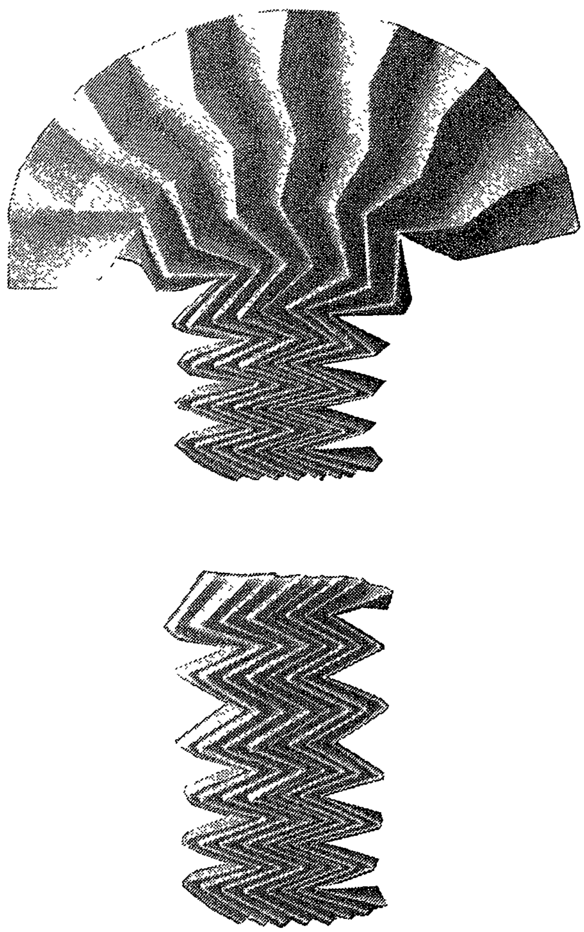
Symmetry: Culture and Science

Origami, 2

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

Volume 5, Number 2, 1994



The *Miura-ori*
opened out like a fan

Symmetry: Culture and Science
Vol. 5, No. 2, 153-165, 1994

MATHEMATICAL REMARKS ABOUT ORIGAMI BASES

Jacques Justin

LITP INSTITUT BLAISE PASCAL
4 place Jussieu, 75252 Paris CEDEX 05, France*

FOREWORD

This paper is based, with only minor modifications, adjunctions and suppressions, on an handwritten manuscript of 1982 which was sent at this time to some folders interested in mathematics. This led to correspondence with at least two of them: professors Kodi Husimi and James Sakoda. The first one has seemingly considered a particular case of what I call a 'perfect bird base' in his book, in Japanese, *Origami no Kikagaku* (The Geometry of Origami, Tokyo: Nippon Hyoronsha, 1979). The second one has made use of perfect bird bases in his magazine *Mac Origami* and in his book *Origami Flowers Arrangement* (published by himself, 1992) and has devised a good approximate method to find the 'origin' of the perfect bird base.

1 INTRODUCTION

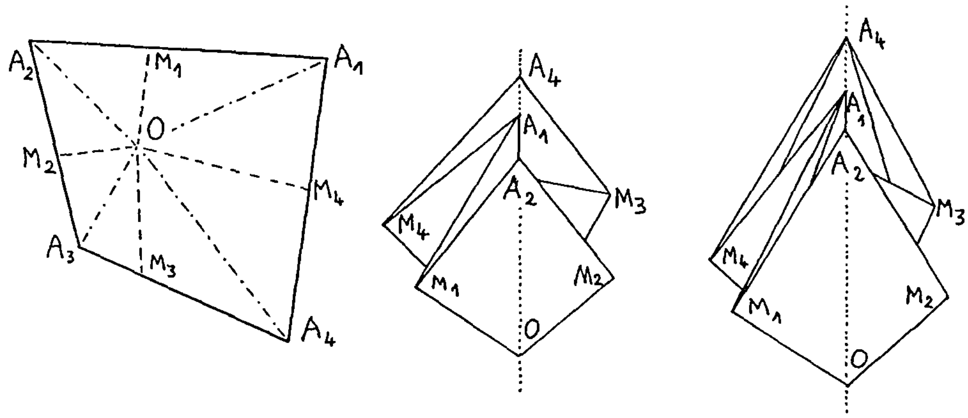
When we fold the traditional bird base (*Orizuru Kiso*, in Japanese) we find that it has agreeable properties, for instance the flaps are easy to move. But try with a rectangle: the result is less satisfactory. We shall examine the possibility of folding 'good' bird bases with polygons of various shapes. Afterwards we shall try a similar approach for the frog and the windmill bases.

