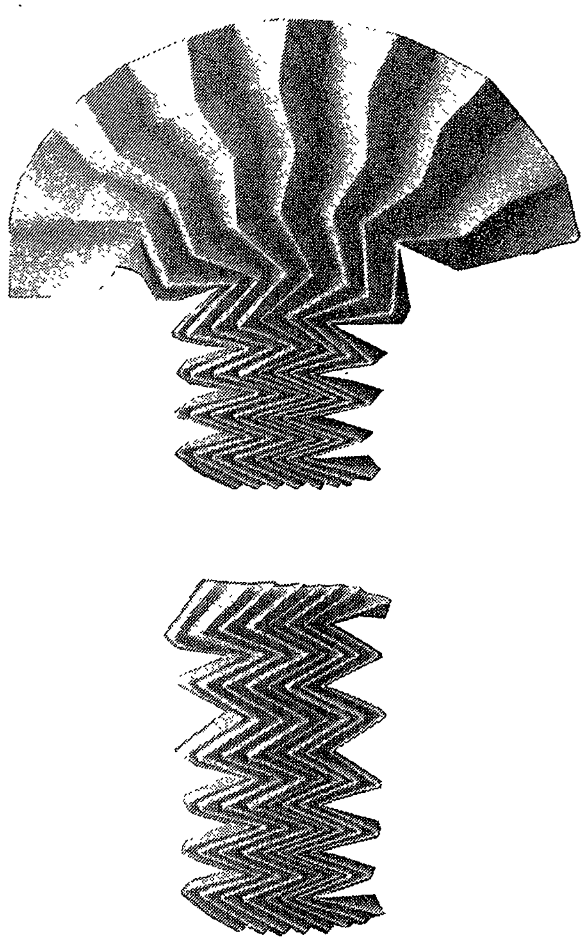
Symmetry: Culture and Science

Origami, 1

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

Volume 5, Number 1, 1994



The *Miura-ori*
opened out like a fan

Symmetry: Culture and Science
Vol. 5, No. 1, 69-83, 1994

DRAWING THE REGULAR HEPTAGON AND THE REGULAR NONAGON BY ORIGAMI (PAPER FOLDING)

Humiaki Huzita

INFN c/o Department of Physics, University of Padova,
Via Marzolo 8, I – 35131 Padova, Italy
E-mail: HUZITA@PADOVA.INFN.IT

Abstract: *The regular heptagon and the regular nonagon are good examples for showing paper folding ability, since neither can be made by Euclidean methods (using ruler and compass). For the convenience, starting with a square diagram, the concentric polygons are demonstrated here. As you see, this does not disturb the generality. Here the simplest method is described first, and then the explanation follows.*

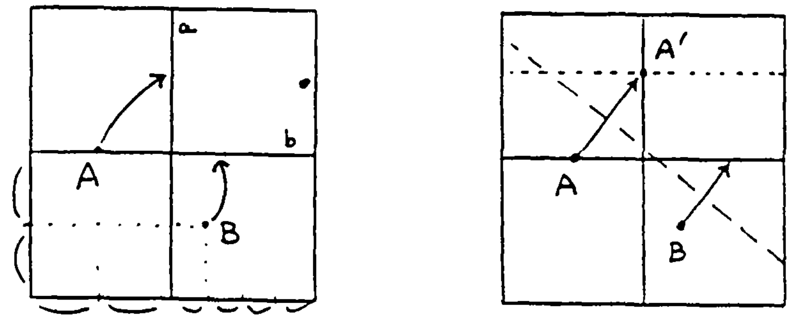
1 HEPTAGON

1.1 How to *draw* a heptagon using origami.

(1) Make two medians, a vertical and b horizontal. Take two points A and B of coordinates (1/4, 1/2) and (5/8, 1/4) respectively, using a rectangular system with (0, 0) at lower left corner and (1, 1) at upper right corner. Subsegments of lengths 1/2, 1/4, 1/8 . . ., etc., are easily made by folding, since each folding step makes a half. Now *fold* such that point A comes onto line a exactly and point B comes onto line b exactly at the same time.

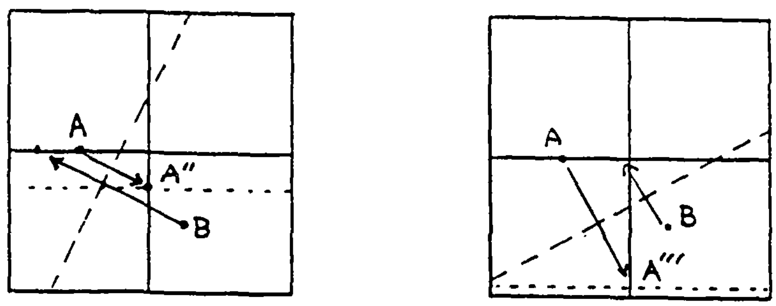
(2) The realisation is not unique: three are obtained. The three displacement positions of A on line a, say A', A'' and A''' correspond to shoulder height, hip height and foot respectively.

