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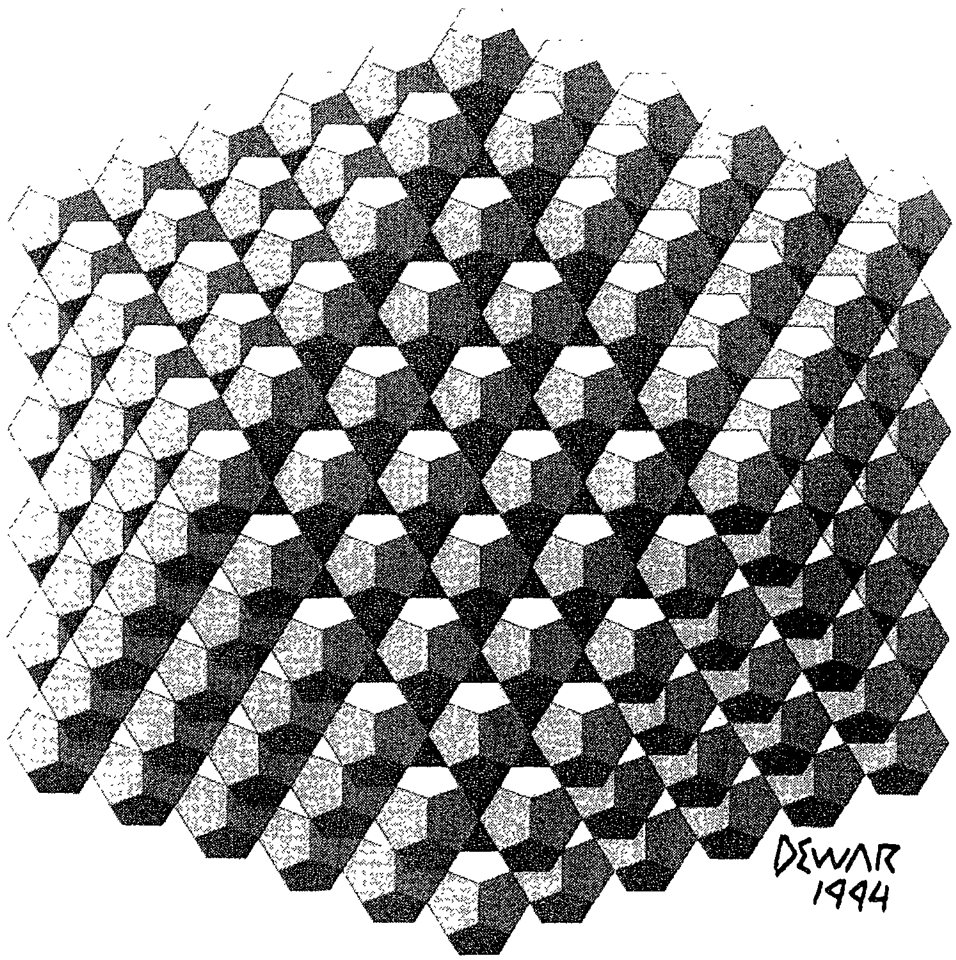

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TOROIDAL SKEW POLYHEDRA

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The five Platonic solids – icosahedron, dodecahedron, octahedron, cube, and tetrahedron – have been used as tools, toys, and decoration since 2000 BC [3]. They are distinguished by the fact that each face is symmetric to every other face, and each vertex is symmetric to every other vertex. What other objects have these properties?

At the beginning of the seventeenth century, Johannes Kepler described two regular "hedgehogs" with five pointed stars for faces [4]. If we allow edges of faces and faces themselves to intersect we can describe a total of four new regular polyhedra, called the Kepler-Poinsot polyhedra.

Kepler and Poinsot found new polyhedra by allowing the edges of faces and vertex figures to intersect. In 1937, H. S. M. Coxeter described regular polyhedra in which the vertex figures were non-planar "skew polygons". He went on to introduce skew polyhedra; polyhedra in higher dimensional Euclidean spaces. One example of such an object, discovered by J. F. Petrie, can be described as follows. Identify opposite sides of a square of side length n to form a torus. If the n^2 unit squares covering this torus are considered as a subset of the faces of the double n-gonal prism $\{n\} \times \{n\}$, the resulting regular skew polyhedron is a realization of the map $\{4,4\}_{n,0}$ [1, 2].

In 1977 Branko Grünbaum described several regular polyhedra with skew faces, further broadening the definition of regular skew polyhedra [5]. Skew polyhedra are interesting because of their relationship to the beautiful Platonic solids, because the set of vertices of a regular skew polytope is a family of evenly spaced points on a high dimensional sphere, and because they provide us with hints about smooth embeddings of surfaces in Euclidean space.

In our search for regular skew polyhedra, we are seeking geometric objects whose vertices and faces are symmetric. If we omit the condition that our polyhedra must have geometric realizations, such objects are known as "regular maps".

