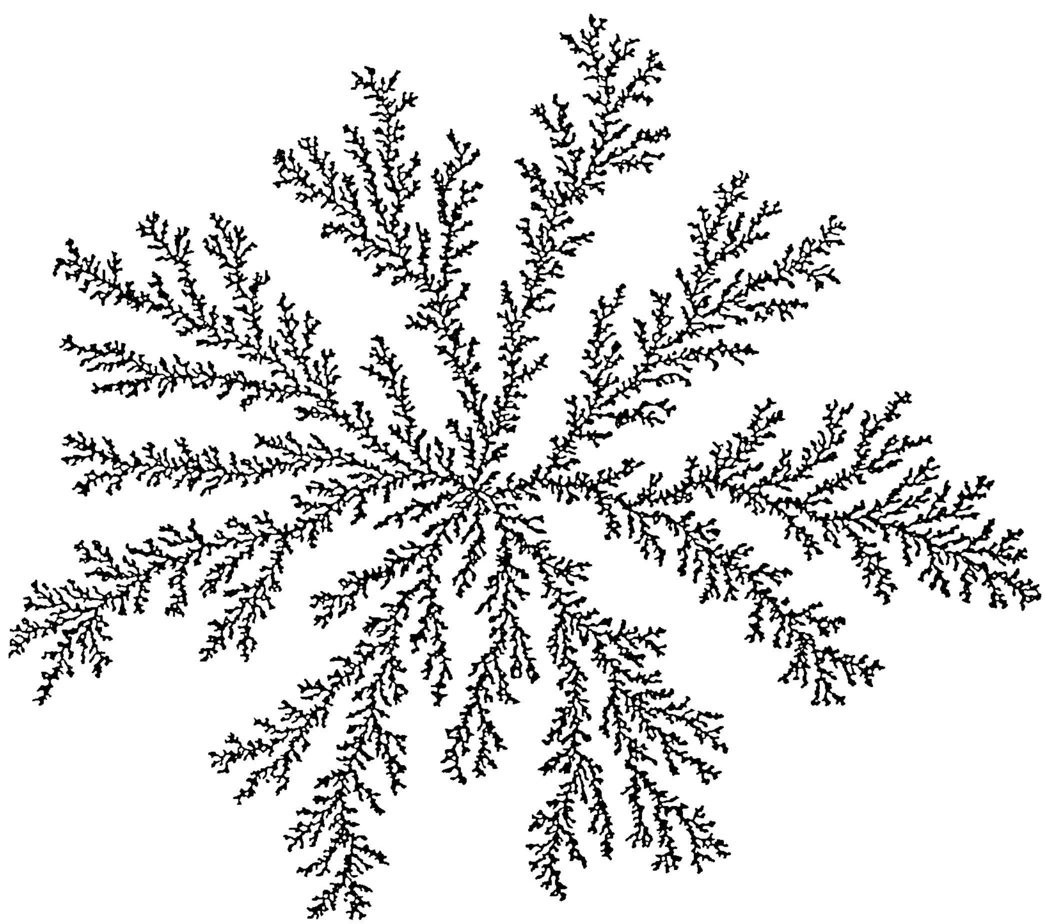
Symmetry: Culture and Science

Symmetry and Topology in Evolution

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

Volume 4, Number 1, 1993



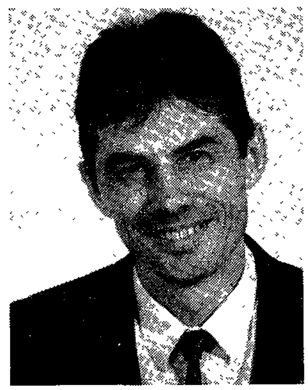
DLA fractal cluster
of 10^6 particles

Symmetry: Culture and Science
Vol. 4, No. 1, 1993, 49-68

SYMMETRY CHANGES BY CELLULAR AUTOMATA IN TRANSFORMATIONS OF CLOSED DOUBLE-THREADS AND CELLULAR TUBES WITH MÖBIUS-BAND, TORUS, TUBE-KNOT, AND KLEIN-BOTTLE TOPOLOGIES

Szaniszló Bérczi

Physicist, astronomer, (b. Szeged, Hungary, 1950).
Address: Eötvös Loránd University, Department of Astronomy, Budapest, Ludovika tér 2, H-1083, Hungary.



