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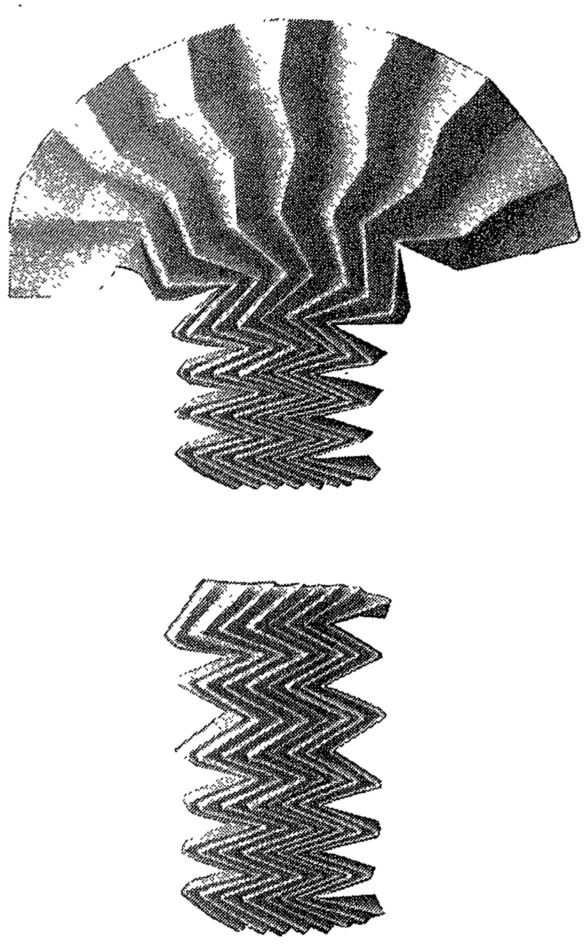

The *Miura-ori*
opened out like a fan

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FOLDS – THE BASIS OF ORIGAMI

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Abstract: *The physical and mathematical aspects of folds, which characterize the shape of artifacts made of a shin sheet of paper, are studied. The present aim is to clarify the reason why shapes of origami works are generally characterized by ragged surfaces consisting of convex and concave surfaces divided by sharp edges. The following conclusions are obtained: first, the ragged surface consisting of complex sharp folds is the natural characteristic deformation of a thin sheet of paper; second, the area of the spherical image for a complex of folds, which is the integral curvature of the domain, must be vanished; third, the above condition is satisfied only by compensation of positive and negative areas of spherical images corresponding to convex and concave areas on the origami surfaces.*

INTRODUCTION

The purpose of this paper is to study physical and mathematical aspects of a fold and of a complex of folds. Folds are found in abundance in natural creatures as in artifacts around us. However, the origin of this common geometric feature may be different from one thing to another. In this paper, the theoretical basis of folds for artifacts, which are fabricated basically through inextensional deformation from a thin flat sheet, is sought. Origami is, of course, the most typical one in this category (Miura, 1989a).

OBSERVATION OF A CRUSHED SHEET OF PAPER

Figure 1 is a photograph of a sheet of paper crushed arbitrarily by hand. It represents the characteristic shape of surfaces deformed from a sheet of paper. The most remarkable feature observed is its ragged shape consisting of many sharp ridges (folds). It can never be a smooth surface. Why is the shape so ragged?

The second feature of the surface is that it can be divided into convex and concave regions. And these two regions are usually divided by sharp ridges.

Although the present example cannot be an origami work, it represents a common and natural feature of an artifact made of a sheet of paper. The present author is especially interested in the shape around the vicinity of folds, which constitutes basic elements of origami. In the following section, its physical aspect (formation of folds) and mathematical aspect (geometry at the vicinity of folds) are investigated.