(3) Considering the upper-most point of line a in the square (the point with coordinate (1/2, 1)) as the head and the center, or (the point with coordinates (1/2, 1/2)) as the center of heptagon, the completion is easily made.



Folds Realization I

Figure 1



Realization II Realization III

Figure 2

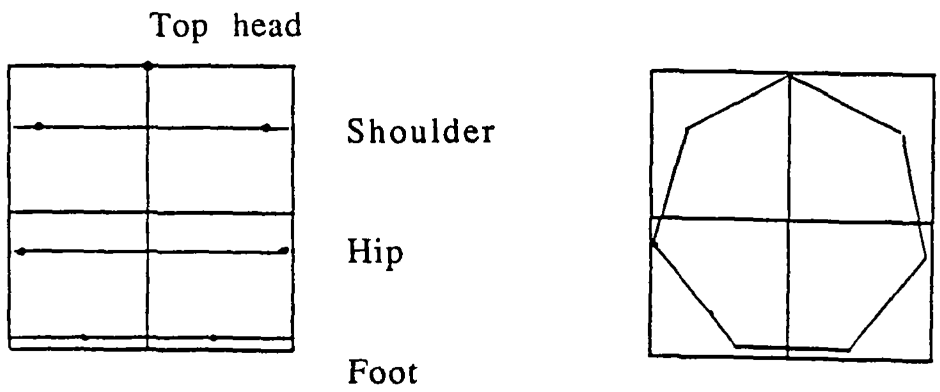


Figure 3

1.2 Explanation of the Method

Setting the origin of coordinates in the center of the heptagon, the seven points of the diagram can be represented in the complex plane,

$$1, \quad e^{(2\pi/7)i}, \quad e^{2(2\pi/7)i}, \quad e^{3(2\pi/7)i}, \quad e^{4(2\pi/7)i}, \quad e^{5(2\pi/7)i}, \quad e^{6(2\pi/7)i}$$

For simplicity, replacing $e^{(2\pi/7)i}$ by A, the problem is now to solve the equation:

$$A^6 + A^5 + A^4 + A^3 + A^2 + A + 1 = 0$$

Knowing the fact:

$$A^6 = 1/A, \; A^5 = 1/A^2, \; A^4 = 1/A^3, \; \text{etc.}$$

the equation can be rewritten as,

$$1/A + 1/A^2 + 1/A^3 + A^3 + A^2 + A + 1 = 0$$

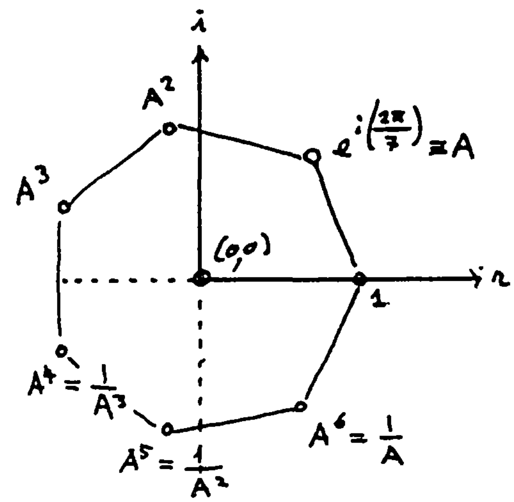


Figure 4

By simple relations:

$$(A + 1/A)^2 = A^2 + 2 + 1/A^2 \qquad \text{then} \qquad A^2 + 1/A^2 = (A + 1/A)^2,$$

and

$$(A + 1/A)^3 = A^3 + 3A + 3(1/A) + 1/A^3$$

then

$$A^3 + 1/A^3 = (A + 1/A)^3 - 3(A + 1/A)$$

replacing $A + 1/A$ by Z, the equation is reduced to a third order:

$$Z^3 + Z^2 - 2Z - 1 = 0.$$

The graphical meaning of Z is as shown in Figure 5.

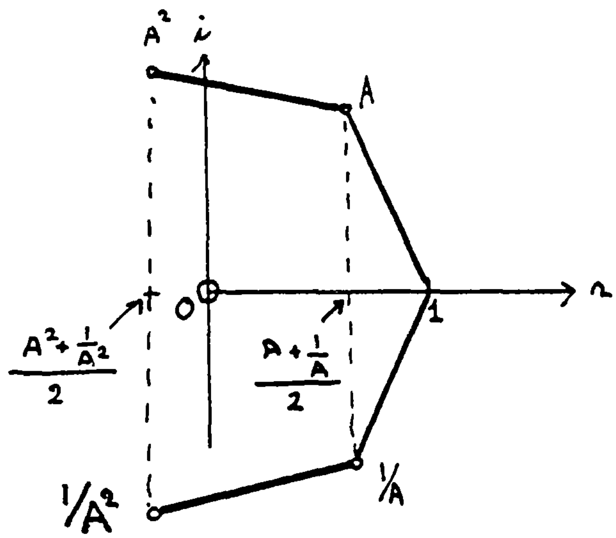


Figure 5

To solve this, let us remember the classic geometric representation for finding a zero of a polynomial. To solve an equation