Fields of interest: Planetology and cosmopetrography (Lunar rocks), symmetry of structures in botany, mineralogy, archaeology, and molecular chemistry, cellular automata.

Publications: Escherian and non-Escherian developments of new frieze groups in Hanti and old-Hungarian communal art, In: Coxeter, H. S. M. *et al.*, eds., *M. C. Escher: Art and Science*, Amsterdam: North-Holland, 1986, pp. 349-358; *Szimmetria és Strukturaépités*, [Symmetry and Structure Building, in Hungarian], Lecture Note Series J3-1441, Budapest, 1990; *Kristályoktól Bolygótestekig*, [From Crystals to Planetary Bodies, in Hungarian], Budapest: Academic Press, 1991; Local and global model of Fibonacci plant symmetries, In: Gruber, B., and Zopp, J. H., eds., *Symmetries in Science IV: Biological and Biophysical Systems*, New York: Plenum Press, 1990; The periodic system of Platonic and Archimedean solids and tessellations, *Acta Geologica Academiae Scientiarum Hungaricae*, 1980 (23), 1-4. 184-200.

Abstract: *The definition and classification of double-frieze structures in ornamental art from archaeology (Bérczi, 1985, 1986, 1989), and the introduction of symmetry operations as local type, cellular automatic operations (Bérczi, 1985, 1987, 1989) opened the possibility of using these concepts in the crystallography of different surface-mosaic structures.*

The first application of these concepts was in the transformations of the double-thread cellular Möbius-band to torus. This transformation preserves the half of the double thread in the form of cellular band, but rearranges its neighbourhood in the direction perpendicular to the direction of the cellular band. This transformation – from Möbius-band to torus, to and back – rearranges the pattern of the double-thread system and it is invariant to the knot-structure of tube-knots. Surface structure of Klein-bottle built from Möbius-bands is also discussed.

INTRODUCTION

To develop the topics of this paper it was necessary to unify the achievements in three different directions of investigations. One direction was: *symmetry as a local* (cellular automatic) *operation* (Bérczi, 1976, 1985, 1986, 1989). The second one was: the recognition of the role of double-threads in ornamental constructions (mainly in archaeological finds; Bérczi, 1986, 1989). The third one was the intuitive rediscovery of the Möbius-band to torus transformation (Bérczi, 1990). Construction of a cellular automatic model to these developments formed a framework to build together and summarise them in this paper.

SYMMETRY BY LOCAL OPERATION

The rich set of Avar-Onogurian ornamental structures (Fig. 3) in which there were frequently double friezes, suggested to the author, that for the classification of these double friezes a new meaning of the classical symmetry concept should be needed. The classical symmetry concept used global-local connections: symmetry was the order of the ordered *whole* on its repeating, congruent *elements* (represented by symmetry operations). The symmetry concept connected with Avar-Onogurian structures modified the role of operations. Double friezes required a local and one-step generator type operation concept. (We may call it technological symmetry concept or cellular automatic operation concept because of its step-by-step effect in structure building.) This symmetry concept was a local one, which recognised the global order (the cells were conscious of the global order), but considered operations as generators of the 'state' of the neighbourhood.

