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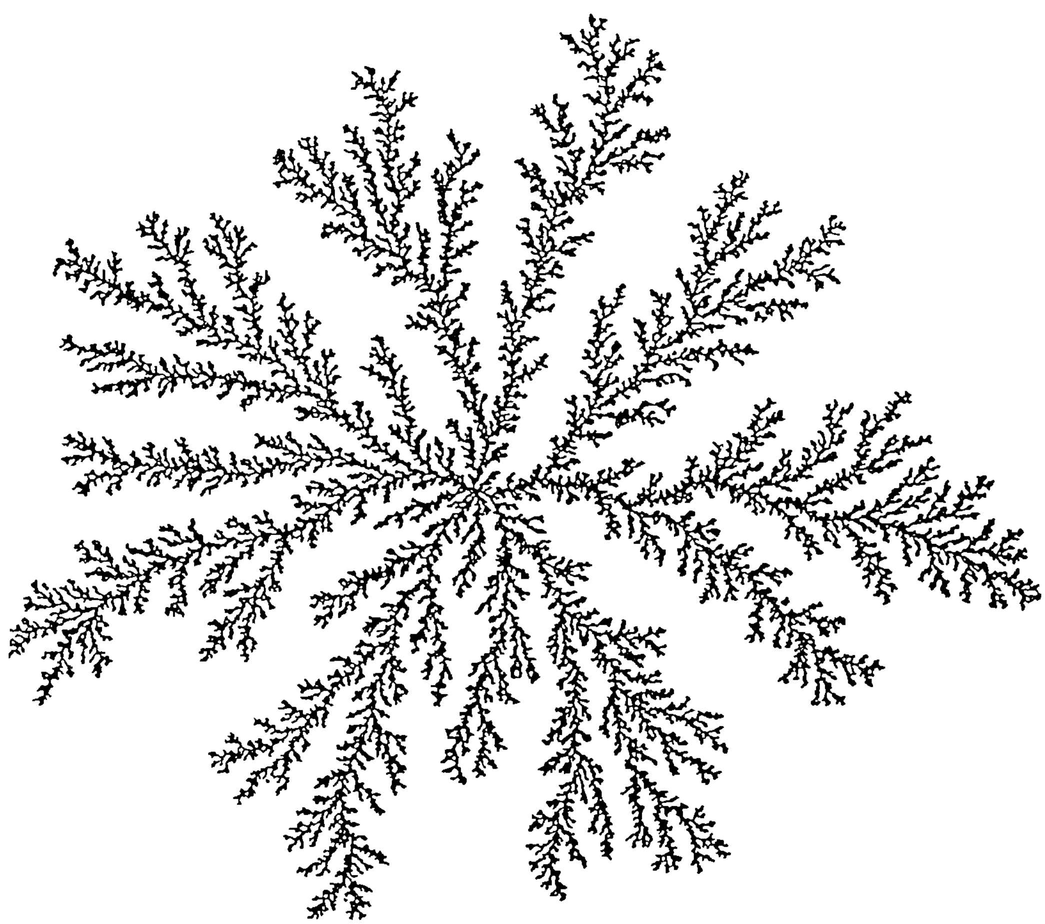

DLA fractal cluster
of 10^6 particles

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ON THE MATHEMATICS OF SYMMETRY BREAKINGS

Béla Lukács

Physicist, (b. Budapest, Hungary, 1947).
Address: Central Research Institute for Physics, Budapest 114, P.O.B. 49, H-1525 Hungary, E-mail: lukacs@rmk530.rmki.kfki.hu .
Fields of Interest: Cosmology; General Relativity; Heavy Ion Physics; Speech Acoustics; Economics.

Abstract: *Deviations from symmetries are discussed. We distinguish two cases: (i) symmetric laws + asymmetric external disturbances lead to approximately symmetric actual states; (ii) symmetric laws without other influences lead to states asymmetric in an extent determined by the laws themselves via spontaneous symmetry breaking.*

INTRODUCTION

This paper is devoted to the mathematics of the different kinds of symmetry breaking, considering the aims of the present issue. Our goals are limited: we do not want to go into fine details of definitions, but try to demonstrate that a symmetry may be broken for different reasons and to show how the extent in which it is broken can be quantified.

