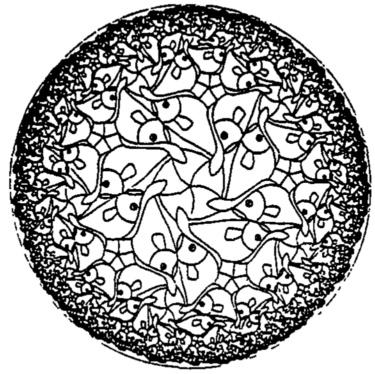

Figure 4. A pattern with symmetry group $[5,5]^+$.

The Hyperbolic Pattern-Creation Process

The present version of the computer program allows for the design of repeating patterns with color symmetry whose symmetry group is a subgroup of $[p,q]$ and whose motif lies within a p-gon of the corresponding tessellation $\{p,q\}$. The pattern-creation process consists of two parts: (1) design of the motif, and (2) replication of the whole pattern from the motif. The design of the motif is done most easily with a computer graphics input device such as a data tablet or mouse — the motif is outlined by a sequence of points entered by the input device and connected by line segments.

To replicate a pattern from a motif, first note that it is easy to replicate that part of a pattern within a p-gon of $\{p,q\}$ if that p-gon already has a copy of the motif within it — the copy of the motif is simply rotated about the center or reflected across lines through the center of the p-gon. The algorithm for replicating the whole pattern depends on the fact that the p-gons of $\{p,q\}$ form "layers": the first layer is a p-gon centered in the bounding circle, and each subsequent layer is defined inductively as the set of p-gons having a common vertex (only) or edge (only) with a p-gon from a previous layer. Then it is merely a matter of moving a copy of the motif from one p-gon to another (either from one layer to the next or within a layer), using appropriate elements of the symmetry group. For more details on the pattern-creation process, see [Dunham, 1986a].

Color Symmetry and Future Work

A pattern is said to have *n-color symmetry* if each of its motifs is drawn with one of n colors and each symmetry of the pattern maps all motifs of one color onto motifs of another (possibly the same) color; i.e. each symmetry permutes the n colors. The pattern of Figure 1 has 2-color symmetry (as does the Euclidean checkerboard pattern): reflection of the pattern across the side of any triangle interchanges black and white; rotation about a triangle vertex through twice its angle produces the identity permutation — black triangles go to black triangles and white triangles go to white triangles. Figure 5, Escher's *Circle Limit II* pattern, exhibits 3-color symmetry: 120-degree rotation about meeting points of outer crosses permutes the colors black, white, and gray. Figure 6 shows a pattern with 4-color symmetry. Escher was a pioneer in n-color symmetry (for $n \geq 3$) for Euclidean, hyperbolic, and even spherical patterns. His *Circle Limit III* pattern (not shown here) exhibits 4-color symmetry; his *Circle Limit IV* pattern of white angels and black devils has no (non-trivial) color symmetry, nor does *Circle Limit I* (Figure 2 above) — the black have "sharper" noses than the white fish. Hyperbolic color symmetry would seem to be a fruitful area for future research. For more on color symmetry, see [Senechal, 1983], and [Shubnikov and Koptsik, 1974].

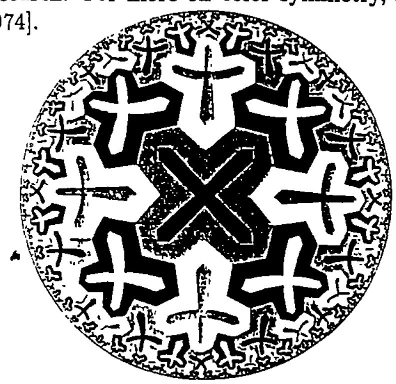

Figure 5. A computer generated rendition of M. C. Escher's *Circle Limit II* pattern.

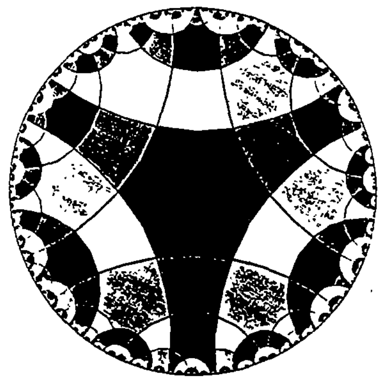

Figure 6. A repeating hyperbolic pattern with 4-color symmetry.

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