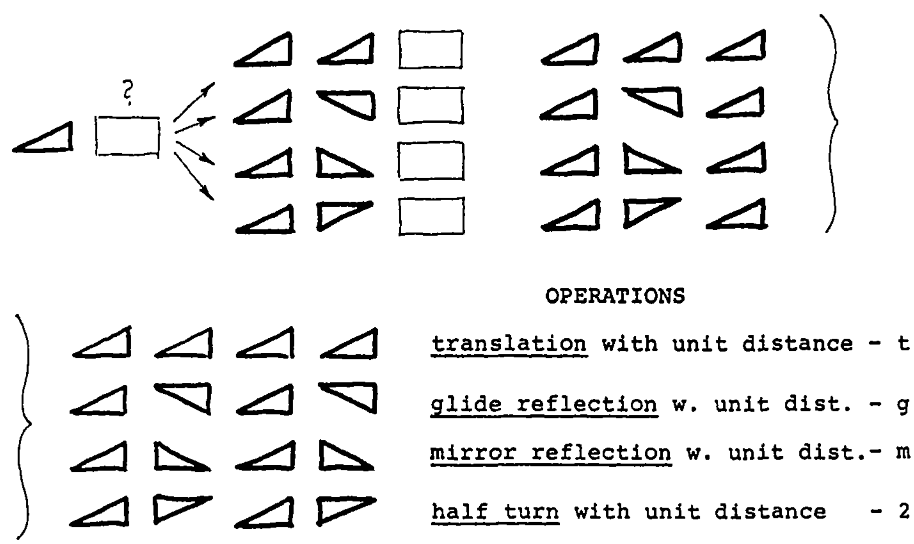


Figure 1: Combinatorical construction of the four congruencies which work as operations to generate the neighbour pattern in order to build a line in the plane. A combination of the three last of them results in a fifth one: *mg*. These five line-patterns form a set of basic frieze patterns(*t*, *g*, *m*, 2, *mg*).

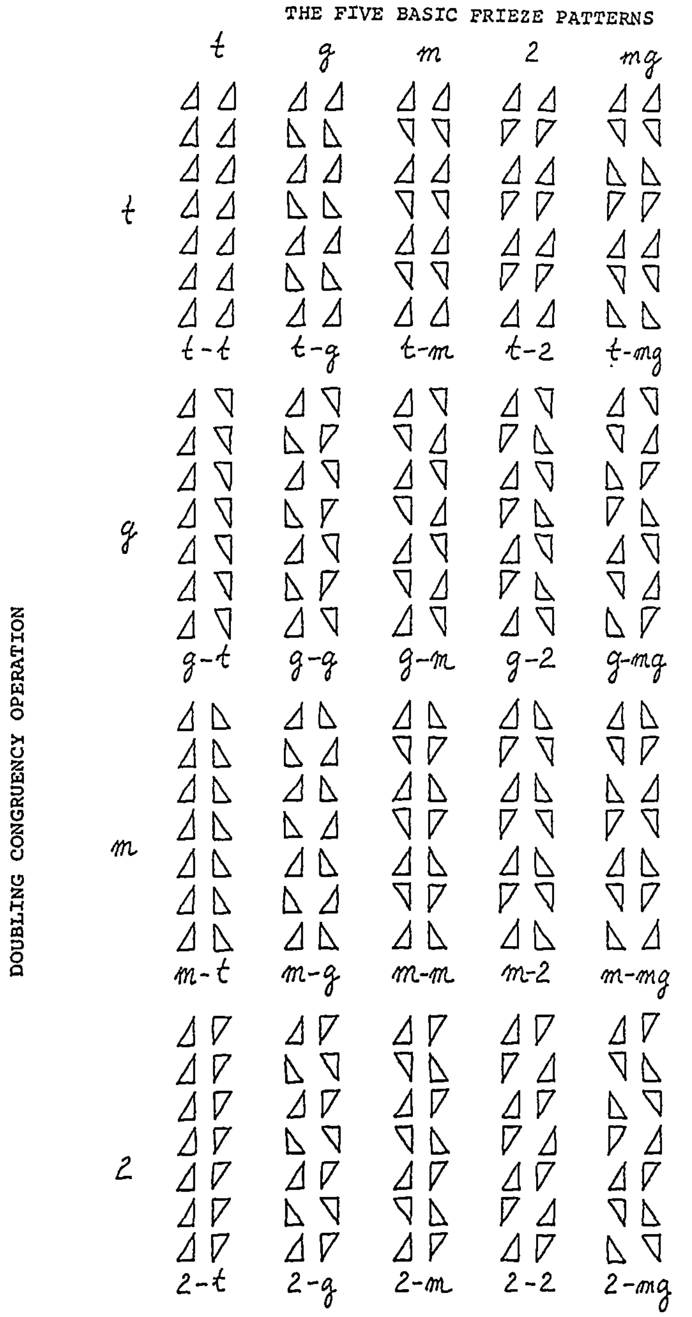


Figure 2: The matrix of the 20 double frieze patterns (Bérczi 1986, 1989.) The matrix is organized (woven) from the five basic frieze patterns: *t*, *g*, *m*, 2, and *mg*. doubled by the four simple frieze generating operations: *t*, *g*, *m*, 2, applied as local operations. Avar-Onogurian representatives are shown in Figure 3. Celtic representatives are shown in Figure 4.