1. ON SYMMETRIES

Instead of a complete definition let us use the following one:

A SYMMETRY IS A TRANSFORMATION LEAVING SOMETHING INVARIANT.

Example: consider a sphere. By rotating it remains the same.

For a precise definition of *geometrical* symmetries see another paper in this issue (Lukács, 1993).

2. ON BREAKING

If there is no symmetry at all, nobody speaks about broken symmetry. We speak about a *broken* symmetry if

1) the situation is *almost* symmetric;
2) the situation *ought* to be symmetric still it is *not*.

The symmetry can be broken

1) by external asymmetric influences;
2) spontaneously.

3. ON THE EXTENT OF AN APPROXIMATE SYMMETRY

The measure is slightly subjective. However, if there is approximate symmetry, then the ideal symmetric situation can be visualized, and the actual one can be compared with it.

Example: the ellipsis. Its equation is

$$r = r(\varphi) = R/(1 + e \cos \varphi) \tag{1}$$

where e *is the excentricity. For ellipses* $0 \leq e \leq 1$; *for* $e << 1$ *the ellipsis is almost a circle. So the extent of the deviation from the symmetric circle (rotational symmetry) is measured by the dimensionless* e.

If the symmetry breaking is caused by an external influence then *generally* it is proportional to the influence (and then one can compare the acting forces). If the breaking is spontaneous, this is not necessary at all.

4. AN EXAMPLE FOR NON-SPONTANEOUS SYMMETRY BREAKING: OBLATENESS

A rotating fluid body is often an oblate spheroid (although other solutions of the hydrostatic equations exist as well (Chandrasekhar, 1969)). For small angular momentum Ω one can get simple approximate formulae as follow:

The rotation results in an inertial 'force'. Including its 'potential' one gets

$$V(r, \varphi) = -GM/r + ½\Omega^2 r^2 \sin^2 \varphi \tag{2}$$

Then the equipotential surfaces (of which one is the surface of the fluid (Thorne, 1971)) are given by roots of a cubic equation. Again for small Ω they can be approximated in a transparent way. Consider a value $V_0 < 0$. Then $r_0 = r_0(\varphi)$:

$$r_0 = (-GM/V_0)\{1 + ½[(GM\Omega)^2/(V_0^3)] \sin^2 \varphi\} \tag{3}$$

Then the extent of asymmetry or symmetry breaking is the ratio of the two terms in the bracket {} for, say, $\varphi = \pi/2$. Substituting terrestrial surface data, the ratio is 0.0017, rather moderate.

5. ON SPONTANEOUS SYMMETRY BREAKING

Again instead of a precise mathematical definition here we adopt the following transparent one:

THE SYMMETRY BREAKING IS SPONTANEOUS IF THE LAWS ARE SYMMETRIC STILL THE MOST NATURAL CONFIGURATION (SAY GROUND STATE) FOR THEM IS ASYMMETRIC.

The simplest example is mechanical equilibrium in a potential V(x) = V(-x). *The potential is even, i.e., mirror-symmetric: this will be the symmetry. The most natural configuration is now the equilibrium one; if that is not unique, then the ground state among them. Here come the details.*

The condition for equilibrium is

$$x = x_o; \quad (\mathrm{d}V/\mathrm{d}x)_o = 0 \tag{4}$$

For simplicity we restrict ourselves to polynomials, such that

$$\begin{aligned} &V(0) = 0 \\ &V(\pm\infty) = +\infty \\ &V(-x) = V(x) \end{aligned} \tag{5}$$

We are going *up* with the degree of the polynomial.

P_o

The only possibility compatible with (5) is

$$V_o(x) = 0 \tag{6}$$

Then (4) holds *everywhere*; the ground state is degenerate. This case is trivial.

P_1

Conditions (5) cannot be satisfied: this class is empty.