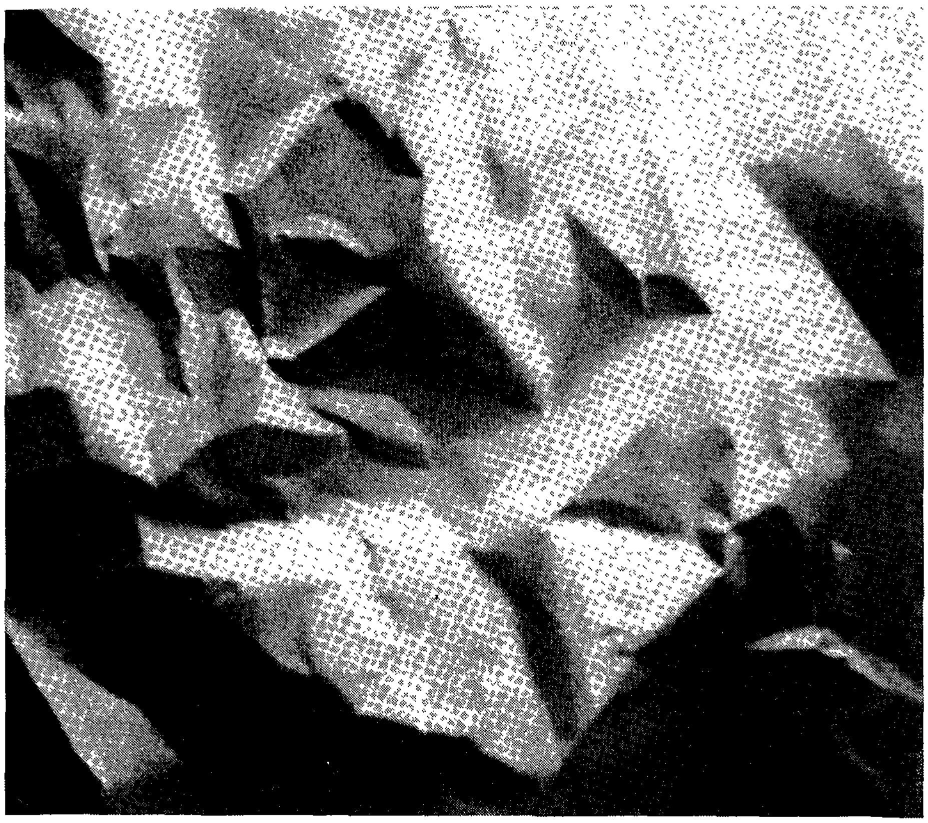

Figure 1: A crushed sheet of paper.

PHYSICAL ASPECTS OF FOLDS

Paper is not an abstract surface, it is a real material having physical properties such as elastic, plastic, and other behaviors.

A conventional method for describing deformations of a thin sheet type of material is to divide them into bending (out-of-plane of the sheet), stretching/contracting (in-plane), and shear (in-plane) deformations. Because of the thinness of sheets, other components of deformation are generally negligible. If the thickness decreases further, the strain energy of out-of-plane deformation decreases much faster than that of in-plane deformation. According to the minimum principle of strain energy, the sheet deforms in a way in which the total strain energy is minimized. Therefore, for a very thin sheet, the deformation should be dominated by bending deformation. This is the general view of physicists about a thin sheet of material.

The planar deformation of a piece of piano wire subject to terminal compression is considered now. Piano wire is a very thin elastic one-dimensional medium and thus, due to the strain energy theory, the deformation will be bending dominant and it is virtually inextensional along its axis. Thus it provides some information about the deformation of bending dominant media.

As a matter of fact, this problem of the elastic thin bar is famous, because this was one of the first examples of the variational principle which the great mathematician Leonhard Euler conceived.

Euler obtained the very beautiful and smooth curves as shown in Figure 2. Now these curves are called *Elastica*.