2 DEFINITIONS

The polygons to be studied need not be convex. Let $A_1A_2...A_n$ (Figure 1) be a polygon, and O a point inside it such that the triangles A_1OA_2 and so on do not

* Mailing address: J. Justin, 19 rue de Bagneux, 92330 Sceaux, France.

overlap each other (the polygon is said to be star-shaped from O; if it is concave, some points O do not work). We call *preliminary base* the result of bringing OA_1, OA_2, ..., OA_n together by mountain folds, and *origin of the base* the point O (Figure 2). Of course, when flattening the model, valley folds must appear, say OM_1, OM_2, ... and we have (Figure 1) $\widehat{A_1OM_1} = \widehat{M_1OA_2}$ and so on. In the preliminary base, O and the vertices A_1, A_2, ... lie on a line, and the flaps like $OA_2M_1A_1O$ can be bookfolded with that line as a hinge. Now, let us make a reverse fold on each flap, with the condition that the creases pass through the corresponding vertices, for instance through A_1 and A_2 for the flap $OA_2M_1A_1O$. We shall call the result a *bird base* (Figure 3). See Figure 4 for the creases on the unfolded paper, and remark that the sides A_iA_{i+1} of the polygon have been folded at points E_i that differ, generally, from the M_i's (for ease of notation we consider that $A_{n+1} = A_1$ and so on). The bird base retains the above-mentioned property of the preliminary base, that O and the A_i's lie on the same line Δ, which is an axis of rotation for the flaps like $OP_1A_1A_2O$. Now, let us say:



Figures 1-3

Definition: A bird base (Figure 5) is perfect if:
(a) the triangular flap $P_1A_2P_2$ *can be bookfolded with* P_1P_2 *as a hinge, and so on for the other flaps.*
(b) when $P_1A_2P_2$ *has been folded 'downward',* A_2 *is in a new position* A' *which lies on* Δ*, and so on.*
(c) the flaps like $P_1A_2P_2$ *consist everywhere of, exactly, two layers of paper, that is the triangles* $A_2P_1E_1$ *and* $A_2P_2E_2$ *of Figure 4 are adjacent without gap or overlap.*

Of course, such properties are among the good ones of the standard bird base, some others being let aside because they are too specific of the square.

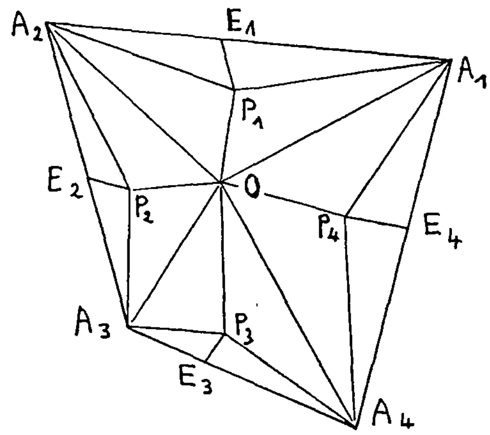
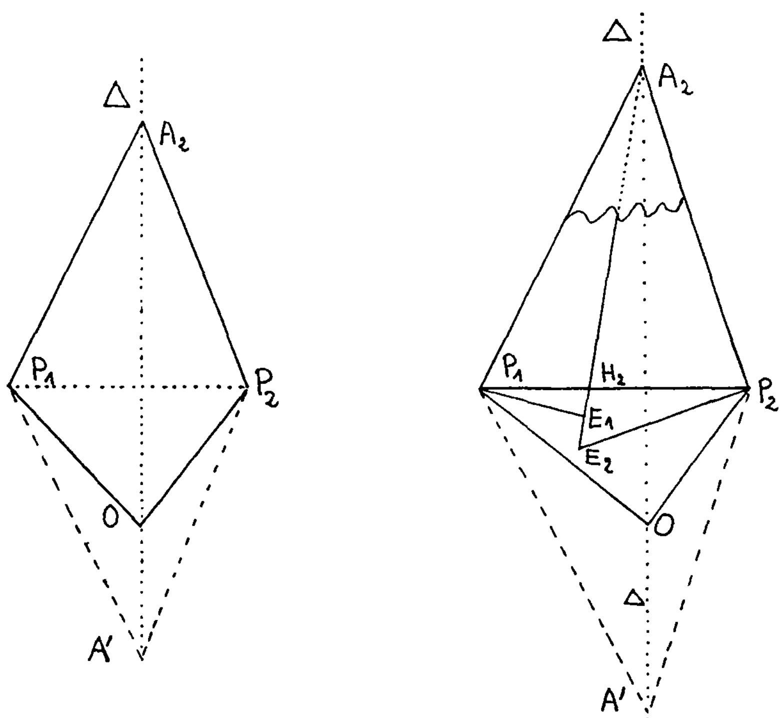


Figure 4



Figures 5-6

3 PERFECT BIRD BASES

Figure 6 shows the hidden reverse folds. By property (c), the points A_2, E_1, E_2 lie on the same line. Let H_2 be its intersection with P_1P_2. By property (a), E_1 and E_2 must be on the side of P_1P_2 which does not contain A_2. So we have: $P_1E_1A_2 \leq P_1H_2A_2$ (triangle $P_1E_1H_2$), and $P_2E_2A_2 \leq P_2H_2A_2$, hence

$$P_1E_1A_2 + P_2E_2A_2 \leq \pi ; \tag{1}$$

with equality only if E_1 and E_2 are H_2. Adding up the n inequalities like (1) and remarking that $P_iE_iA_i + P_iE_iA_{i+1} = \pi$, we get: $n\pi \leq n\pi$. That inequality being a true equality, (1) must be also an equality. Hence E_1 and E_2 coincide with H_2. Now, fold the flap $P_1A_2P_2$ downward, giving $P_1A'P_2$. The side A_1A_2 of the polygon lies now at the position $A_1H_2A_0$. But it is now completely unfolded, and so H_2 lies on the (straight) line A_1A', that is on Δ by property (b). Of course, also, P_1P_2 is perpendicular to A_2H_2 at H_2.

