The Miura-ori opened out like a fan
MATHEMATICAL REMARKS ABOUT ORIGAMI BASES

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FOREWORD

This paper is based, with only minor modifications, adjunctions and suppressions, on an handwritten manuscript of 1982 which was sent at this time to some folders interested in mathematics. This led to correspondence with at least two of them: professors Kodi Husimi and James Sakoda. The first one has seemingly considered a particular case of what I call a 'perfect bird base' in his book, in Japanese, Origi

no Kikagaku (The Geometry of Origami, Tokyo: Nippon Hyoronsha, 1979). The second one has made use of perfect bird bases in his magazine Mac Origami and in his book Origami Flowers Arrangement (published by himself, 1992) and has devised a good approximate method to find the 'origin' of the perfect bird base.

1 INTRODUCTION

When we fold the traditional bird base (Orizuru Kiso, in Japanese) we find that it has agreeable properties, for instance the flaps are easy to move. But try with a rectangle: the result is less satisfactory. We shall examine the possibility of folding 'good' bird bases with polygons of various shapes. Afterwards we shall try a similar approach for the frog and the windmill bases.

2 DEFINITIONS

The polygons to be studied need not be convex. Let $A_1A_2\ldots A_n$ (Figure 1) be a polygon, and $O$ a point inside it such that the triangles $A_1OA_2$ and so on do not

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overlap each other (the polygon is said to be star-shaped from $O$; if it is concave, some points $O$ do not work). We call preliminary base the result of bringing $OA_1$, $OA_2$, ..., $OA_n$ together by mountain folds, and origin of the base the point $O$ (Figure 2). Of course, when flattening the model, valley folds must appear, say $OM_1$, $OM_2$, ... and we have (Figure 1) $A_1M_1O = M_1O_2A_2$ and so on. In the preliminary base, $O$ and the vertices $A_1, A_2, ...$ lie on a line, and the flaps like $OA_2M_1A_1O$ can be bookfolded with that line as a hinge. Now, let us make a reverse fold on each flap, with the condition that the creases pass through the corresponding vertices, for instance through $A_1$ and $A_2$ for the flap $OA_2M_1A_1O$. We shall call the result a bird base (Figure 3). See Figure 4 for the creases on the unfolded paper, and remark that the sides $A_iA_{i+1}$ of the polygon have been folded at points $E_i$ that differ, generally, from the $M_i$'s (for ease of notation we consider that $A_n+1 = A_1$ and so on). The bird base retains the above-mentioned property of the preliminary base, that $O$ and the $A_i$'s lie on the same line $\Delta$, which is an axis of rotation for the flaps like $OP_1A_1A_2O$. Now, let us say:

**Figures 1-3**

**Definition:** A bird base (Figure 5) is perfect if:

(a) the triangular flap $P_1A_2P_2$ can be bookfolded with $P_1P_2$ as a hinge, and so on for the other flaps.

(b) when $P_1A_2P_2$ has been folded "downward", $A_2$ is in a new position $A'$ which lies on $\Delta$ and so on.

(c) the flaps like $P_1A_2P_2$ consist everywhere of, exactly, two layers of paper, that is the triangles $A_2P_1E_2$ and $A_2P_2E_2$ of Figure 4 are adjacent without gap or overlap.
Of course, such properties are among the good ones of the standard bird base, some others being let aside because they are too specific of the square.
3 PERFECT BIRD BASES

Figure 6 shows the hidden reverse folds. By property (c), the points $A_2, E_1, E_2$ lie on the same line. Let $H_2$ be its intersection with $P_1P_2$. By property (a), $E_1$ and $E_2$ must be on the side of $P_1P_2$ which does not contain $A_2$. So we have:

$$P_1E_1A_2 \leq P_1H_2A_2 \quad (\text{triangle } P_1E_1H_2), \quad \text{and } P_2E_2A_2 \leq P_2H_2A_2,$$

hence

$$P_1E_1A_2 + P_2E_2A_2 \leq \pi; \quad (1)$$

with equality only if $E_1$ and $E_2$ are $H_2$. Adding up the $n$ inequalities like (1) and remarking that $P_iE_iA_i + P_iE_iA_{i+1} = \pi$, we get: $n \pi \leq n \pi$. That inequality being a true equality, (1) must be also an equality. Hence $E_1$ and $E_2$ coincide with $H_2$. Now, fold the flap $P_1A_2P_2$ downward, giving $P_1A_2A_2$. The side $A_2A_2$ of the polygon lies now at the position $A_1H_2A_2$. But it is now completely unfolded, and so $H_2$ lies on the (straight) line $A_1A$, that is on $\Delta$ by property (b). Of course, also, $P_1P_2$ is perpendicular to $A_2H_2$ at $H_2$.