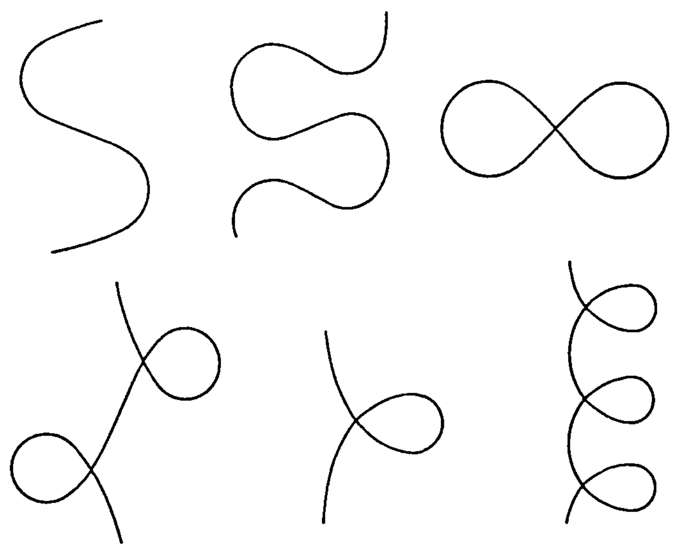

Figure 2: Euler's Elastica.

Since these smooth deformations are derived from the minimum principle, it can be said that these deformations are natural.

Now, the solution for a similar problem of Euler's for two-dimensional medium, that is, a thin sheet of paper is sought. This problem can be formulated as "to obtain the deformation for a thin sheet of material of an infinite extension subjected to uniform end shortening". Studies by Miura (1970) and Tanizawa-Miura (1978a) have revealed the solution. The ten numerical solutions for the problem are shown in Figure 3. The figures represent the out-of-plane deformations for a fundamental region by contour line maps. In these figures, the numerals represent the heights of deformations.

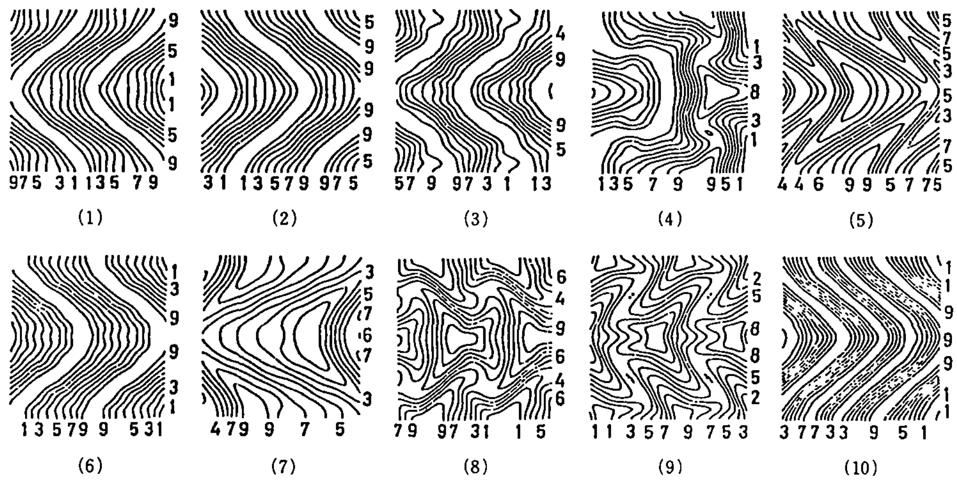

Figure 3: Out-of-plane deformations for thin plate subject to uniform end shortening; contour lines.

It can be observed that these deformations are characterized by ragged surfaces having sharp ridges (fold), which makes a sharp contrast with the smooth curves obtained in the one-dimensional medium.

When the least energy solution is sought among these solutions, and the thickness of the sheet is decreased to infinitesimal small, a polyhedral surface having a beautiful symmetry is obtained. It is composed of repetition of a fundamental region, which is further composed of four congruent parallelograms as shown in Figure 4.

As is well-known, the particular properties of this surface was applied to origami and maps by Miura (1978b and 1989b). Since this 'origami' is formed by natural forces and not by hand, it can be called 'natural origami'. It is interesting to note that this is the simplest origami of all.