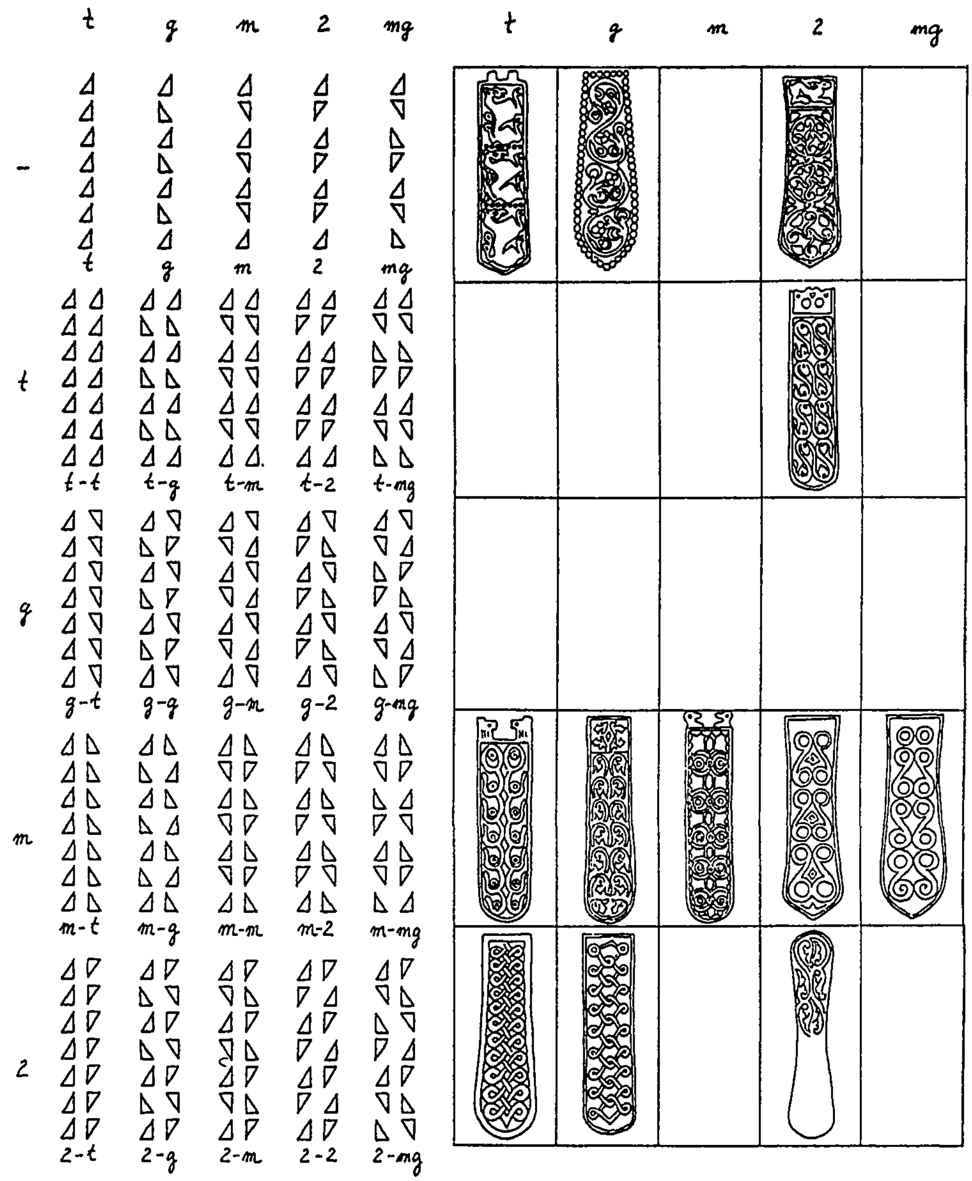


Figure 3: The rich set and classification of Avar-Onogurian double friezes according to their pattern forming operations (Bérczi, 1986, 1989.).