Now, let us look at Figure 7 which shows the above results on a part of the unfolded paper. We see that $P_4$, for instance, is the incenter of the triangle $O \setminus A_1A_2$, that is the intersection of its inner bisectors. The circle inscribed in $O \setminus A_1A_2$ has $P_4$ for center and touches the sides of the triangle at $H_1, E_1, H_2$. Actually that is exactly the consequence of folding $O \setminus A_1A_2$ by a rabbit ear procedure, but here we have the important fact that the circles inscribed in $O \setminus A_1A_2$ and $O \setminus A_2A_3$ touch $O \setminus A_2$ at the same point $H_2$. We are now in a position to state:
Theorem: Given a polygon $A_1A_2...A_n$,
(a) there exists a perfect bird base with origin $O$ if and only if there exist positive numbers $r$ and $e_1, ..., e_n$ such that

$$e_i + e_{i+1} = A_iA_{i+1} \text{ for } 1 \leq i \leq n \text{ and } e_i + r = OA_i \text{ for } 1 \leq i \leq n$$  \hspace{1cm} (2)

(b) the conditions in (a) are equivalent to the following one:

$$OA_i - OA_{i+2} = A_iA_{i+1}A_j - A_{i+1}A_{i+2} \text{ for } 1 \leq i \leq n$$ \hspace{1cm} (3)

(c) the polygon has at most one perfect bird base.

Proof of (a): If the base with origin $O$ is perfect, Figure 7 shows that the circle inscribed in $OAr42$ touches the sides at $E_1, H_1, H_2$. So we have $OH_1 = OH_2 = OH_3 = ... = r$, and $A_1H_1 = A_1E_1, A_2E_1 = A_2H_2 = A_2E_2$. So, putting $A_iH_i = e_i$ for $1 \leq i \leq n$, we have the relations (2).

Reciprocally, if the relations (2) are satisfied, the three circles $(O; r), (A_1; e_1), (A_2; e_2)$ (where the first letter is the center and the second one is the radius) touch each other at points $H'_1$ on $OA_1, H'_2$ on $OA_2, E'_1$ on $A_1A_2$. It is easy to see that these points are on the circle inscribed in $OA_1A_2$, and so, we have the configuration for a perfect bird base as given in Figure 7.

Proof of (b): If the relations (2) hold, we have: $OA_i - OA_{i+2} = e_i - e_{i+2} = (e_i + e_i+1) - (e_{i+2} + e_{i+1}) = A_iA_{i+1}A_j - A_{i+1}A_{i+2}$, so relations (3) are satisfied. Reciprocally, suppose that (3) holds. Put $(OA_1 + OA_2 - A_1A_2)/2 = r_1, (OA_3 + OA_4 - A_2A_3)/2 = r_2$, and so on. Then $r_1 = r_2 = ... = r_n$ is an immediate consequence of (3). Let $r$ be the common value of the $r_i$ and put $e_i = OA_i - r$. Then, for instance: $e_1 + e_2 = OA_1 + OA_2 - 2r = OA_1 + OA_2 - (OA_1 + OA_2 - A_1A_2) = A_1A_2$. So $r$ and the $e_i$'s satisfy (2).

Proof of (c): Suppose that $O$ and another point $U$ are the origins of perfect bird bases. Let $\Sigma$ be the perpendicular bisector of $OU$. As $O$ and $U$ are in the interior of the polygon $A_1A_2...A_n$, there is at least one vertex that lies in each of the open (that is not containing $\Sigma$) half-planes ($\Sigma; O$) and ($\Sigma; U$). On the other hand, the relations (3) in the above theorem give: $OA_i - OA_{i+2} = A_iA_{i+1}A_j - A_{i+1}A_{i+2}$, so we have: $OA_i - UA_i = OA_{i+2} - UA_{i+2}$. Then $A_i$ and $A_{i+2}$ are on the same side of $\Sigma$. So, $A_1, A_3, A_5, ...$ are in one of the half-planes ($\Sigma; O$), ($\Sigma; U$) and $A_2, A_4, A_6, ...$ are in the other one. So $\Sigma$ crosses every side of the polygon. If this one is convex, this is impossible because $S$ crosses the border at two points in this case. If the polygon is concave, it must have a re-entrant angle, like $A_1A_2A_3$ in Figure 8. But then, $O$ and $U$ must lie in the angle $\Sigma$, because $OA_1, OA_2, OA_3, UA_4, UA_2, UA_3$ are in the interior of the polygon. Then $A_2, A_1$, or $A_2, A_3$ are on the same side of $\Sigma$, a contradiction (for brevity's sake, some details have been omitted).
So there is at most one perfect bird base for any polygon. We propose, if it exists, to call its origin the 'point of Loiseau' of the polygon (L'oiseau: French words for: the bird).