P_2

From (5) we get

$$V_2(x) = ax^2, \qquad a>0 \tag{7}$$

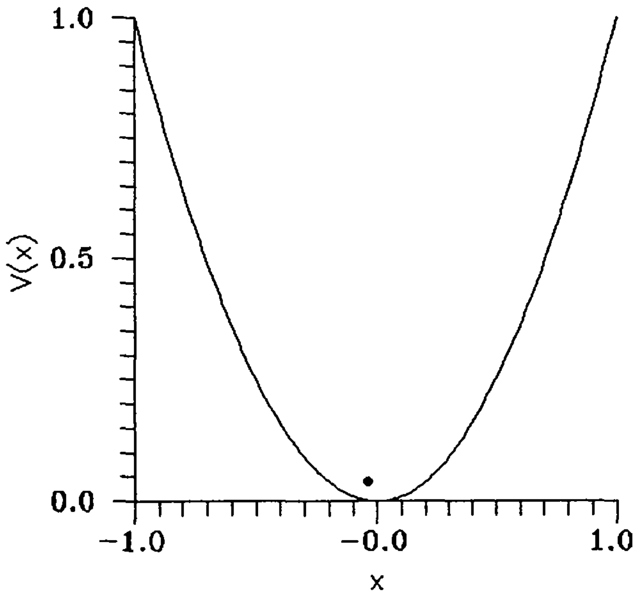

Figure 1: Quadratic potential. Stable equilibrium at the center.

This potential, for $a=1$, is shown as Figure 1. The only solution of (4) is $x=0$ (heavy dot).

SYMMETRIC EQUILIBRIUM STATE FOR SYMMETRIC LAWS.

P_3

Conditions (5) cannot be satisfied: this class is empty.

P_4

From (5) we get

$$V_4(x) = ax^2 + bx^4, \qquad b>0 \tag{8}$$

This potential, for some values of a and for $b=1$, is shown on Figure 2. The equilibrium condition (4) leads to

$$x(a + 2bx^2) = 0 \tag{9}$$

Now, $x_{00}=0$ is always a root, but for $a<0$ two other ones exist as well:

$$x_{0\pm} = \pm(-a/b)^{1/2} \tag{10}$$

Then, as shown on Figure 2, for $a\geq 0$ $x_{00}=0$ is the only equilibrium point, and is an energy *minimum* (heavy dot). However, for $a<0$ the situation is *qualitatively* different.

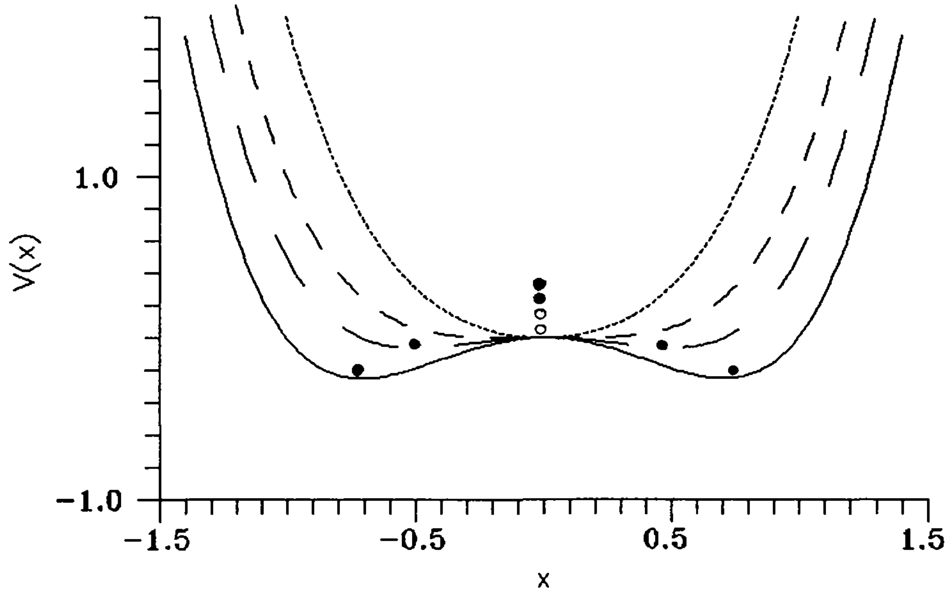

Figure 2: Quartic potentials; $a = +1$ (dot), 0 (short dash), -0.5 (long dash), -1 (solid). Heavy dots at stable *minima*, circles at unstable *maxima*.

