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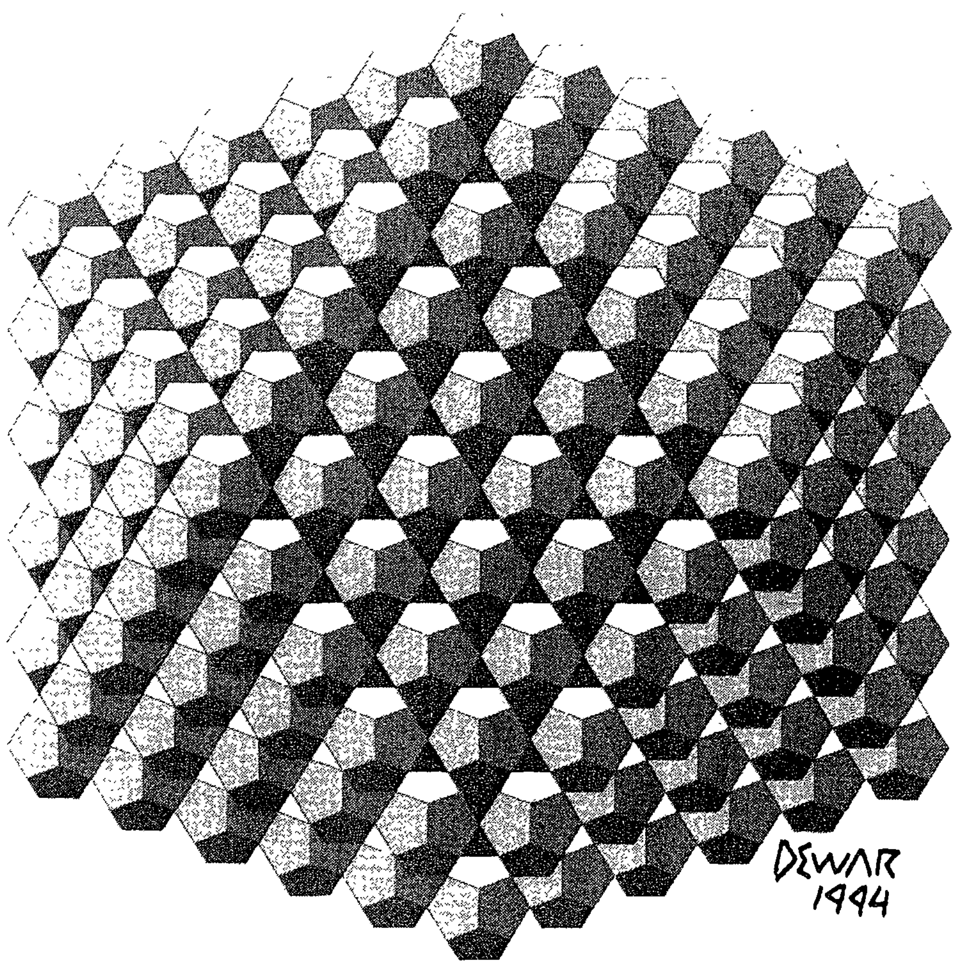

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SYMMETRY AND IRREVERSIBILITY IN PAUL HINDEMITH'S *LUDUS TONALIS*

(LECTURE-RECITAL)