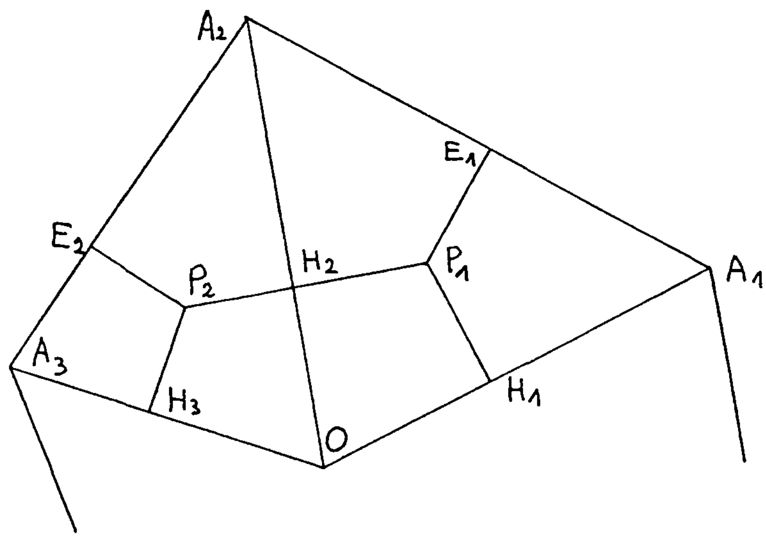


Figure 7

Now, let us look at Figure 7 which shows the above results on a part of the unfolded paper. We see that P_1, for instance, is the incenter of the triangle OA_1A_2, that is the intersection of its inner bisectors. The circle inscribed in OA_1A_2 has P_1 for center and touches the sides of the triangle at H_1, E_1, H_2. Actually that is exactly the consequence of folding OA_1A_2 by a rabbit ear procedure, but here we have the important fact that the circles inscribed in OA_1A_2 and OA_2A_3 touch OA_2 at the same point H_2. We are now in a position to state:

Theorem: *Given a polygon* $A_1A_2...A_n$,
(a) there exists a perfect bird base with origin O *if and only if there exist positive numbers* r *and* $e_1, ..., e_n$ *such that*

$$e_i + e_{i+1} = A_iA_{i+1} \text{ for } 1 \leq i \leq n \text{ and } e_i + r = OA_i \text{ for } 1 \leq i \leq n \quad (2)$$

(b) the conditions in (a) are equivalent to the following one:

$$OA_i - OA_{i+2} = A_{i+1}A_i - A_{i+1}A_{i+2} \text{ for } 1 \leq i \leq n \quad (3)$$

(c) the polygon has at most one perfect bird base.

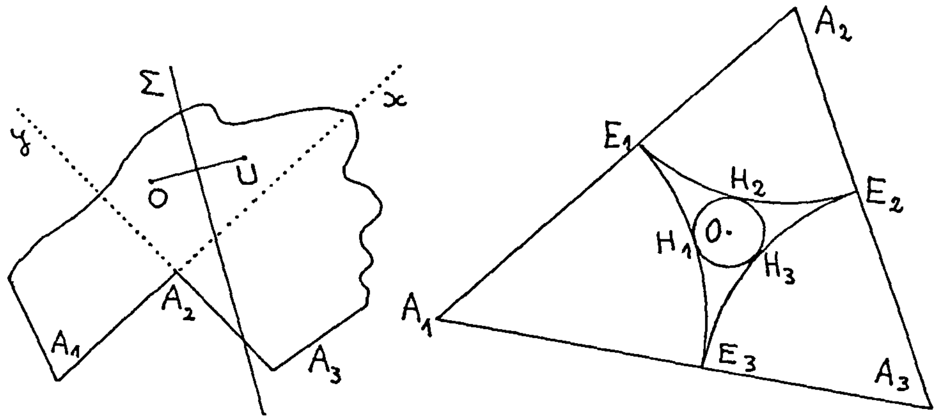
Proof of (a): If the base with origin O is perfect, Figure 7 shows that the circle inscribed in OA_1A_2 touches the sides at E_1, H_1, H_2. So we have $OH_1 = OH_2 = OH_3 = ... = r$, and $A_1H_1 = A_1E_1, A_2E_1 = A_2H_2 = A_2E_2$. So, putting $A_iH_i = e_i$ for $1 \leq i \leq n$, we have the relations (2).