Since this problem is an extension of a one-dimensional medium to a two-dimensional medium, the solution is by analogy considered to be *Plate Elastica*. It is surprising that the result actually obtained is quite different from our imagination of Elastica which is a symbol of 'smoothness'. The ragged surface consisting of complex sharp folds is the inherent and natural characteristic deformation for a thin sheet of paper.

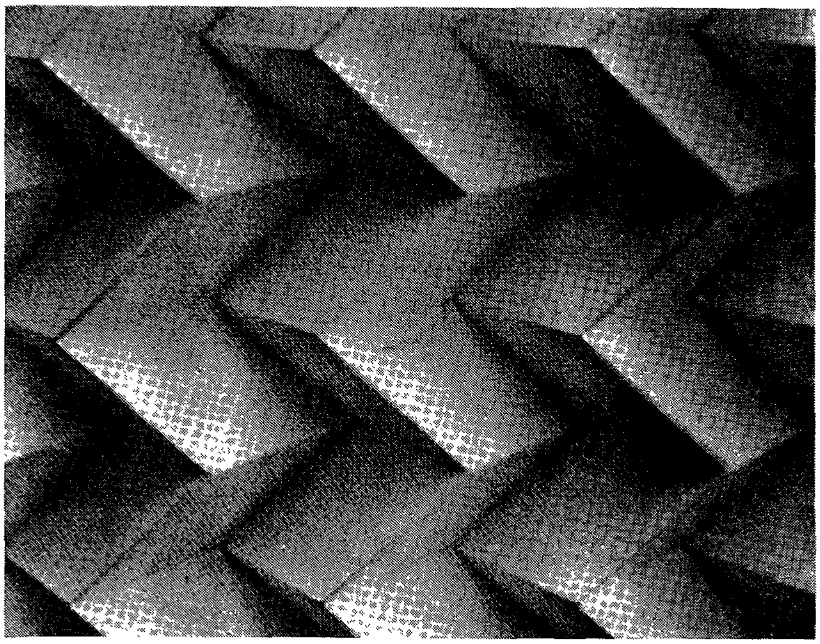

Figure 4: Polyhedral surface, *Plate Elastica*

THE BRIDGE BETWEEN PHYSICAL VS. MATHEMATICAL ASPECTS

The above study on folds indicates that, for a very thin sheet, the strain is concentrated in the vicinity of folds and there are small amounts of in-plane stretching and shear strains.

For the extremum case of vanishing thickness, in-plane strains vanish. This means that the middle plane of the sheet is absolutely inextensional. In other words, the resulting deformation is an isometric transfer of the initially flat surface of zero Gaussian curvature (the product of the principal curvatures). Because the Gaussian curvature is invariant under the isometric transfer, the zero Gaussian curvature is kept throughout the deformation.

$$K = 0 \tag{1}$$

Since a sheet of paper, which we used for origami works, is initially flat ($K=0$), and almost inextensional within its plane, origami works can be described mathematically as the isometric transformations of the initially flat ($K=0$) surface. In this sense, surfaces of origami works belong to a group of developable surfaces.

MATHEMATICAL EXPRESSION OF FOLDS

Now, the mathematical expression is sought over a domain enclosing a complex of folds, for example, a domain somewhere on the surface of Figure 1. Miura (1989c) applied the method of spherical images (Gaussian representation) successfully to express the complex. Therefore, a similar but slightly different approach is used in this study.

The integral curvature $G(U)$ is obtained by integrating the Gaussian curvature K, multiplied by the element of area dA over a domain U of the surface S, of the region (Fig. 5).