$$ax^3 + bx^2 + cx + d = 0$$

first we have to make a grid of coefficients using the following rule: from the starting point O make a segment OA to the right horizontally of length a, then turn to the right at a right angle if coefficients a and b are of same sign, if they are not of same sign, then turn left. Similarly draw c and d making points B, C and D. Let us call these straight lines a, b, c and d. Now we start from the initial point and hit a billiard ball so that, when the ball hits a new line (wall), it bounces or deviates from its path at right angles towards the next line, finally hitting the target, in this case point D.

If the target is hit, then it is easy to understand the initial slope of the trajectory is an actual solution. All the triangle formed by the trajectory and the grid lines are similar and the following relations can be obtained:

$$x/a = y/(x + b) = z/(y + c)$$

Therefore

$$y = (x/a)(x + b)$$

then,

$$(x/a)[(x/a)(x + b) + c] = z = -d.$$

That is,

$$a(x/A)^3 + b(x/A)^2 + c(x/A) + d = 0$$

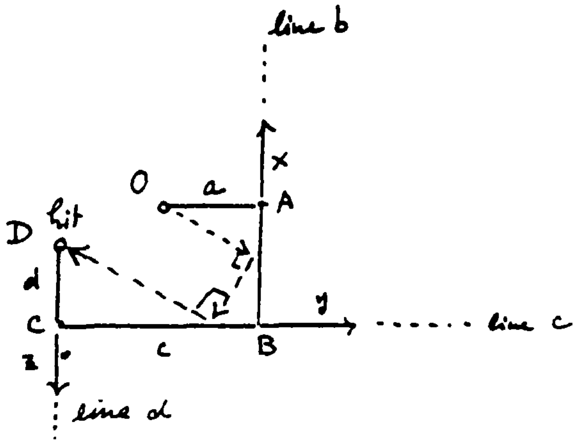


Figure 6

The complexity of coefficient signs can be reduced by changing the sign of definition of d to:

$$ax^3 + bx^2 + cx = d$$

Then the diagram becomes

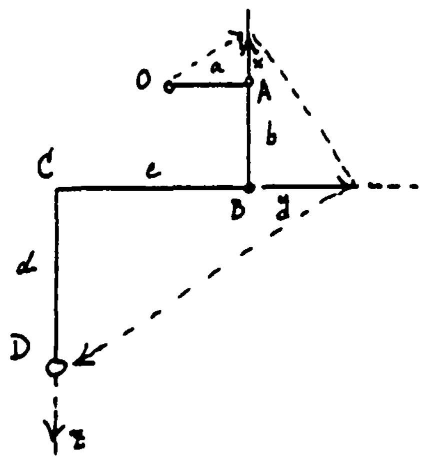


Figure 7

The problem is now how to hit the target and how to decide on the initial condition (that is, the slope of the direction of the ball destined to hit the target D). The origami solution to solve an equation of third order is easy following the Beloch method. Make a line parallel to line b at an equal distance from segment a but on the opposite side, call it line u. Now make a line parallel to line c at an equal distance from segment d but on the opposite side, call it line v.

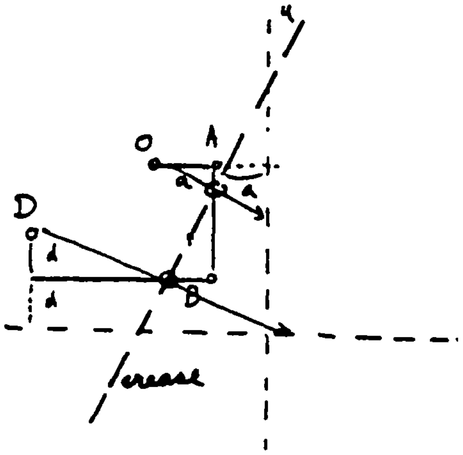


Figure 8a

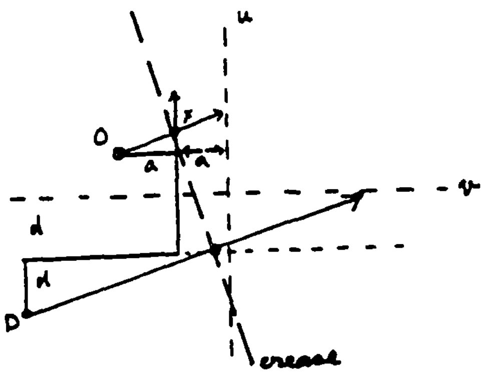


Figure 8b

Fold superposing point O onto line u, and at the same time point D onto line v. The crease determines the bounce points at line b and line c as crossing points with the crease U and V. OU is the initial direction of the ball. The slope of OU is, therefore, the zero of the equation.