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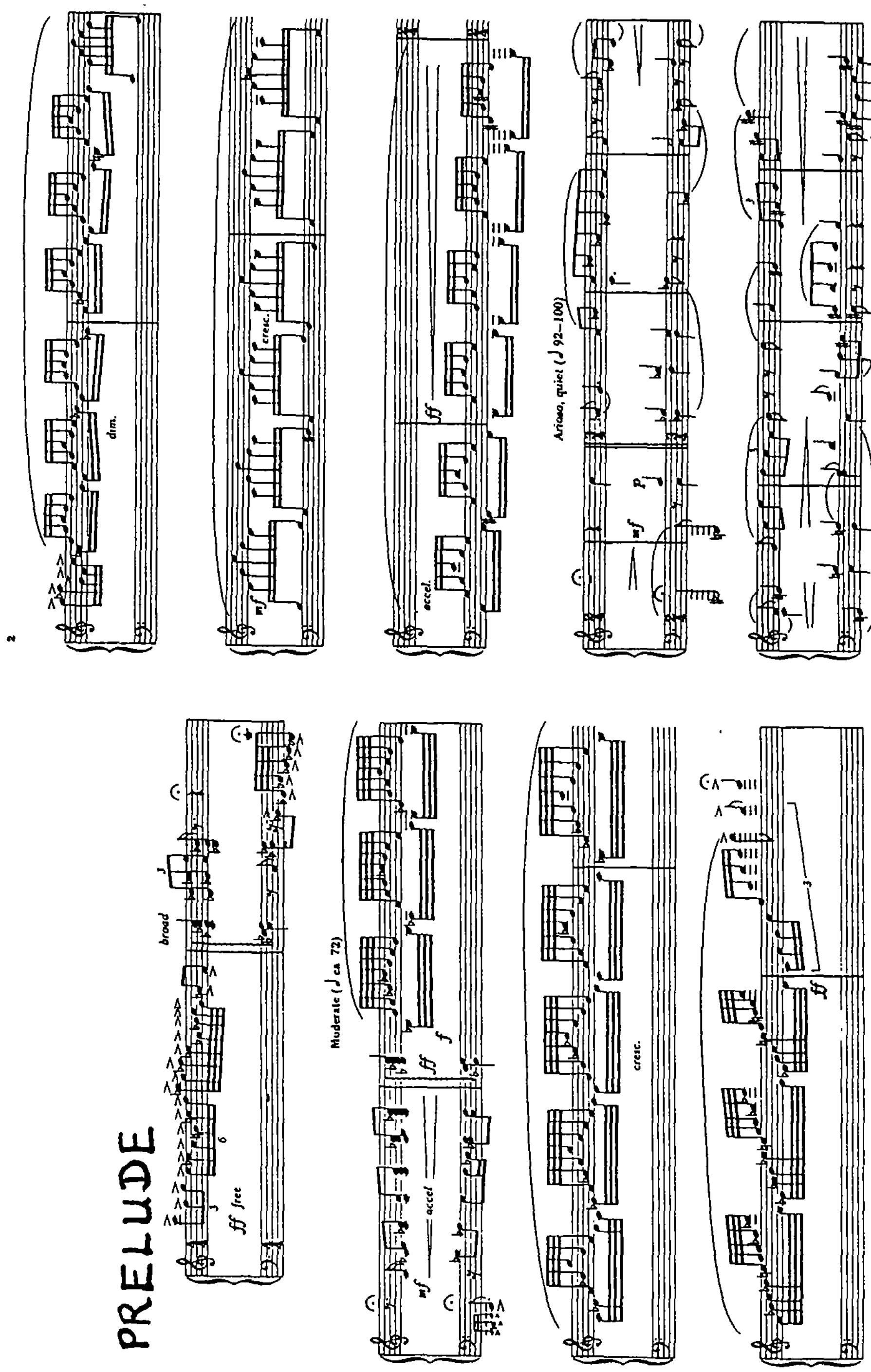
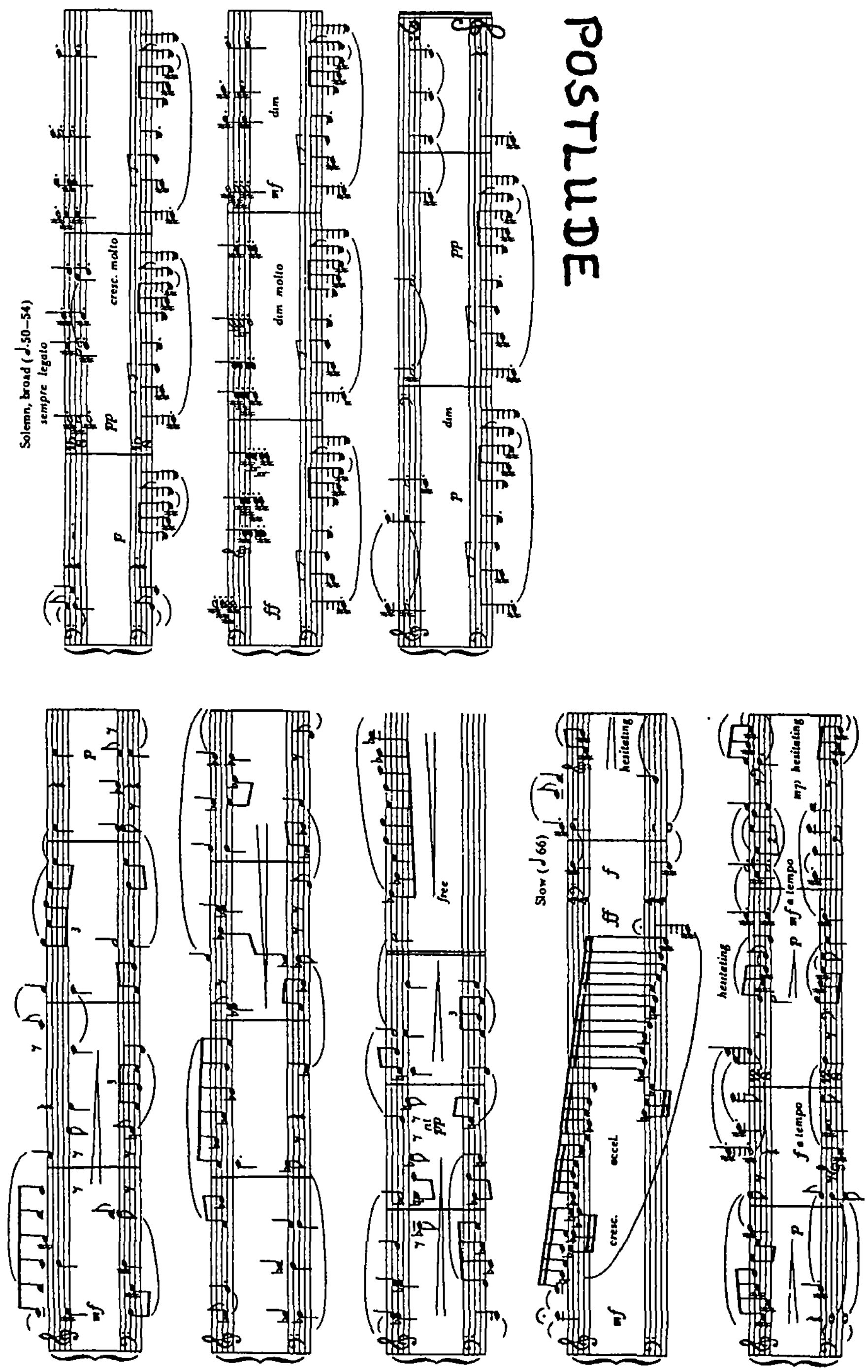
The title LUDUS TONALIS could be translated as "Tonal game". However, such a rendition gives only part of the meaning implied in the Latin wording. The term ludus (from Latin *ludere* = to play) can refer to three different scenarios: the playing of an instrument, the playing of a drama on stage, and the playing of games. (Historically, the word *ludus* often described medieval liturgical dramas.) Hindemith probably had all three meanings in mind when he chose this particular title: the work contains what appears to be an almost complete array of keyboard techniques and performance 'colors'; the capturing characteristics of many of the fugues and interludes are suggestive of dramatic characters; and the entire cycle most certainly expresses wonderful fun—fun for the composer who wrote this significant work within only a few weeks' time, and fun for performers, especially for those who undertake to play the entire cycle.