Corollary 1: For any triangle there is exactly one perfect bird base. Its origin can be constructed with ruler and compasses.

Proof: In Figure 9 we have $e_1 + e_2 = A_1A_2$, and so on. Then $E_1, E_2, E_3$ are the points where the circle inscribed in $A_1A_2A_3$ touches the sides. Describe the circle $(A_1; e_1)$, that is the circle with center $A_1$ and radius $e_1$, and in the same way, the circles $(A_2; e_2), (A_3; e_3)$. Then by the relations $e_i + r = OA_i, O$ must be the center of a circle that touches the three preceding ones. Therefore, $O$ can be obtained by elementary geometry. Here is one possible construction (deduced from inversions with centers $E_i$).

Let $G_1$ on the line $A_1A_2$ be such that $(A_1; A_2; E_1; G_1)$ is harmonic. Describe the circle $(G_1; G_1E_1)$. It meets the circle $(A_3; e_3)$ at $H_3$ within $A_1A_2A_3$. Similarly, construct $H_1, H_2$. Then $O$ is the point of intersection of the lines $A_1H_1, A_2H_2, A_3H_3$.

When the triangle is isosceles, $A_1A_2 = A_1A_3$ (Figure 10), it is easier to construct the point of Loiseau. Describe the circle with center $A_1$ and radius $|A_1A_2 - A_2A_3|$. It meets the median $A_1E_2$ at two points, $X$ and $Y$. Trace the perpendicular bisector either of $A_2X$ if $A_1A_2 > A_2A_3$, or of $AY$ if not. It meets $A_1E_2$ at $O$.

Corollary 2: The quadrangle $A_1A_2A_3A_4$ has a perfect bird base if and only if $A_1A_2 + A_3A_4 = A_2A_3 + A_4A_1$. Then the origin $O$ lies at the intersection of two
branches of hyperbolas. In the case where $A_1A_3$ is an axis of symmetry, $O$ is the point where the circle inscribed in $A_1A_2A_3$ touches $A_1A_3$.

Proof: If $O$ is the origin of a perfect bird base (Figure 11) we have $A_2A_1 - A_2A_3 = OA_1 - OA_3 = A_4A_1 - A_4A_3$. This implies $A_1A_2 + A_3A_4 = A_2A_3 + A_4A_1$. Reciprocally, if that condition holds, then $A_2$ and $A_4$ lie on the same branch of some hyperbola with focuses $A_1$, $A_3$. Similarly $A_1$ and $A_3$ lie on a branch of hyperbola with focuses $A_2$ and $A_4$. These two curves intersect at some point $O$ within the quadrangle. As $O$ satisfies the conditions (3) of the theorem, it is the origin of a perfect bird base.

In the case where $A_1A_3$ is an axis of symmetry (Figure 12), the hyperbola passing through $A_1$ and $A_3$ becomes the line $A_1A_3$, and then, $OA_1 - OA_3 = A_2A_1 - A_2A_3$ shows that $O$ is the point of contact of $A_1A_3$ with the circle inscribed in $A_1A_2A_3$.

Remark: If the quadrangle is convex the condition on the sides is equivalent to the condition that it is circumscribed to some circle.

Corollary 3: (a) If the polygon $A_1A_2...A_n$, with $n$ even, has a perfect bird base, we must have:

$$A_1A_2 - A_2A_3 + A_3A_4 - ... + A_{n-1}A_n - A_nA_1 = 0.$$  

(b) If the polygon $A_1A_2...A_n$, with $n$ odd, has a perfect bird base with origin $O$, then $O$ can be constructed with rules and compasses.

(c) Given a point $O$ it is easy to construct irregular polygons with $n$ sides having a perfect bird base with origin $O$. 
Proof: (a) If \( n \) is even, the system of equations \( e_i + e_{i+1} = A_iA_{i+1} \) \((1 \leq i \leq n)\) has solutions only if (4) is true.

(b) If \( n \) is odd, we can find the \( E_i \)'s by solving the preceding system for the \( e_i \)'s. After, the point \( O \) can be constructed almost as we did for the triangle.