Now let solve the equation

$$Z^3 + Z^2 - 2Z - 1 = 0$$

The grid and lines u and v turn out:

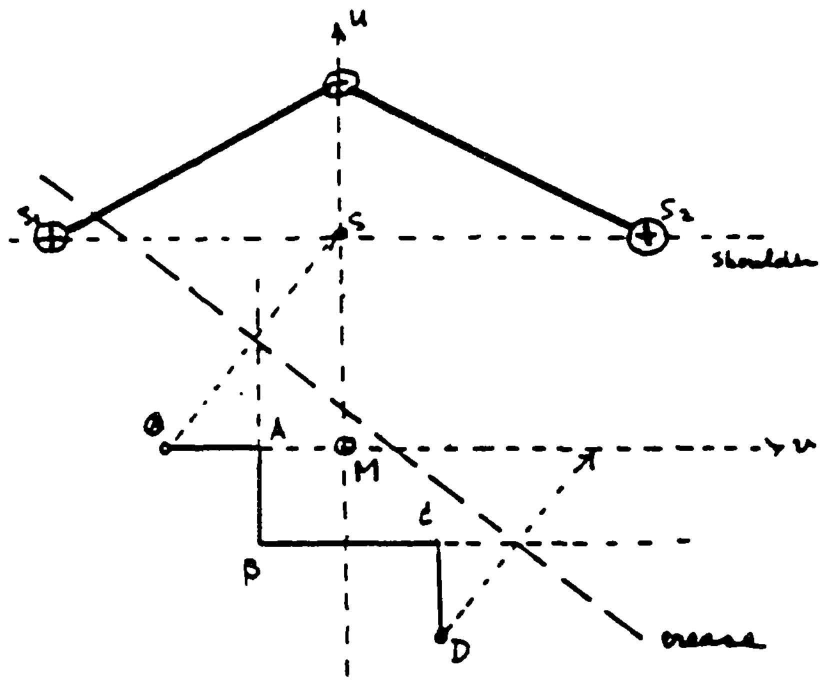


Figure 9

Now let us solve the equation:

(1) fold, superimposing O onto line u, and D onto line v at the same time. The new position of point O, or S, should be the shoulder height of a heptagon of radius 4 (since unit $OA = 1$). We now have all the information to draw a heptagon of radius 4;

(2) call the crosspoint of lines u and v the point M. This is the center of the heptagon of radius 4;

(3) make a point, called T the top point, 4 units upward from M, or twice of OM;

(4) make line s that passes point S and is parallel to line v;

(5) fold, making point O the pivot, superimposing T onto line s. We get the shoulder points, $S1$ and $S2$;

(6) fold, superimposing T onto the S's points. The new positions of S are hip points $H1$ and $H2$;

(7) repeating same procedure we get the foot point.

Here is an interesting characteristics of the polygon. Let us take into consideration the hip height. Let us again call a heptagon of radius 2, Z. That is

$$A^2 + 1/A^2 \equiv Z$$

$$(A^2 + 1/A^2) = A^4 + 2 + 1/A^4 = 1/A^3 + 2 + A^3 \text{ then } 1/A^3 + A^3 = Z^2 - 2$$

and

$$(A^2 + 1/A^2)^3 = A^6 + 3A^2 + 3/A^2 + 1/A^6 = 1/A + 3A^2 + 3/A^2 + A$$

then

$$A + 1/A = Z^3 = 3Z$$

Finally we get the same original equation,

$$Z^3 + Z^2 - 2Z - 1 = 0$$

For the foot level,

$$A^3 + 1/A^3$$

satisfies the equation. This means for given radius (or top point) and the center position, all the other points of a heptagon are directly given by three real roots of the equation

$$Z^3 + Z^2 - 2Z - 1 = 0$$

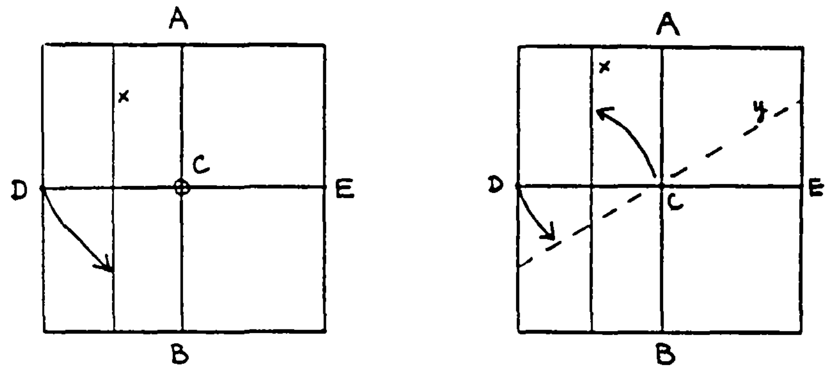
These phenomena are very natural if you consider that the number 7 is a prime number and multiplication corresponds a rotation of one section of the heptagon.