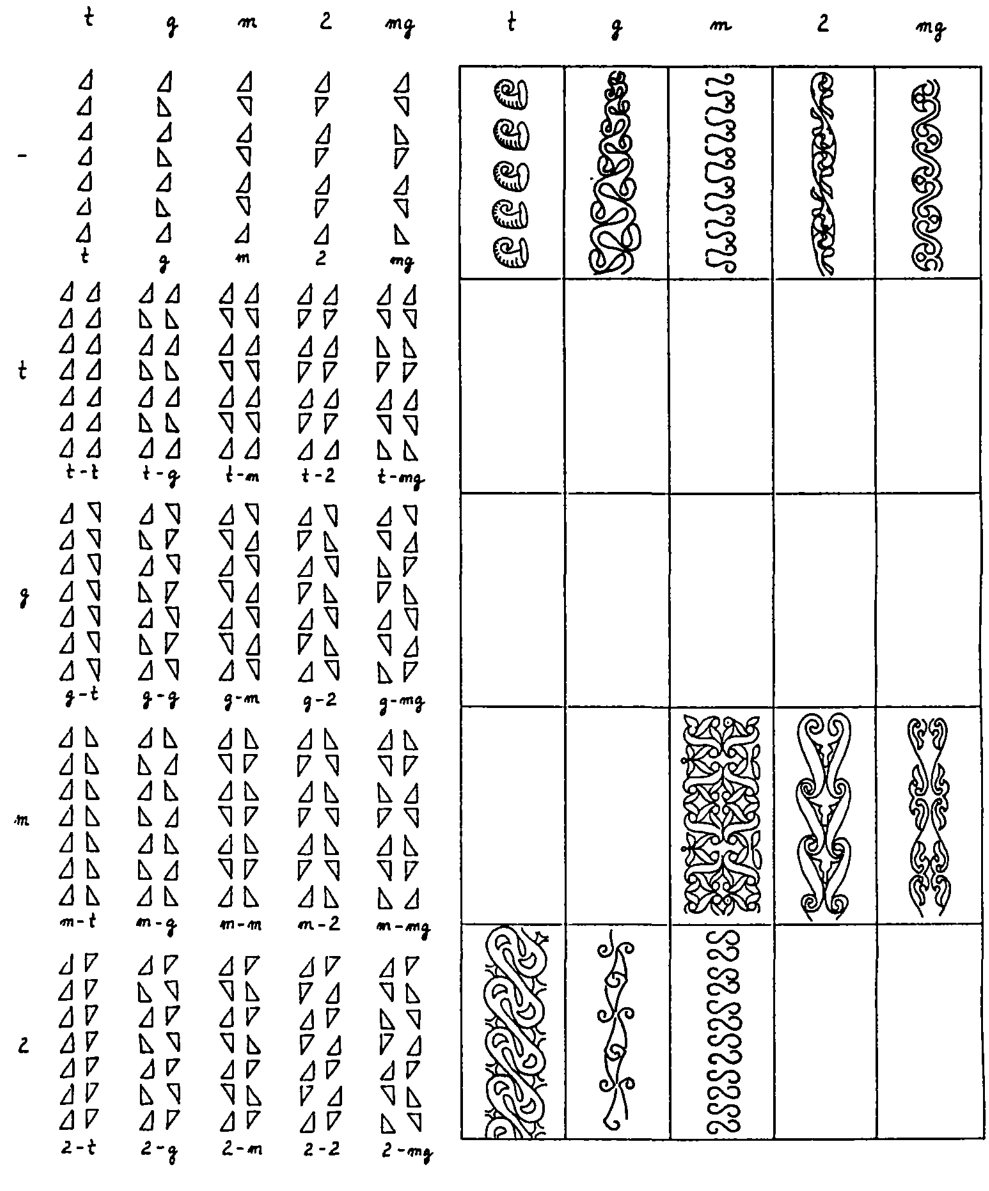


Figure 4: Classification of pattern-representatives from the Celtic ornamental art from archaeological finds.

There are four simple congruency operations which may generate different frieze patterns (with unit width) along the line (embedded into the plane). They are the following ones: translation, mirror reflection, glide reflection and half turn (Fig. 1). The basic frieze patterns are those which were generated by these congruency operations plus one more frieze: that which was their combination: *mg*. The doubling of them needs a local generator of the neighbourhood thread from the given frieze pattern. This operation can be carried out by the four line-generator simple congruencies (Fig. 1). Local congruency operations determine the neighbourhood positions of a repeating element in a net of perpendicular and horizontal rows of the patterns of the matrix in Figure 2.