Reciprocally, if the relations (2) are satisfied, the three circles $(O; r)$, $(A_1; e_1)$, $(A_2; e_2)$ (where the first letter is the center and the second one is the radius) touch each other at points H'_1 on OA_1, H'_2 on OA_2, E'_1 on A_1A_2. It is easy to see that these points are on the circle inscribed in OA_1A_2, and so, we have the configuration for a perfect bird base as given in Figure 7.

Proof of (b): If the relations (2) hold, we have: $OA_i - OA_{i+2} = e_i - e_{i+2} = (e_i + e_{i+1}) - (e_{i+2} + e_{i+1}) = A_{i+1}A_i - A_{i+1}A_{i+2}$, so relations (3) are satisfied. Reciprocally, suppose that (3) holds. Put $(OA_1 + OA_2 - A_1A_2)/2 = r_1$, $(OA_2 + OA_3 - A_2A_3)/2 = r_2$, and so on. Then $r_1 = r_2 = ... = r_n$ is an immediate consequence of (3). Let r be the common value of the r_i and put $e_i = OA_i - r$. Then, for instance: $e_1 + e_2 = OA_1 + OA_2 - 2r = OA_1 + OA_2 - (OA_1 + OA_2 - A_1A_2) = A_1A_2$. So r and the e_i's satisfy (2).

Proof of (c): Suppose that O and another point U are the origins of perfect bird bases. Let Σ be the perpendicular bisector of OU. As O and U are in the interior of the polygon $A_1A_2...A_n$, there is at least one vertice that lies in each of the open (that is not containing Σ) half-planes $(\Sigma; O)$ and $(\Sigma; U)$. On the other hand, the relations (3) in the above theorem give: $OA_i - OA_{i+2} = A_{i+1}A_i - A_{i+1}A_{i+2} = UA_i - UA_{i+2}$, so we have: $OA_i - UA_i = OA_{i+2} - UA_{i+2}$. Then A_i and A_{i+2} are on the same side of Σ. So, $A_1, A_3, A_5, ...$ are in one of the half-planes $(\Sigma; O)$, $(\Sigma; U)$ and $A_2, A_4, A_6, ...$ are in the other one. So Σ crosses every side of the polygon. If this one is convex, this is impossible because S crosses the border at two points in this case. If the polygon is concave, it must have a re-entrant angle, like $\widehat{A_1A_2A_3}$ in Figure 8. But then, O and U must lie in the angle $\widehat{xA_2y}$ because OA_1, OA_2, OA_3, UA_1, UA_2, UA_3, are in the interior of the polygon. Then A_2, A_1, or A_2, A_3 are on the same side of Σ, a contradiction (for brevity's sake, some details have been omitted).

So there is at most one perfect bird base for any polygon. We propose, if it exists, to call its origin the 'point of Loiseau' of the polygon (*L'oiseau*: French words for: the bird).



Figures 8-9

Corollary 1: *For any triangle there is exactly one perfect bird base. Its origin can be constructed with ruler and compasses.*

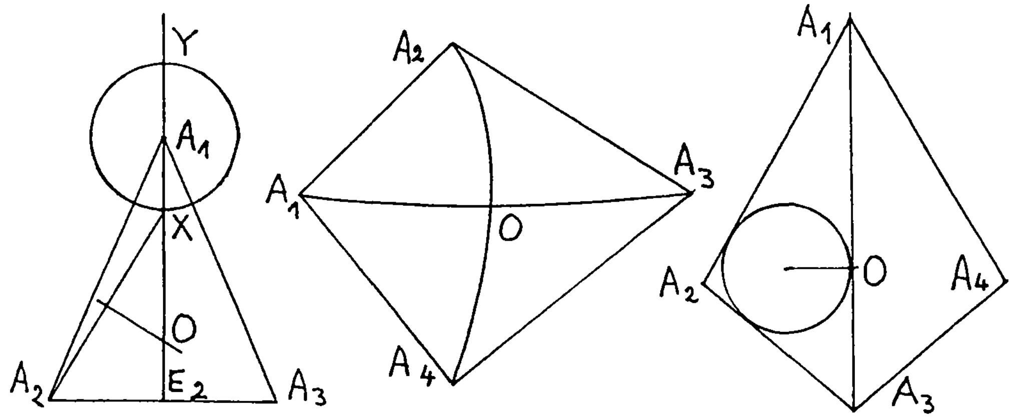
Proof: In Figure 9 we have $e_1 + e_2 = A_1A_2$, and so on. Then E_1, E_2, E_3 are the points where the circle inscribed in $A_1A_2A_3$ touches the sides. Describe the circle $(A_1;\, e_1)$, that is the circle with center A_1 and radius e_1, and in the same way, the circles $(A_2;\, e_2)$, $(A_3;\, e_3)$. Then by the relations $e_i + r = OA_i$, O must be the center of a circle that touches the three preceding ones. Therefore, O can be obtained by elementary geometry. Here is one possible construction (deduced from inversions with centers E_i).