In conclusion, a polygon of $2^n3^m + 1$ can be made by folding paper. On the other hand, Euclidian methods can make only polygons of $2^n + 1$, since its capacity is only up to second order. As you see, the monodecogon (11 sided polygon) is the problem. At present origami cannot solve it. It is a problem of the fifth order. As you will see later, it is not impossible to realise by origami, since the origami geometry is not a closed system like the Euclidean methods but completely open. You can just invent a new way of folding! One possibility is shown in the Proceedings of the First International Meeting of Origami Science and Technology held at Ferrara, Italy 1989, page 53. (Those who are interested in this, please contact the author.)

2 NONAGON

2.1 How to *draw* a nonagon using origami

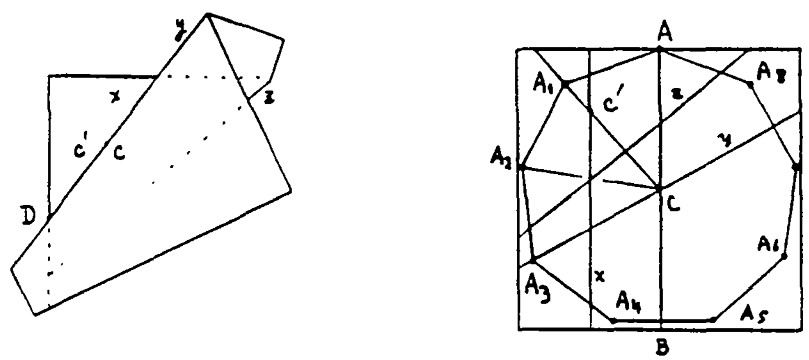
(1) Make two medians called AB (vertical), DE (horizontal) and their crossing (center C). Fold, moving point D onto C and call the crease line x. Fold, making point C a pivot and moving point D onto crease x. Call the new crease line y.



Fold 1 Fold 2

Figure 10

(2) Fold moving point C onto line x and point D onto line y at the same time and call the crease line z.



Fold 3 Get the wanted apexes

Figure 11

(3) Call the new position of point C on line x, point C', and call the crossing of line x and line z point F. Make a line through C and C' and another through C and F. On these lines and on line y make points $A1, A2$ and $A3$ an equal distance of $|CA|$

from the center C. Fold along line y to obtain $A4, A5$ and $A6$ at new positions of $A2, A1$ and A. By the reflection at AB, obtain $A7$ and $A8$. $A, A1, A2, A3, A4, A5, A6, A7$, and $A8$ are apexes of the wanted regular nonagon.

2.2 Explanation of the method

First of all, the number 9 is not a prime number but can be factorized into prime numbers 3×3. This means that the nonagon can be made by a regular triangle and a trisection of the angle of 120 degrees. To make an angle of 120 degrees is easy, as Euclidean methods can also make it. The problem is the trisection of 120 degrees. Let me introduce the elegant method of Hitoshi Abe of 1980 in a little detail, since it seems an important and also an interesting topic.

2.3 Trisection of an angle

It is the famous multimillennial problem, apparently easy to solve but in fact very difficult, which remained unsolved until 1837 when M. L. Wanzel in Paris showed the impossibility of trisecting an angle into equal parts by using ruler and compass or the Euclidean method. If you use other apparatus, it is certainly possible. In one hour you can invent several of them – something like carpenter's tools. In origami geometry which we have so far followed, it is possible merely by folding actions, that is without any tool.

Here is *H. Abe's method*, the earliest (published 1980) in the world and most elegant. An angle is given at point A by half lines a and b as below.

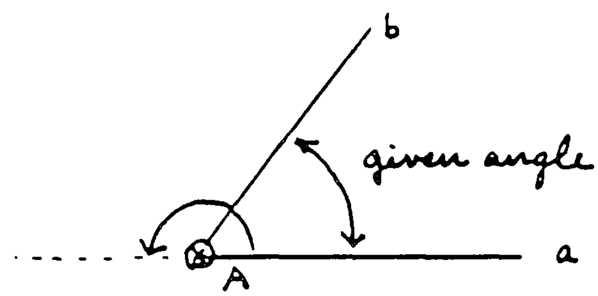


Figure 12

(1) Make line r vertical to line a at A by folding ($A \longrightarrow A$, $a \longrightarrow a$), then take any point B on line r and make the bisector c by folding ($A \longrightarrow B$). Call the crossing point of lines r and c, C.