The LUDUS TONALIS consists of twelve fugues which are linked by eleven interludes and wrapped by a praeludium and a postludium. This layout, with pieces on each of the twelve semitones, recalls several forerunners. However, the tonal organization of the fugues is neither chromatic (as in Bach's Well-Tempered Clavier) nor in fifths and their relative minors (as in Chopin's or Scryabine's 24 Préludes). Instead, Hindemith uses a tonal organization in which the succession of twelve pitches is determined by their continually lessening relationship to the central note C. Exploring this irreversible aspect of the work's layout, and discovering the patterns contained in it, will be a first step in approaching the cycle.

The framing pieces, "Praeludium" and "Postludium" respectively, act as mediators between the contrasting aspects of irreversibility and symmetry. The Praeludium is built on two contrasting central notes: C (bars 1-32) and F# (bars 34-47). This piece thus anticipates, as it were, the entire tonal argument of the composition in contraction; it leaves us where the fugues will eventually leave us—at the tritone. Having launched the LUDUS TONALIS with a piece so fraught with allusions to the main body of the work, the question arises—and must have arisen to Hindemith—what kind of *finale* would be a match, rounding off the cycle in a meaningful way. Hindemith's solution is ingenious; he composed the Postludium as a special kind of retrograde inversion of the Praeludium: one in which the page can literally be turned upside down (see the music excerpt on the following pages) and read backwards! While this may seem as a fancy game, it constitutes in fact one of the most haunting compositional tasks—quite certainly a good reason why, since Bach's Art of the Fugue, no work of similar dimensions has been written in this technique.