(c) Describe a circle \( C \) with center \( O \) (Figure 13), then a circle \( C_1 \) externally tangent to \( C \), then a circle \( C_2 \) tangent to \( C \) and \( C_1 \), then a circle \( C_3 \) tangent to \( C \) and \( C_2 \), always externally, and so on; at last \( C_n \) tangent to \( C \), \( C_{n-1} \) and \( C_1 \). Join together the centers \( A_1, A_2, \ldots, A_n \) of the circles. We get a polygon having a perfect bird base with center \( O \).

4 FROG BASES

Let \( A_1A_2\ldots A_n \) be a polygon, and \( B_i \) a point on the side \( A_iA_{i+1} \) for \( 1 \leq i \leq n \). We shall say that a bird base for the polygon \( A_1B_1A_2\ldots A_nB_n \) is a frog base for the polygon \( A_1A_2\ldots A_n \). A frog base is said perfect if the corresponding bird base is perfect. We have the following

**Theorem:** If \( O \) is the origin of a perfect frog base for a polygon \( A_1A_2\ldots A_n \), then \( B_1 \) is the point where the circle inscribed in \( A_1OA_2 \) touches \( A_1A_2 \) and so on.

**Proof:** (Figure 14). The relation (3) applied to the associated perfect bird base gives \( B_1A_1 - B_1A_2 = OA_1 - OA_2 \), so that the circle inscribed in \( A_1OA_2 \) touches \( A_1A_2 \) at \( B_1 \).
Corollary: For a rhombus, the center of symmetry is the origin of a perfect frog base.

Proof: (Figure 15). With $O$ at the center, choose the $B_i$'s as above (for instance fold the perfect bird base of the rhombus, then the $B_i$'s are the $E_i$'s of Figure 7). Then by symmetry $OB_1 - OB_2 = A_2B_1 - A_2B_2 (= 0)$ so that $O$ is the origin of a perfect bird base for the octagon $A_1B_1A_2...B_4$, that is a perfect frog base for the rhombus.

5 A PROBLEM
The following property was found when trying to make irregular frog bases.

**Problem:** Let \( OAB \) be a triangle and \( M \) be an arbitrary point on \( AB \). Let \( I \) and \( J \) be the incenters of the triangles \( AOM \) and \( BOM \). Show that the circle with diameter \( IJ \) meets \( AB \) at \( M \) (of course) and at a second point, \( P \), which does not depend of \( M \) (Figure 16).

**Hint for a geometrical proof.** Project \( I \) and \( J \) orthogonally on \( AB \) at \( H \) and \( K \), compute \( AH \) and \( AK \) and remark that \( HK \) and \( PM \) have the same middle.

**Origami solution of the problem.** Mountain fold along \( OI \) and \( OJ \), then petal fold along \( IJ \), which gives Figure 17. As \( PB \) and \( PA \) are adjacent, \( JPI = \pi/2 \). So \( P \) lies on the circle with diameter \( IJ \). But \( PA + PB = AB \), and also \( PA - PB = OA - OB \), so the point \( P \) of \( AB \) does not depend of \( M \).

### 6 WINDMILL BASES

In current terminology of folders the windmill base made from a square is either the windmill itself, or the quadruple preliminary base obtained by squashing the four points of the windmill. Here we use the first meaning. Let \( A_1A_2...A_n \) be a convex polygon (Figure 18), \( O \) a point in its interior and \( M_i \) a point of the side
Let us valley fold the polygon in such a way that all the \( M_i \)'s come to \( O \). If we pinch the corners and fold them flat (Figure 19) we shall call the result a windmill base defined by \( O \) and the \( M_i \)'s. The main property is that the flap containing \( A_1 \) can be bookfolded with \( OD_1 \) as a hinge, and so on. Figure 18 shows that \( D_1D_2 \) is the perpendicular bisector of \( OM_1 \). The creases \( D_2M_1 \) and \( D_2M_2 \) correspond to the hinge of the flap \( A_2 \), as they coincide with \( D_2O \) when the base is folded. Last, when we have flattened the flap \( A_2 \), an extracrease has appeared, say \( D_2X_2 \), which is the perpendicular bisector of \( M_1M_2 \).