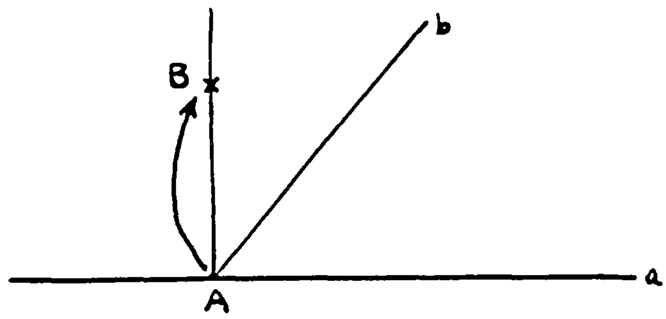


Figure 13

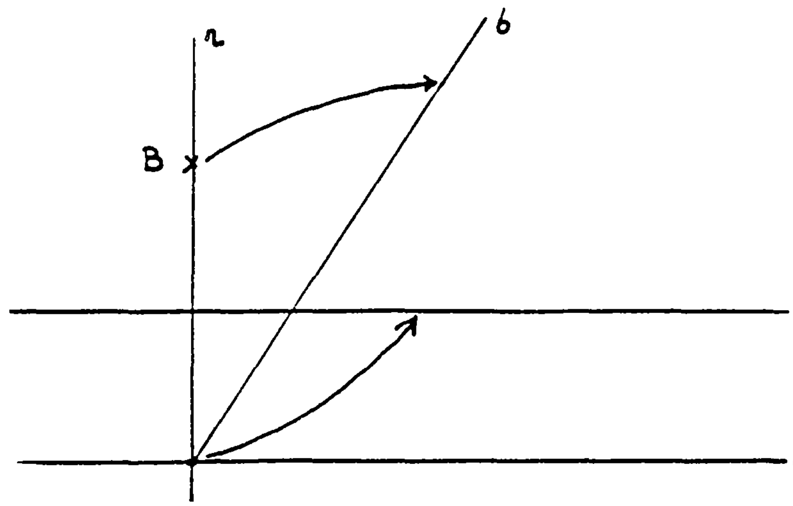


Figure 14

Now fold ($A \longrightarrow a, B \longrightarrow b$) (fold type # 6, see Fig. 15).

Let us call the folded position of points A, B and C, A', B' and C' respectively, and the crossing point of line c and the new crease d, D. Then point A' and point D (i.e., the half lines OA' and OD) trisect the given angle. The proof is simple. Let us call the folded position of line r line r', and the crossing points of line r' with line a and line r, E and F respectively. Then

$$\angle FAA' = \angle AA'C$$

since they are internal alternate angles of the parallel lines a and c. And

$$\angle AA'C = \angle BA'C$$

since triangle $A'BA$ is isosceles ($A'A = A'B$).

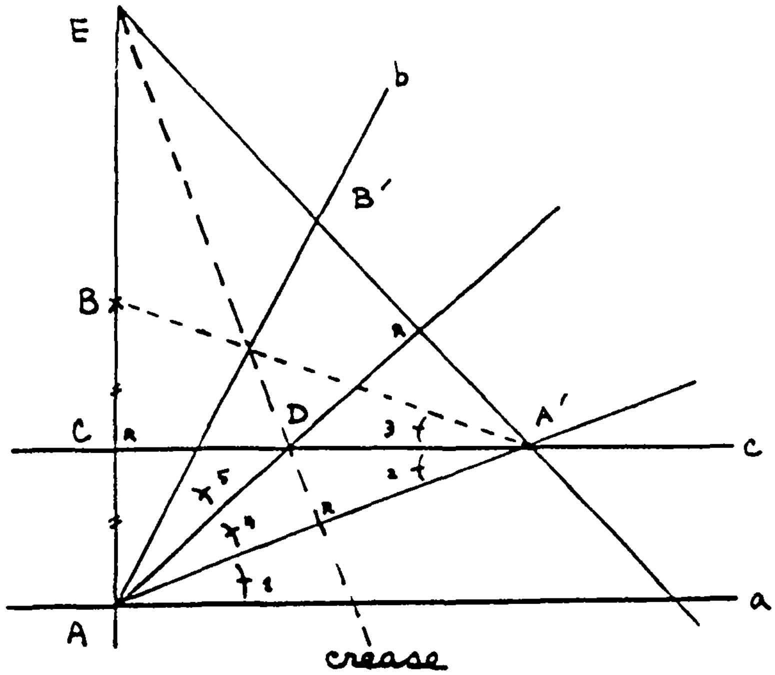


Figure 15

Finally

$$\angle A'AD = \angle AA'D$$

and

$$\angle B'AD = \angle BA'D$$

because of the superposition by the last folding. Therefore,

$$\angle FAA' = \angle A'AD = \angle DAB'$$

i.e., angle *FAB'* (angle of *a* and *b*, or the given angle) is trisected into equal parts by *AA'* and *AD*.