Let G_1 on the line A_1A_2 be such that $(A_1;\, A_2;\, E_1;\, G_1)$ is harmonic. Describe the circle $(G_1;\, G_1E_1)$. It meets the circle $(A_3;\, e_3)$ at H_3 within $A_1A_2A_3$. Similarly, construct H_1, H_2. Then O is the point of intersection of the lines A_1H_1, A_2H_2, A_3H_3.

When the triangle is isosceles, $A_1A_2 = A_1A_3$ (Figure 10), it is easier to construct the point of Loiseau. Describe the circle with center A_1 and radius $|A_1A_2 - A_2A_3|$. It meets the median A_1E_2 at two points, X and Y. Trace the perpendicular bisector either of A_2X if $A_1A_2 > A_2A_3$, or of AY if not. It meets A_1E_2 at O.

Corollary 2: *The quadrangle* $A_1A_2A_3A_4$ *has a perfect bird base if and only if* $A_1A_2 + A_3A_4 = A_2A_3 + A_4A_1$. *Then the origin* O *lies at the intersection of two*

branches of hyperbolas. In the case where A_1A_3 is an axis of symmetry, O is the point where the circle inscribed in $A_1A_2A_3$ touches A_1A_3.



Figures 10-12

Proof: If O is the origin of a perfect bird base (Figure 11) we have $A_2A_1 - A_2A_3 = OA_1 - OA_3 = A_4A_1 - A_4A_3$. This implies $A_1A_2 + A_3A_4 = A_2A_3 + A_4A_1$. Reciprocally, if that condition holds, then A_2 and A_4 lie on the same branch of some hyperbola with focuses A_1, A_3. Similarly A_1 and A_3 lie on a branch of hyperbola with focuses A_2 and A_4. These two curves intersect at some point O within the quadrangle. As O satisfies the conditions (3) of the theorem, it is the origin of a perfect bird base.

In the case where A_1A_3 is an axis of symmetry (Figure 12), the hyperbola passing through A_1 and A_3 becomes the line A_1A_3, and then, $OA_1 - OA_3 = A_2A_1 - A_2A_3$ shows that O is the point of contact of A_1A_3 with the circle inscribed in $A_1A_2A_3$.

Remark: If the quadrangle is convex the condition on the sides is equivalent to the condition that it is circumscribed to some circle.

Corollary 3: *(a) If the polygon $A_1A_2...A_n$, with n even, has a perfect bird base, we must have:*

$$A_1A_2 - A_2A_3 + A_3A_4 - ... + A_{n-1} A_n - A_n A_1 = 0 \tag{4}$$

(b) If the polygon $A_1A_2...A_n$, with n odd, has a perfect bird base with origin O, then O can be constructed with rules and compasses.

(c) Given a point O it is easy to construct irregular polygons with n sides having a perfect bird base with origin O.

Proof: (a) If n is even, the system of equations $e_i + e_{i+1} = A_iA_{i+1}$ $(1 \leq i \leq n)$ has solutions only if (4) is true.

(b) If n is odd, we can find the E_i's by solving the preceding system for the e_i's. After, the point O can be constructed almost as we did for the triangle.

(c) Describe a circle C with center O (Figure 13), then a circle C_1 externally tangent to C, then a circle C_2 tangent to C and C_1, then a circle C_3 tangent to C and C_2, always externally, and so on; at last C_n tangent to C, C_{n-1} and C_1. Join together the centers $A_1, A_2, \ldots, A_n$ of the circles. We get a polygon having a perfect bird base with center O.