The twelve fugues and eleven preludes framed by these remarkable examples of musico-visual symmetry are arranged in such a way as to establish various kinds of intricately mirroring patterns, all the while attentive to the other aspects of the title word "ludus." Of the many personas put on stage in this "play", only a few can be mentioned here. In the first fugue, each of the three subjects featured presents a distinct character which has considerable impact on its surroundings. Subject 1 appears as serene and composed; it envelopes itself with very harmonious chords. Subject 2 is sorrowful, expressing itself in a series of sighs followed by gradual appeasement; it is wrapped in 'unresolved' intervals and diminished chords. Subject 3 is aggressive; correspondingly it creates strong dissonances. When all three subjects finally meet, laments and aggression seem absorbed by the soothing quiet of subject 1. In another scenario, found in the fourth fugue, the outcome is quite different. A distinctly "male", somewhat rough-hewn first subject dominates the first section, while the second section exposes a very graceful, soft and fragile second subject. When these two intertwine in the third section, the gentle "female" turns into an angry bitch, causing some of the worst clashes in the entire cycle, clashes which only subside as "she" leaves the scene and "he" regains sole control. Other fugues features a seeker, a dancer, a Rococo courtier, a jester...


PRELUDE
ff free
broad
Moderato (♩ ca 72)
accel
cresc.
dim.
accel.
Arioso, quiet (♩ 92–100)


POSTLUDE
Solemn, broad (♩.50–54)
sempre legato
p
pp
cresc. molto
ff
dim molto
mf
dim
p
dim
pp
mf
free
pp
mf
cresc.
accel.
ff
f
Slow (♩66)
hesitating
f a tempo
p
mf a tempo
mp hesitating
hesitating

TOROIDAL SKEW POLYHEDRA

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The five Platonic solids – icosahedron, dodecahedron, octahedron, cube, and tetrahedron – have been used as tools, toys, and decoration since 2000 BC [3]. They are distinguished by the fact that each face is symmetric to every other face, and each vertex is symmetric to every other vertex. What other objects have these properties?

At the beginning of the seventeenth century, Johannes Kepler described two regular "hedgehogs" with five pointed stars for faces [4]. If we allow edges of faces and faces themselves to intersect we can describe a total of four new regular polyhedra, called the Kepler-Poinsot polyhedra.

Kepler and Poinsot found new polyhedra by allowing the edges of faces and vertex figures to intersect. In 1937, H. S. M. Coxeter described regular polyhedra in which the vertex figures were non-planar "skew polygons". He went on to introduce skew polyhedra; polyhedra in higher dimensional Euclidean spaces. One example of such an object, discovered by J. F. Petrie, can be described as follows. Identify opposite sides of a square of side length n to form a torus. If the n^2 unit squares covering this torus are considered as a subset of the faces of the double n-gonal prism $\{n\} \times \{n\}$, the resulting regular skew polyhedron is a realization of the map $\{4,4\}_{n,0}$ [1, 2].

In 1977 Branko Grünbaum described several regular polyhedra with skew faces, further broadening the definition of regular skew polyhedra [5]. Skew polyhedra are interesting because of their relationship to the beautiful Platonic solids, because the set of vertices of a regular skew polytope is a family of evenly spaced points on a high dimensional sphere, and because they provide us with hints about smooth embeddings of surfaces in Euclidean space.

In our search for regular skew polyhedra, we are seeking geometric objects whose vertices and faces are symmetric. If we omit the condition that our polyhedra must have geometric realizations, such objects are known as "regular maps".

A *map* is the decomposition of a two-dimensional surface (without boundary) into non-overlapping simply-connected regions (faces) by arcs (edges) joining pairs of vertices in such a way that edges meet only at vertices and every edge belongs to two faces, or two sides of the same face [2]. A map is said to be *regular* if its automorphism group is flag-transitive. The Platonic solids, the Kepler-Poinsot polyhedra, and the tiling of the torus by n^2 squares described above all provide examples of regular maps.