It should be interesting to note that there are always three realizations (solutions in algebra) and as you can easily see in the figures below the other two correspond to the trisection of (2**R** − given angle) and the trisection of (2**R** + given angle). The former has a classic name, supplementary angle, but the latter has no name in the Euclidean methods. Since in the terminology of navigation it is commonly called the ambiguous angle (for example, the direction to a light house from a ship and that of the ship from the light house), let us borrow the term.

Now we can say that *by origami we can trisect an angle. This method, however, gives us always three answers, one is, of course, the trisection of the given angle. The other two correspond to the trisection of its supplementary angle and its ambiguous angle.*

2.4 Why are there three?

It is a good question! However I would like stop here. I hope that you are now very curious about the mysterious secret of folding papers.

The second realization:

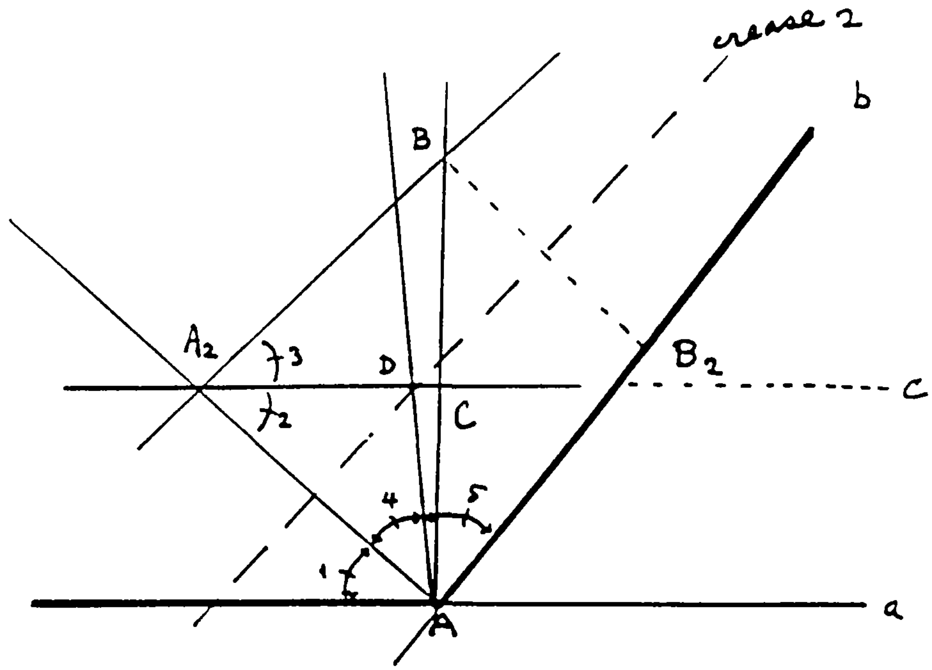


Figure 16

The third realization is shown on Figure 17.

It will be clear what you should do to avoid this ambiguity:

(1) when you define the wanted or given angle, it should be done strictly in terms of half-lines, not complete lines (well defined angle),

(2) also, when you make line c, don't consider the whole line but only the half-line, even in the case of the segment (or sometimes the whole line) which is inside the given angle.

This is well indicated by full lines and dotted lines in the above and below diagrams.

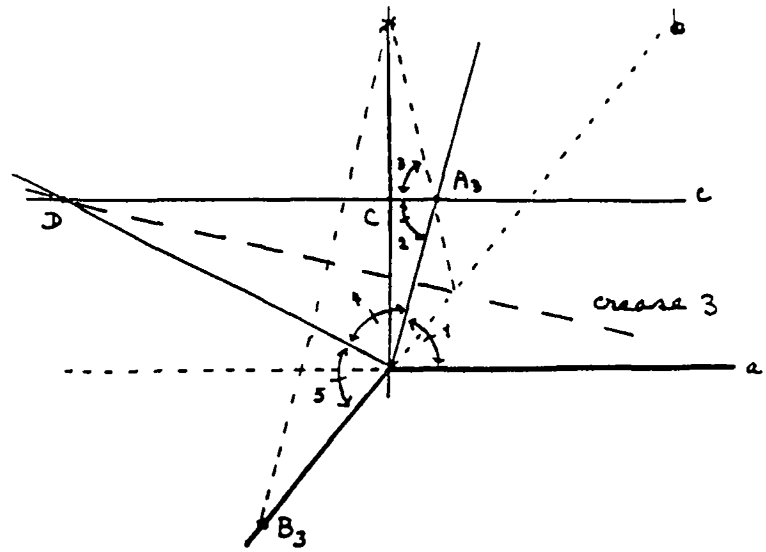


Figure 17

Probably you may point out: why does not the other, more directly related angle (4R – the given angle), come out? You want too much? I limit myself to showing how to obtain it (or to trisect this angle). You will see the clear difference. Just follow the Abe method with the case noted above.