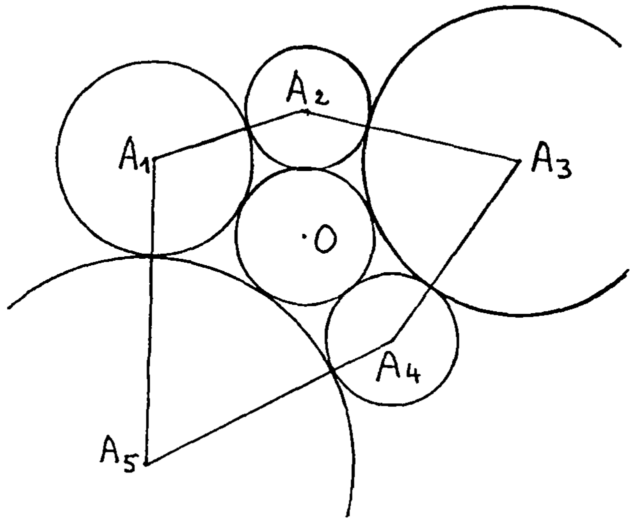


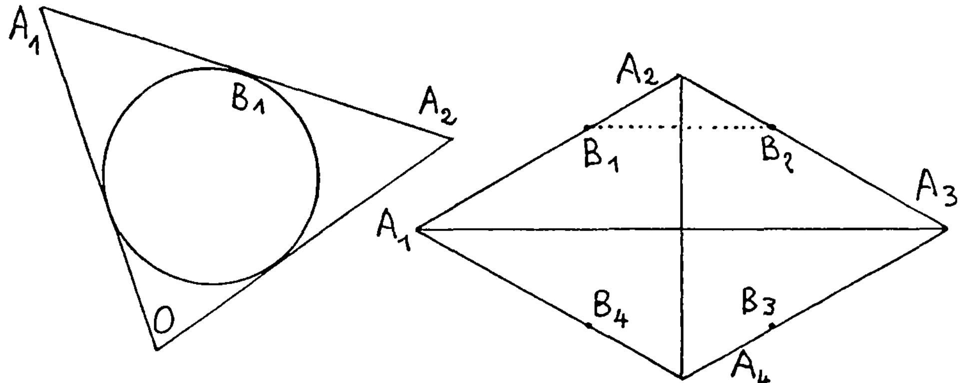
Figure 13

4 FROG BASES

Let $A_1A_2...A_n$ be a polygon, and B_i a point on the side A_iA_{i+1} for $1 \leq i \leq n$. We shall say that a bird base for the polygon $A_1B_1A_2...A_nB_n$ is a *frog base* for the polygon $A_1A_2...A_n$. A frog base is said *perfect* if the corresponding bird base is perfect. We have the following

Theorem: *If O is the origin of a perfect frog base for a polygon* $A_1A_2...A_n$, *then* B_1 *is the point where the circle inscribed in* A_1OA_2 *touches* A_1A_2, *and so on.*

Proof: (Figure 14). The relation (3) applied to the associated perfect bird base gives $B_1A_1 - B_1A_2 = OA_1 - OA_2$, so that the circle inscribed in A_1OA_2 touches A_1A_2 at B_1.

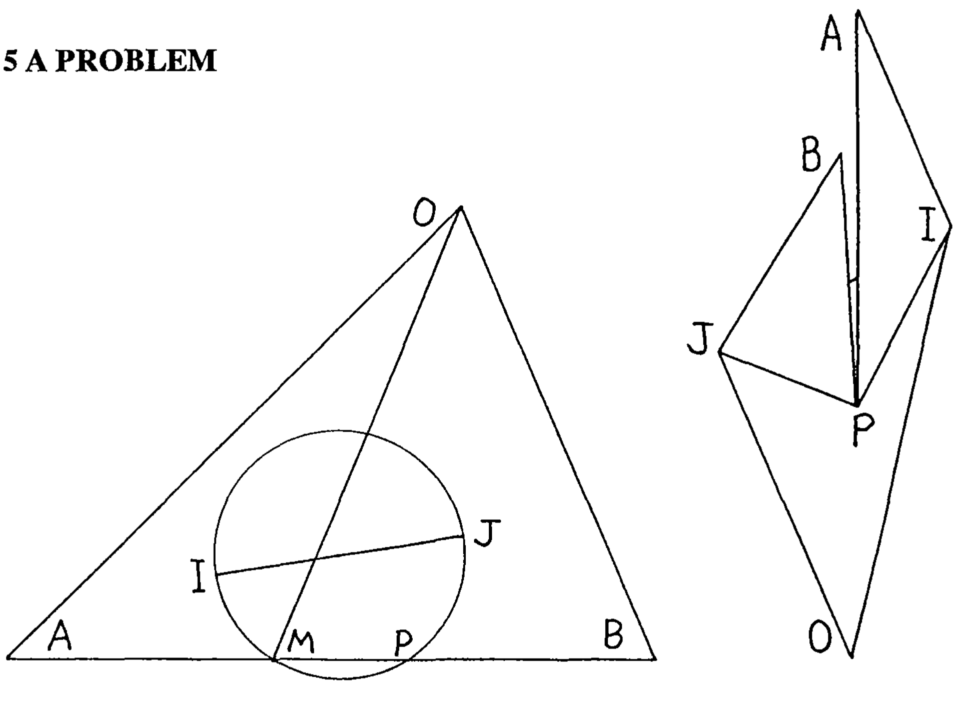


Figures 14-15

Corollary: *For a rhombus, the center of symmetry is the origin of a perfect frog base.*

Proof: (Figure 15). With O at the center, choose the B_i's as above (for instance fold the perfect bird base of the rhombus, then the B_i's are the E_i's of Figure 7). Then by symmetry $OB_1 - OB_2 = A_2B_1 - A_2B_2$ $(= 0)$ so that O is the origin of a perfect bird base for the octagon $A_1B_1A_2 ... B_4$, that is a perfect frog base for the rhombus.