CELLULAR AUTOMATIC FRAMEWORK

The cellular automaton model has a characteristic framework of description. It is composed from two parts of conditions. The first one gives the structure of the *cellular background*, the second one gives the *transitional functions*. Both parts of conditions form a pair of approach: a local and a global one, as follows:

A. CELLULAR BACKGROUND

(*Aa*) Local characteristics of the cell-mosaic system give the form of cells, their connections and neighbourhood relations.
(*Ab*) Global characteristics of the cell-mosaic system give the surface and the enclosure of the local relations to form a whole.

B. TRANSITIONAL FUNCTIONS

(*Ba*) Local transitional function for cell mosaic elements which are individual automata (discrete function in space and time, step by step transforming cell-states).
(*Bb*) Global transitional function for the whole surface populated by the cell-mosaic system (it forms a sequence of stages of the surface taken step by step, as a consequence of summarised – for all cells – local transitional functions).

Although the points *a* and *b* are not independent of each other, the advantages of the cellular automaton model come from this separability of local and global picture: both for conditions and operations, and from the expressed connections between the local and global characteristics of the phenomenon.

THE INDIRECT WAY OF CONSTRUCTION OF CELLULAR AUTOMATON MODEL: THE INDIRECT VON-NEUMANN PROBLEM

The classical way of the development of a cellular automaton model was the construction of *Aa* and *Ab* background and the *Ba* local transitional function, at first. Then followed the deduction of the global transitional function *Bb*, which hold the primary goal of the construction. We may call this way of model construction to the direct way. (The principal aim of von Neumann's cellular automaton construction was to build a self-reproducing structure on the level of global transitional functions.)

Our way of construction in this paper is the indirect way in respect of von Neumann's direction of construction (von Neumann, 1966). After stretching the background by the determination of points of *Aa* and *Ab* we formulate *Bb* global transitional function as a sequence of stages of discrete steps of transformations of the cell-mosaic system: and finally we construct the *Ba* local transitional function for the cells themselves.

In the recent paper the global transitional function has two steps and three stages of state. We introduce to symbol them as follows: V – □ – R (V – for cut, □ – for motions, R – for glue parts together). But before cellular automaton model development we show the problem in a classical way.