$$G(U) = \iint_{U} K dA. \qquad (2)$$

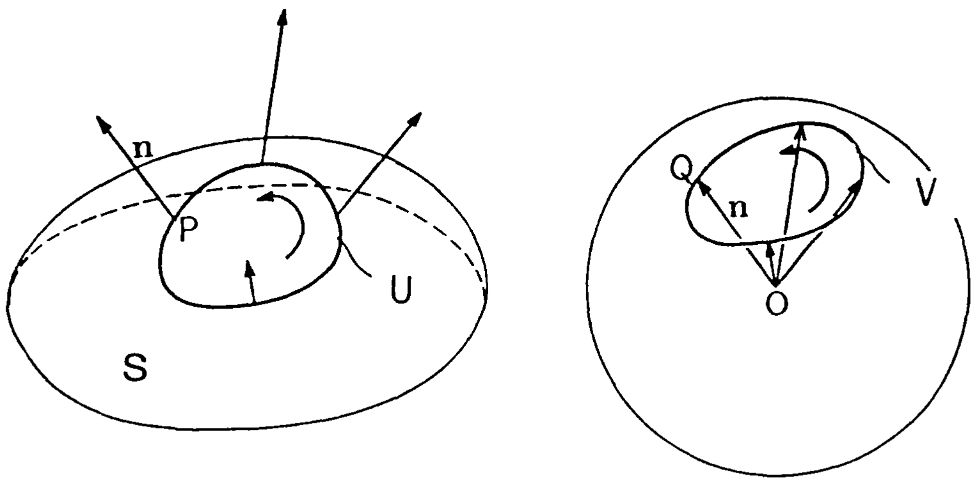

Figure 5: Spherical image of a surface.

One obtains an intuitive interpretation of the integral curvature G, and therefore of the Gaussian curvature K, by investigating the spherical image of a domain U in the surface S. Drawing the normal unit vector **n** of a point P of U from a fixed point, say the origin O, one obtains the spherical image. The ends of these vectors then describe a domain V in the unit sphere, which is the spherical image of U. Apart from the sign the area of the spherical image is then equal to the integral curvature G of U. Intuitively, it is obvious that this area is the larger the more sharply the surface S curves.

Because the surface of origami is generally ragged, as discussed before, the area of the spherical image of origami should be large. However, since $K = 0$ everywhere on the surface of origami, the integral curvature G should also be zero.

$$G(U) = \iint_U K dA = 0 \tag{3}$$

One has to explain this apparent contradiction. Also, there is another difficulty concerning the mathematical singularity at the fold. Because the above mathematical procedure is valid only for a smooth surface, one cannot define the spherical image for a fold.

Figure 6 shows a single-folded sheet of paper. The spherical image for arbitrary curves on the planes P and Q are shown in Figure 6 as p' and q'. Then consider an arbitrary curve PQ which connects P and Q by crossing the fold at R, which is apparently the singular point (angular point). It is natural to think that a fold can be taken as the infinitesimal limit of a bend with a finite curvature. The same can be said about their spherical images. Thus, we can make a reasonable definition of the spherical image of a fold as follows:

> "the spherical image of a fold R dividing a pair of flat surfaces P and Q is the great circular arc r' connecting the images p' and q' of these surfaces".

The definition also holds for a point on a curved fold.

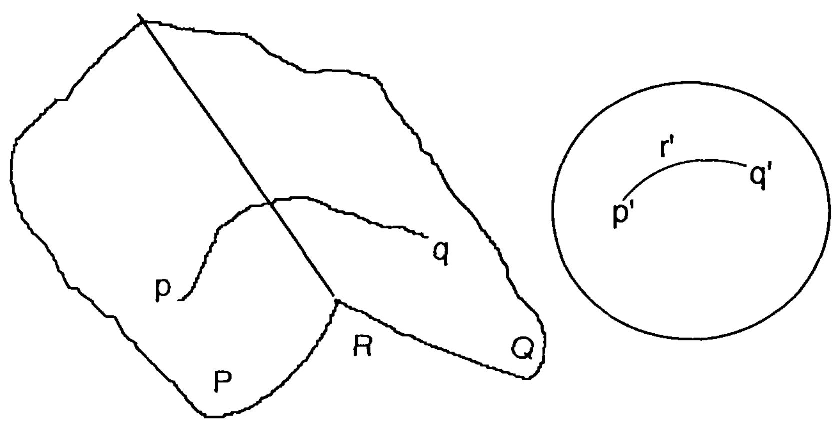

Figure 6: Spherical image of a fold.