A *map* is the decomposition of a two-dimensional surface (without boundary) into non-overlapping simply-connected regions (faces) by arcs (edges) joining pairs of vertices in such a way that edges meet only at vertices and every edge belongs to two faces, or two sides of the same face [2]. A map is said to be *regular* if its automorphism group is flag-transitive. The Platonic solids, the Kepler-Poinsot polyhedra, and the tiling of the torus by n^2 squares described above all provide examples of regular maps.

Regular maps describe topological surfaces, not skew polyhedra. To get a skew polyhedron from a regular map, we need to find a set of vertices in Euclidean space such that the symmetries permuting those vertices correspond to symmetries of the map. Faces and edges of the skew polyhedron are defined by connecting these vertices as dictated by the faces and edges of the regular map [6]. (Note that the faces and vertex figures of this skew polyhedron need not be planar.)

Petrie's skew polyhedron was formed by identifying opposite sides of a square to get a torus. The vertices on this torus must somehow be embedded in Euclidean space in such a way that the symmetries permuting those vertices match the symmetry group of $\{4,4\}_{n,0}$.

If our square has edge-length 2π, the parametrization $P_2 : (x,y) \mapsto (e^{ix}, e^{iy})$ sends it to a torus in $\mathbb{C}^2$, the Cartesian product of two circles. (The function $P_1 : t \mapsto e^{it}$ wraps a line segment of length 2π around a circle. The function P_2 wraps a square, the product of two line segments, around a torus, which is the product of two circles.) Each vertex of $\{4,4\}_{n,0}$ is sent to a point of the form $(e^{2\pi i(j/n)}, e^{2\pi i(k/n)})$ with $0 \leq j,k < n$.

The skew polyhedron described by connecting those vertices is regular because the tiling of the plane by squares of side length $2\pi/n$ is regular. Composing symmetries of the tiling with P_2, we see that the symmetries of our skew polyhedron correspond to automorphisms of $\{4,4\}_{n,0}$.

Another regular map of squares on the surface of a torus is $\{4,4\}_{n,n}$. This is obtained from the regular tiling the plane by unit squares by identifying opposite edges of the square with vertices at $(n,0), (0,n), (-n,0)$ and $(0,-n)$. We can again use P_2 and an appropriately subdivided square of side length 2π to get a skew polyhedron corresponding to the regular map $\{4,4\}_{n,n}$ for each n.

As we have seen, there are several ways to regularly tile a torus with squares. We can

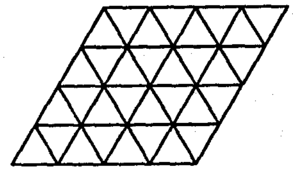

Figure 1: Rhombus tiled by triangles.

also tile tori with triangles and hexagons.

The regular maps of type $\{3,6\}_{b,0}$ are formed by identifying opposite sides of a rhombus tiled by $2b^2$ equilateral triangles. Identifying opposite sides of the rhombus in Figure 1 gives us the regular map $\{3,6\}_{4,0}$.

To make a torus out of a square of squares, we could roll the squares up into a tube then roll the tube into a torus. If we roll our rhombus into a tube in the same way, either the opposite sides of the rhombus do not meet exactly or the equilateral triangular faces get bent out of shape. We avoid these difficulties by embedding the rhombus in a cube and "rolling up" the cube.

Figure 2 shows a rhombus cut in half and embedded in a cube. As before, a parametric equation is used to identify opposite faces of the cube. When the top and bottom are identified, the two triangles shown will connect to form a rhombus. Identifying the other pairs of opposite faces will bring opposite sides of the rhombus together to form a torus.

The parametric equation $P_3 : (x,y,z) \mapsto (e^{ix}, e^{iy}, e^{iz})$ sends a cube with edge length 2π to the product of three circles in $\mathbb{C}^3$, identifying opposite faces. The vertices of our subdivided rhombus are sent to the vertices a skew polyhedron of type $\{3,6\}_{4,0}$.

The same techniques used to construct regular skew polyhedra of type $\{3,6\}_{b,0}$ can be applied to find polyhedra corresponding to the dual map $\{6,3\}_{b,0}$. Similar techniques give skew polyhedra corresponding to regular maps of type $\{3,6\}_{b,b}$ and $\{6,3\}_{b,b}$.

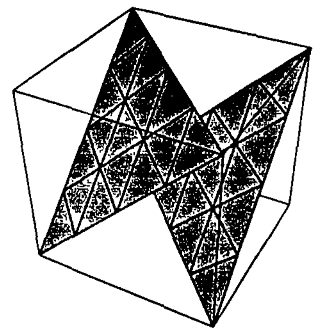

Figure 2: Rhombus embedded in a cube.

We can construct six infinite families of regular skew polyhedra by using the parametric equations P_2 and P_3 to identify opposite edges of squares and rhombi. There are infinitely many more regular maps to study; our search for regular skew polyhedra has hardly begun.

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