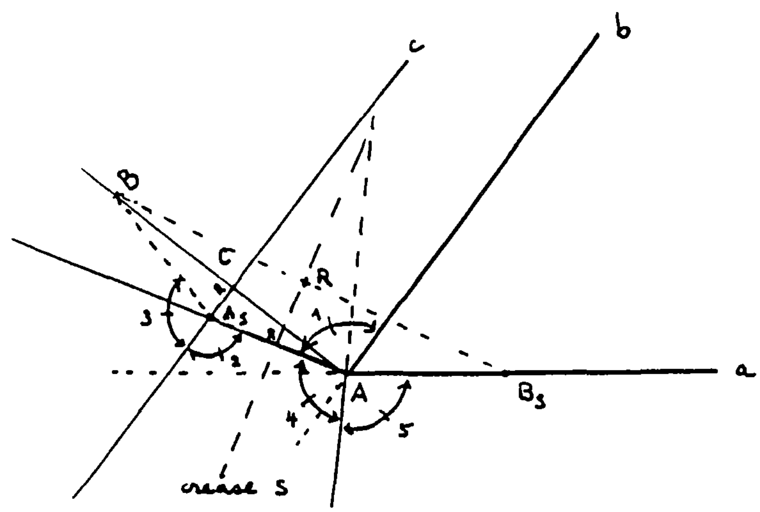
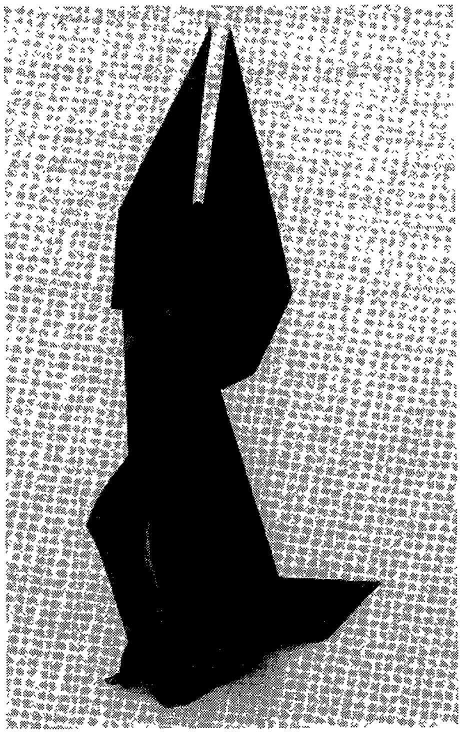


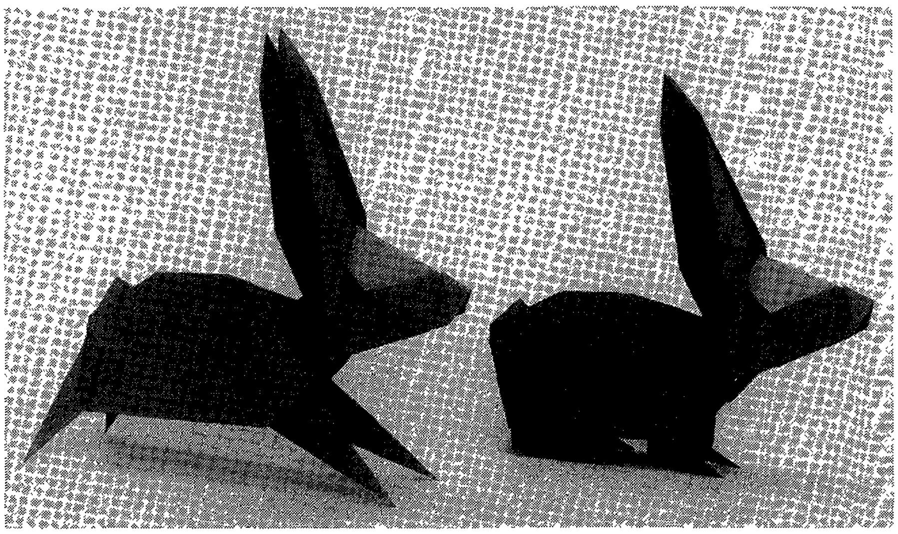
Figure 18

Now you know what the two other hidden (or unwanted) solutions are.

It is very popular now to say that software is often more powerful than hardware. We can understand these words as intelligence and mechanical tools. Origami geometry (or better origami science in general) is purely based on software having no hardware at all. Euclidean methods are strictly limited by their tools (ruler and compass) and there is no possibility of expansion. Its system is closed. On the contrary, origami is a completely open system of the intelligence and there are possibilities for developing it to a higher level by inventing new folds and admitting them as new fold types that will not necessarily be as simple as folds so far treated.



Fox



Rabbit