As seen, x_{00} is an equilibrium point, but energy *maximum* (circle). A local maximum is an *unstable* equilibrium: for small perturbations the state goes *away* and returns not. On the other hand, the new $x_{0\pm}$'s are *stable minima* (heavy dots). After some transient evolution the mass point settles down at one of $x_{0\pm}$.

ASYMMETRIC GROUND STATE(S) FOR SYMMETRIC LAWS.

Now, notice that the actual final state will lack the mirror-symmetry. In the same time the set of ground states is still symmetric: there is one on the right, one on the left. Similarly, both are equally probable (50-50 per cent) as final states. *This* symmetry is a consequence of the symmetry of the potential. But the symmetric potential V_4 leads to asymmetric final states if $a<0$.

This phenomenon is called *spontaneous* symmetry breaking, since in a dynamical evolution from symmetric laws and initial conditions the system goes into an asymmetric actual state without any external asymmetric influence.

The very simple example shows that no very specific or complicated mechanisms are needed for this spontaneous symmetry breaking. Still, one generally expects symmetry from symmetric laws and has the tendency to stop at the symmetric solutions. *So, remember the possibility of spontaneous symmetry breaking.*

The necessary condition for such a breaking is the appearance of a 'double-bottomed potential'. This is known from various parts of natural sciences. Some cases will be seen in the following articles. In addition, we mention here three examples from physics.

1) *Higgs mechanisms in particle physics* (Langacker, 1981). Some hypothetical scalar bosons may have quartic self-potentials $V(\Phi)$, where Φ stands for the wave function, so roughly $|\Phi|^2$ is proportional to Higgs particle number or probability. (If the potential is not polynomial, or higher than quartic, then the theory is not renormalisable, the infinities from divergences cannot be removed, so the theory has no prediction at all, i.e., is not a theory.) So Higgses spontaneously appear in a 'number' determined by the parameters of $V(\Phi)$, except if the average energy is higher than $V(0)$-V_{min}. (I.e., above the central peak of Fig. 2.) If the Higgses are coupled to other particles, the interaction mimics other particle masses $\sim |\Phi|^2$. The mechanism was invented for renormalisable quantum field theories of vector bosons, where particle masses are detected but would prevent renormalisation. In such systems phase transition is expected when the specific energy passes through a value in the order of the central peak: at high energy (temperature) the actual state is symmetric, and going to low temperature this symmetry spontaneously breaks down ("the state rolls down from the hill").

2) *Ferromagnetism* (Kittel, 1961). Sometimes the interaction between electron spins or orbital momenta is attractive and strong enough to prefer identical directions for neighbouring atoms. Then the ground state is asymmetric: magnetic momenta pointing into a definite direction in average, although the laws are isotropic in the 3D-space. This direction is, however, random, because the isotropic laws lead to isotropy in probability. If the environment is not isotropic (e.g., the weak geomagnetic field), then sometimes this anisotropy singles out the direction (but the strength of a ferromagnet is prescribed by its physical parameters and is definitely not proportional to geomagnetism).

3) *Piezoelectricity of* $BaTiO_3$ (Kittel, 1961; von Hippel, 1950). $BaTiO_3$ is a cubic crystal. The expected location of the *Ti* ion is at the center, the actual one is excentric by a fixed shift. In a very simplified way one may say that the *Ti* ion is fairly big compared to *O*'s. The *O*'s are located at the centers of the lateral faces of the cube, so restricting the room for *Ti* just in the center. Then it is 'more convenient' (say, energetically) for the *Ti* ion to be excentric, which means a potential of qualitative form of Figure 2. For a more correct and less transparent explanation see (Kittel, 1961); in an approximation a quartic potential is obtained. The configuration has a hysteresis loop, but above a threshold value the external potential makes the *Ti* jump into an oppositely excentric position, with reversed electric dipole moment. This can be done with external pressure as well, and this is the reason for piezoelectricity.

6. CONCLUSION

The most important message is:

REMEMBER: A SYMMETRIC POTENTIAL MAY HAVE MINIMUM IN THE CENTER BUT MAY HAVE MAXIMUM AS WELL. SYMMETRIC LAWS OFTEN LEAD TO ASYMMETRIC GROUND STATES WITHOUT ANY FURTHER REASON.

Life is not always simple.

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