Now we shall say that the windmill base is perfect if, for \( 1 \leq i \leq n \), the line \( D_iX_i \) coincides with the line \( D_iA_i \). This amounts to saying that the flap \( A_iX_iD_iO \) is a triangle, or consists everywhere of two layers. We have the following

**Theorem:** Given a convex polygon \( A_1A_2...A_n \), the points \( O \) inside and \( M_i \) on the sides \( A_iA_{i+1} \), for \( 1 \leq i \leq n \), define a perfect windmill base if and only if:

1. there exist positive \( e_i \)'s such that \( e_i + e_{i+1} = A_iA_{i+1} \) and \( A_iM_{i+1} = A_iM_i = e_i \);
2. \( O \) lies outside every circle \( (A_i; e_i) \);
3. \( O \) lies inside every circle \( (M_iM_{i+1}M_j) \), for \( 1 \leq i, j \leq n; j \neq i; j \neq i+1 \), where \( (PQR) \) denotes the circle circumscribed to the triangle \( PQR \).

**Proof:** (a) The conditions are necessary. In the unfolded perfect windmill base (Figure 20), \( D_2A_2 \) is the perpendicular bisector of \( M_1M_2 \), so \( A_2M_1 \) and \( A_2M_2 \) have the same length, say \( e_2 \), and the \( e_i \)'s satisfy (1). Now \( A_2 \) and \( O \) must lie on opposite
sides of $D_1D_2$, so $A_2O > A_2M_1 = e_2$, and then conditions (2) are satisfied. Last, the polygon $D_1D_2...D_n$ must be convex with $O$ inside it. So $O$ and $D_i$ lie on the same side of $D_1D_2$ for $i \neq 1, i \neq 2$. So $M_1D_1 > D_iO$. But $D_1$ is the center of the circle ($M_{i-1}M_iO$). So $M_1$ lies outside this circle. But if we remark that $O$ and $M_1$ lie on the same side of $M_{i-1}M_i$ (opposite to $A_i$) this is equivalent to the fact that $O$ lies inside the circle ($M_{i-1}M_iM_1$). So conditions (3) are satisfied.

(b) Reciprocally, if the conditions (1), (2) and (3) are satisfied, then, by reversing the arguments above, we see that the perpendicular bisectors of $OM_1$ and $OM_2$ intersect at some $D_2$ on the inner bisector of $A_1A_2A_3$, and that the polygon $D_1D_2...D_n$ is convex and contains $O$. So, by valley folding along $D_1D_2...D_n$ we obtain a perfect windmill base.

**Corollary:** If a convex polygon is circumscribed to a circle, then it has infinitely many perfect windmill bases.

(Proof left to the reader).

**Remarks:** (a) the conditions (1) can be satisfied if and only if the lengths $A_iA_{i+1}$ satisfy some conditions easy to state (solve $e_i + e_{i+1} = A_iA_{i+1}$ and write that the $e_i$'s are positive). It is also equivalent to say that the polygon $A_1A_2...A_n$ can be deformed by modifying its angles but not its sides, so that it become circumscribed to some circle.

Coordinates: $A_1 (0, 14), A_2 (-16, 8), A_3 (-8, 0), A_4 (8, 0), A_5 (16, 8), O (12, 7)$

(b) when conditions (1) are satisfied, the region corresponding to conditions (2), that is the intersection of the inside of $A_1A_2...A_n$ with the outside of all the circles
(\(A_i; e_j\)) is not empty. Though rather intuitive, this fact is difficult to prove. Maybe it could be proved by a continuous deformation of the polygon in the way said just above. In any case, it follows easily from the proposition hereafter which can be proved by methods of Graph Theory.

**Proposition:** Consider a convex polygon \(A_1A_2...A_n\) and real numbers \(a_i\) associated with its vertices, such that \(A_iA_{i+1} \geq a_i + a_{i+1}\) for \(1 \leq i \leq n\) (with the convention that \(a_{n+1} = a_1\)). Then there exists at least one vertex \(A_i\) such that \(A_iA_j \geq a_i + a_j\) for all \(i \neq j\).

(c) However it is not always possible to satisfy both (2) and (3), or even (3) alone. Try for instance to fold a windmill base defined by the \(M_i\)'s and \(O\) of Figure 21. You will obtain interesting results, but not exactly as wanted.

**7 CONCLUSION**

Many geometrical problems arise when one tries to generalize the traditional origami bases. We have studied here some natural generalizations. Bases with irregular polygons or with polygons with many sides are probably of little use in Origami. However some may be amusing. For instance fold a kite base from a square, then fold the kite into a perfect bird base, then pull out the two corners of the square that had been folded first. With this 'kited bird base' you can fold a flapping bird with long neck, short tail, medium-sized wings and small legs.