Regular maps describe topological surfaces, not skew polyhedra. To get a skew polyhedron from a regular map, we need to find a set of vertices in Euclidean space such that the symmetries permuting those vertices correspond to symmetries of the map. Faces and edges of the skew polyhedron are defined by connecting these vertices as dictated by the faces and edges of the regular map [6]. (Note that the faces and vertex figures of this skew polyhedron need not be planar.)

Petrie's skew polyhedron was formed by identifying opposite sides of a square to get a torus. The vertices on this torus must somehow be embedded in Euclidean space in such a way that the symmetries permuting those vertices match the symmetry group of $\{4,4\}_{n,0}$.

If our square has edge-length 2π, the parametrization $P_2 : (x,y) \mapsto (e^{ix}, e^{iy})$ sends it to a torus in $\mathbb{C}^2$, the Cartesian product of two circles. (The function $P_1 : t \mapsto e^{it}$ wraps a line segment of length 2π around a circle. The function P_2 wraps a square, the product of two line segments, around a torus, which is the product of two circles.) Each vertex of $\{4,4\}_{n,0}$ is sent to a point of the form $(e^{2\pi i(j/n)}, e^{2\pi i(k/n)})$ with $0 \le j,k < n$.

The skew polyhedron described by connecting those vertices is regular because the tiling of the plane by squares of side length $2\pi/n$ is regular. Composing symmetries of the tiling with P_2, we see that the symmetries of our skew polyhedron correspond to automorphisms of $\{4,4\}_{n,0}$.

Another regular map of squares on the surface of a torus is $\{4,4\}_{n,n}$. This is obtained from the regular tiling the plane by unit squares by identifying opposite edges of the square with vertices at $(n,0), (0,n), (-n,0)$ and $(0,-n)$. We can again use P_2 and an appropriately subdivided square of side length 2π to get a skew polyhedron corresponding to the regular map $\{4,4\}_{n,n}$ for each n.

As we have seen, there are several ways to regularly tile a torus with squares. We can

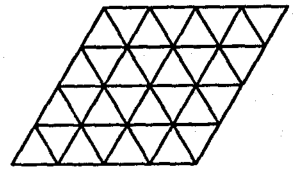

Figure 1: Rhombus tiled by triangles.

also tile tori with triangles and hexagons.

The regular maps of type $\{3,6\}_{b,0}$ are formed by identifying opposite sides of a rhombus tiled by $2b^2$ equilateral triangles. Identifying opposite sides of the rhombus in Figure 1 gives us the regular map $\{3,6\}_{4,0}$.

To make a torus out of a square of squares, we could roll the squares up into a tube then roll the tube into a torus. If we roll our rhombus into a tube in the same way, either the opposite sides of the rhombus do not meet exactly or the equilateral triangular faces get bent out of shape. We avoid these difficulties by embedding the rhombus in a cube and "rolling up" the cube.

Figure 2 shows a rhombus cut in half and embedded in a cube. As before, a parametric equation is used to identify opposite faces of the cube. When the top and bottom are identified, the two triangles shown will connect to form a rhombus. Identifying the other pairs of opposite faces will bring opposite sides of the rhombus together to form a torus.

The parametric equation $P_3 : (x, y, z) \mapsto (e^{ix}, e^{iy}, e^{iz})$ sends a cube with edge length 2π to the product of three circles in $\mathbb{C}^3$, identifying opposite faces. The vertices of our subdivided rhombus are sent to the vertices a skew polyhedron of type $\{3,6\}_{4,0}$.

The same techniques used to construct regular skew polyhedra of type $\{3,6\}_{b,0}$ can be applied to find polyhedra corresponding to the dual map $\{6,3\}_{b,0}$. Similar techniques give skew polyhedra corresponding to regular maps of type $\{3,6\}_{b,b}$ and $\{6,3\}_{b,b}$.

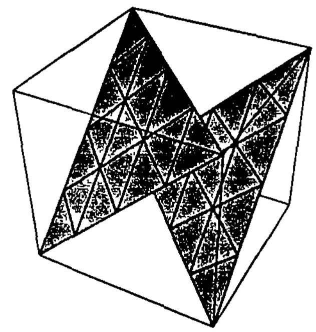

Figure 2: Rhombus embedded in a cube.

We can construct six infinite families of regular skew polyhedra by using the parametric equations P_2 and P_3 to identify opposite edges of squares and rhombi. There are infinitely many more regular maps to study; our search for regular skew polyhedra has hardly begun.

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