5 A PROBLEM



Figures 16-17

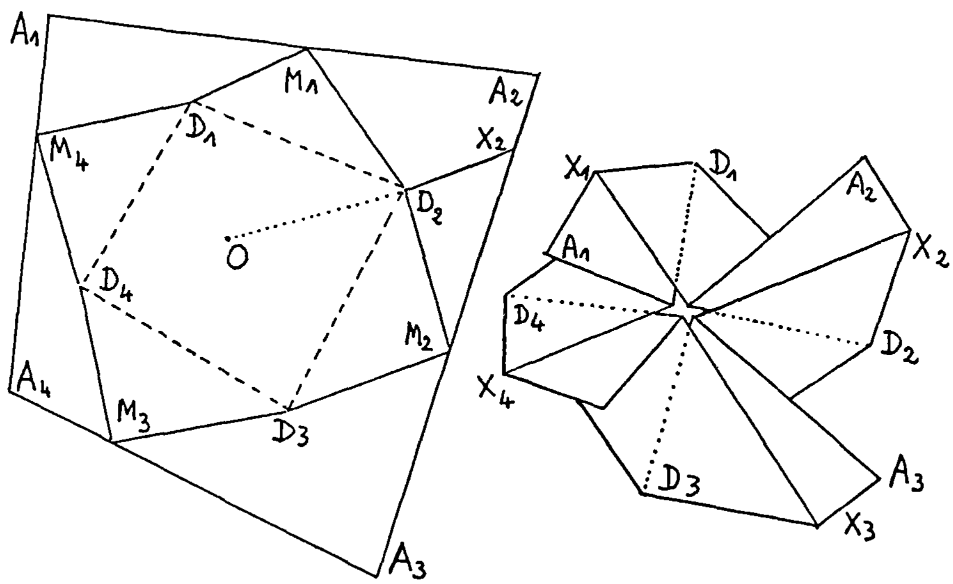
The following property was found when trying to make irregular frog bases.

Problem: Let OAB be a triangle and M be an arbitrary point on AB. Let I and J be the incenters of the triangles AOM and BOM. Show that the circle with diameter IJ meets AB at M (of course) and at a second point, P, which does not depend of M (Figure 16).

Hint for a geometrical proof. Project I and J orthogonally on AB at H and K, compute AH and AK and remark that HK and PM have the same middle.

Origami solution of the problem. Mountain fold along OI and OJ, then petal fold along IJ, which gives Figure 17. As PB and PA are adjacent, $\widehat{JPI} = \pi/2$. So P lies on the circle with diameter IJ. But $PA + PB = AB$, and also $PA - PB = OA - OB$, so the point P of AB does not depend of M.

6 WINDMILL BASES



Figures 18-19

In current terminology of folders the windmill base made from a square is either the windmill itself, or the quadruple preliminary base obtained by squashing the four points of the windmill. Here we use the first meaning. Let $A_1A_2...A_n$ be a convex polygon (Figure 18), O a point in its interior and M_i a point of the side

A_iA_{i+1}, for $1 \leq i \leq n$. Let us valley fold the polygon in such a way that all the M_i's come to O. If we pinch the corners and fold them flat (Figure 19) we shall call the result a *windmill base* defined by O and the M_i's. The main property is that the flap containing A_1 can be bookfolded with OD_1 as a hinge, and so on. Figure 18 shows that D_1D_2 is the perpendicular bisector of OM_1. The creases D_2M_1 and D_2M_2 correspond to the hinge of the flap A_2, as they coincide with D_2O when the base is folded. Last, when we have flattened the flap A_2, an extracrease has appeared, say D_2X_2, which is the perpendicular bisector of M_1M_2.

Now we shall say that the windmill base is *perfect* if, for $1 \leq i \leq n$, the line D_iX_i coincides with the line D_iA_i. This amounts to saying that the flap $A_iX_iD_iO$ is a triangle, or consists everywhere of two layers. We have the following

Theorem: *Given a convex polygon* $A_1A_2...A_n$, *the points* O *inside and* M_i *on the sides* A_iA_{i+1}, *for* $1 \leq i \leq n$, *define a perfect windmill base if and only if:*
(1) there exist positive e_i*'s such that* $e_i + e_{i+1} = A_iA_{i+1}$ *and* $A_iM_{i-1} = A_iM_i = e_i$;
(2) O *lies outside every circle* $(A_i; e_i)$;
(3) O *lies inside every circle* $(M_iM_{i+1}M_j)$, *for* $1 \leq i, j \leq n$; $j \neq i$; $j \neq i+1$, *where* (PQR) *denotes the circle circumscribed to the triangle* PQR.