SPHERICAL IMAGE OF ORIGAMI

Now we have a method to investigate the geometric properties of a fold and a complex of folds. In the following, this method is applied to study a vicinity of a node where three mountain folds and one valley fold meet. This is the basic and the simplest complex of folds in origami (Fig. 7).

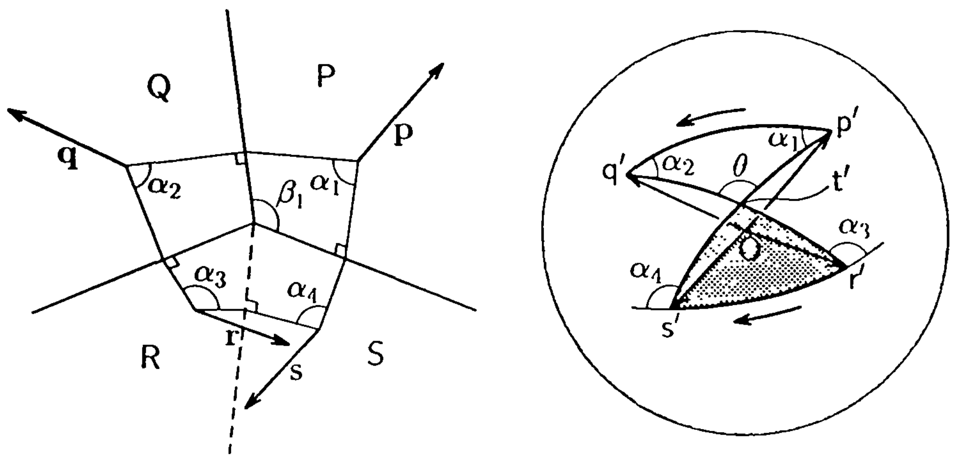

Figure 7: Spherical image of a complex of folds.

The complex of folds is described as:
– it consists of flat surfaces, P, Q, R and S,
– it consists of three mountain fold and one valley fold,
– P and Q form a convex region, R and S form a concave region, or it is divided into a convex and a concave region by $qr \sim sp$ mountain folds.

Here, the convexity of a small domain of a surface is defined so that the line segment joining any two points within the domain does not come out of the surface. It should be noted that this definition is slightly different from the conventional definition of the convexity of a surface.

The spherical image for the complex is described as:
– surfaces P, Q, R, and S are mapped onto points p, q, r, and s, respectively,
– folds pq, qr, rs, and sp are mapped onto great circular arcs pq, qr, rs, and sp,
– the spherical image is an 8-shaped pattern,
– the P, Q convex region is mapped onto the positive triangle tpq, while the R, S concave region is mapped onto the negative triangle trs, (based on the conventional mathematical premise).

It is observed that, for an arbitrary closed curve around the node, the convex region is mapped to the positive spherical triangle, while the concave region is mapped to the negative spherical triangle. Through mathematical manipulation, it is proved that both triangles have the identical absolute area. Therefore the total area of the spherical image vanishes, that is, Equation (3) is satisfied. This logic can be extended to more general cases including multiple folds and curved folds as well.

The following conclusion is obtained. For a complex of folds, the zero integral curvature G is only satisfied by compensation of positive and negative spherical images, which correspond to convex and concave surfaces of origami. This is why the zero of K and G is satisfied even for the ragged surfaces consisting of positive and negative surfaces divided by sharp edges.

CONCLUDING REMARKS

The aim of this study is to clarify the reason why the shapes of origami works are generally characterized by ragged surfaces consisting of convex and concave surfaces divided by sharp edges. The following conclusions are obtained.
(1) The ragged surface consisting of complex sharp folds is the natural characteristic deformation of a thin sheet of paper.
(2) The area of the spherical image of a complex of folds, which is the integral curvature of the domain, must vanished.
(3) The above condition is satisfied only by compensation of positive and negative areas of spherical images corresponding to convex and concave areas on the surface of origami. A more rigorous mathematical approach is necessary to extend the present study.

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