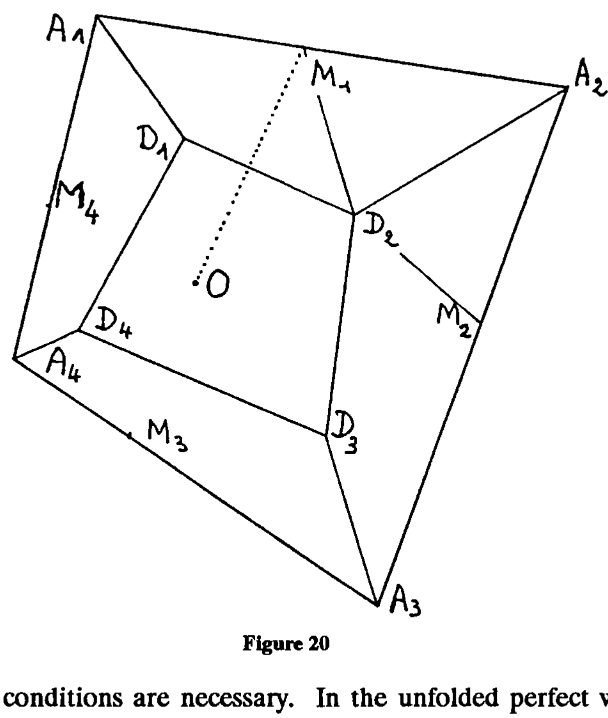


Figure 20

Proof: (a) The conditions are necessary. In the unfolded perfect windmill base (Figure 20), D_2A_2 is the perpendicular bisector of M_1M_2, so A_2M_1 and A_2M_2 have the same length, say e_2, and the e_i's satisfy (1). Now A_2 and O must lie on opposite sides of D_1D_2, so $A_2O > A_2M_1 = e_2$, and then conditions (2) are satisfied. Last, the polygon $D_1D_2...D_n$ must be convex with O inside it. So O and D_i lie on the same side of D_1D_2 for $i \neq 1, i \neq 2$. So $M_1 D_i > D_i O$. But D_i is the center of the circle $(M_{i-1}M_iO)$. So M_1 lies outside this circle. But if we remark that O and M_1 lie on the same side of $M_{i-1}M_i$ (opposite to A_i) this is equivalent to the fact that O lies inside the circle $(M_{i-1}M_iM_1)$. So conditions (3) are satisfied.

(b) Reciprocally, if the conditions (1), (2) and (3) are satisfied, then, by reversing the arguments above, we see that the perpendicular bisectors of OM_1 and OM_2 intersect at some D_2 on the inner bisector of $\widehat{A_1A_2A_3}$, and that the polygon $D_1D_2...D_n$ is convex and contains O. So, by valley folding along $D_1D_2...D_n$ we obtain a perfect windmill base.

Corollary: *If a convex polygon is circumscribed to a circle, then it has infinitely many perfect windmill bases.*
(Proof left to the reader).

Remarks: (a) the conditions (1) can be satisfied if and only if the lengths A_iA_{i+1} satisfy some conditions easy to state (solve $e_i + e_{i+1} = A_iA_{i+1}$ and write that the e_i's are positive). It is also equivalent to say that the polygon $A_1A_2...A_n$ can be deformed by modifying its angles but not its sides, so that it become circumscribed to some circle.

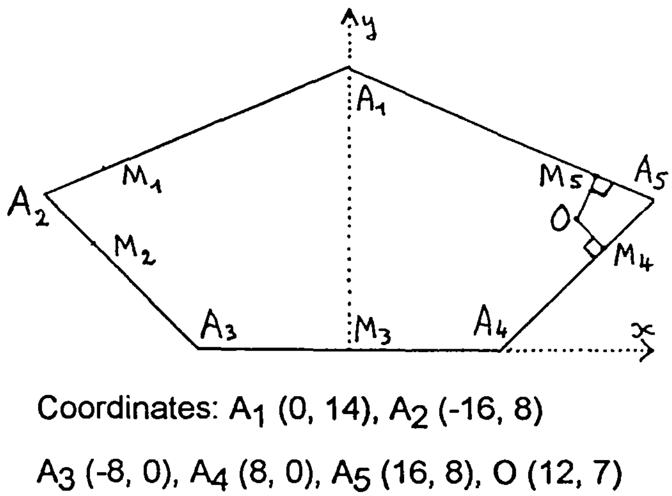


Figure 21

(b) when conditions (1) are satisfied, the region corresponding to conditions (2), that is the intersection of the inside of $A_1A_2...A_n$ with the outside of all the circles

$(A_i; e_i)$ *is not empty*. Though rather intuitive, this fact is difficult to prove. Maybe it could be proved by a continuous deformation of the polygon in the way said just above. In any case, it follows easily from the proposition hereafter which can be proved by methods of Graph Theory.

Proposition: *Consider a convex polygon* $A_1A_2...A_n$ *and real numbers* a_i *associated with its vertices, such that* $A_iA_{i+1} \geq a_i + a_{i+1}$ *for* $1 \leq i \leq n$ *(with the convention that* $a_{n+1} = a_1$*). Then there exists at least one vertice* A_i *such that* $A_iA_j \geq a_i + a_j$ *for all* $i \neq j$.

(c) However it is not always possible to satisfy both (2) and (3), or even (3) alone. Try for instance to fold a windmill base defined by the M_i's and O of Figure 21. You will obtain interesting results, but not exactly as wanted.

7 CONCLUSION

Many geometrical problems arise when one tries to generalize the traditional origami bases. We have studied here some natural generalizations. Bases with irregular polygons or with polygons with many sides are probably of little use in Origami. However some may be amusing. For instance fold a kite base from a square, then fold the kite into a perfect bird base, then pull out the two corners of the square that had been folded first. With this 'kited bird base' you can fold a flapping bird with long neck, short tail, medium-sized wings and small legs.