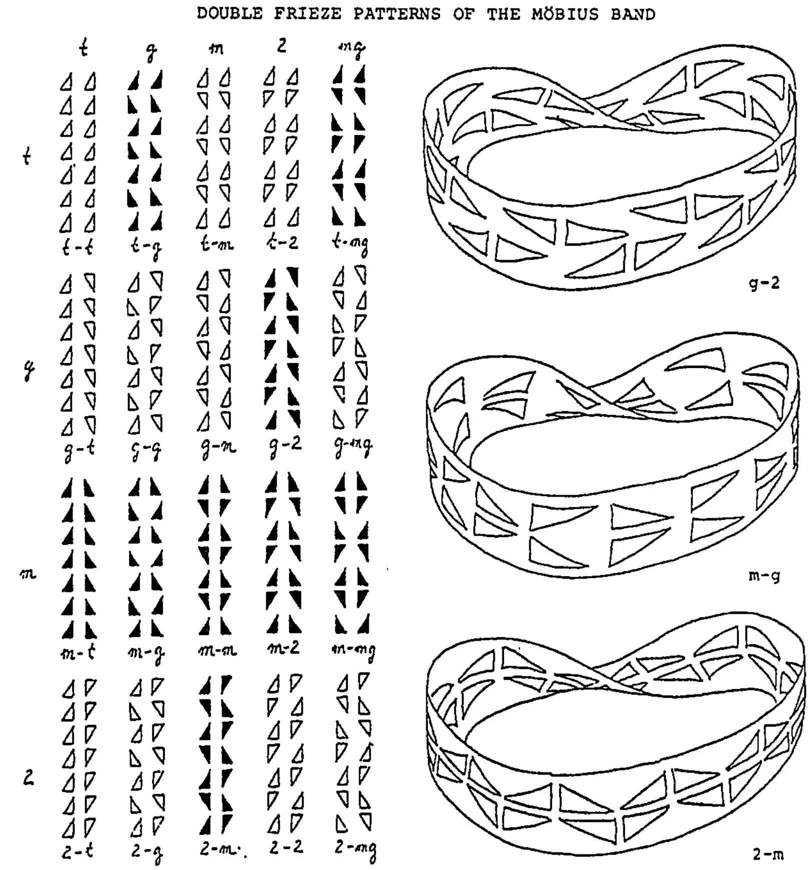


Figure 5: Double frieze patterns which are suitable to fit onto the Möbius-band (black coloured cases in the double frieze matrix) and three of their representatives are shown. The drawings does not mark the edges of the cells only the repeating abstract element of the pattern represent the cells themselves. (Compare this figure with Figure 7, where rearrangements during the Möbius-band – to – torus transformation are summarised, also in matrix form.)

CRYSTALLOGRAPHY OF THE MÖBIUS-BAND

It is well known that Möbius band can be constructed from a finite long, unit wide band (belt) in the following way. Cut this normal band (belt) perpendicular to its edges and attach the two ends after a half turn rotation of one of its end around the middle axis of the band. This half turn transforms normal beltband to Möbius-band. Let us assume that the belt band was adorned with a frieze pattern. What kind of frieze patterns may remain invariant after the transformation shown earlier?

It is important to notice, that the band is built up by transparent pattern of the cells, what means: that both sides of a cell are shown with the same pattern figure. During the given normal-belt to Möbius-band-belt transformation, half turn acts as if it were a glide reflection for the neighbouring cells at attachment position. As a consequence of the construction of the Möbius-band from a normal belt-band we can conclude, that frieze patterns should have glide reflection generator in order to they should be fitted onto the Möbius-band. This is a local condition for the cells. But there is a global condition, too. It is a number condition. Normal belt-band should have a pair number of cells in order to fit its pattern with glide reflection. The transformation procedure (cutting and attachment) shown earlier to construct Möbius-band from a normal-belt-band needs the elimination of one cell at the position of attachment after half turn, because the operation referred destroys the order there: after half turn there will be two cells with the same (i.e., only translational) position, so lacking the needed glide reflection for these two neighbours. To correct this failure of the order of the pattern we have to 'cut out' one of these two cells, during an *in situ* production of the transformation described. Therefore the global condition for the number of cells in a belt with Möbius-band structure is the following: Möbius-band should contain odd number of cells with *g* (glide reflection type) generators. Considering the case of *mg* structure, too, we may conclude that not the number of cells, but the number of units suitable for glide reflection (now pairs of cells) should have the number odd, on a Möbius-band. Of the double-frieze and basic frieze patterns given in Figure 2, those which are suitable to fit onto a Möbius-band are given with black colour in Figure 5. (The transformations of these nine double frieze patterns in the further parts of our paper will be always given in the matrix of double frieze patterns, first given in Figure 2.)

THE MÖBIUS-BAND TO TORUS TRANSFORMATION

As we mentioned in advance, the Möbius-band – to – torus transformation consists of three steps: V – [illegible] – R. Instead of classic description we give these steps as a global transitional function.

(*Bb-1*) Cut the Möbius-band at middle between its edges, along its circle. Colouring of half-band helped to see this operation in Figure 6. (Comment: this cutting separates the two half bands locally, but does not do it globally: the L-long Möbius-band becomes a 2L-long one, 4 times half-turn-twisted belt band.)

(*Bb-2*) Move the opposite sections (cells) of the 2L-long, twisted band, so, that coloured sides be the outer surface, and the two edges of the opposite sections (cells) may be attached. (Comment: this operation may be substituted in a way as follows: do not remove far the cut half-bands (with unit width) from each other, but instead of it, slide one of the half bands behind the other, and so form the position to contact opposite edges of the slidden cells.)

(*Bb-3*) Attach and glue the contacting edges. (Comment: two gluing lines are resulted in locally; the two opposite edges of any cells, but globally only one glued line run around the torus. This glued single line globally turns π in cross section circumference of the torus when it run 2π along the great circumference. The full length of this single line is 4π:2π – small:great circumferences. This is the characteristics to be generalized when used for tube-knots in the inverse transformations: torus (or tube-knot) – to – Möbius-band.)

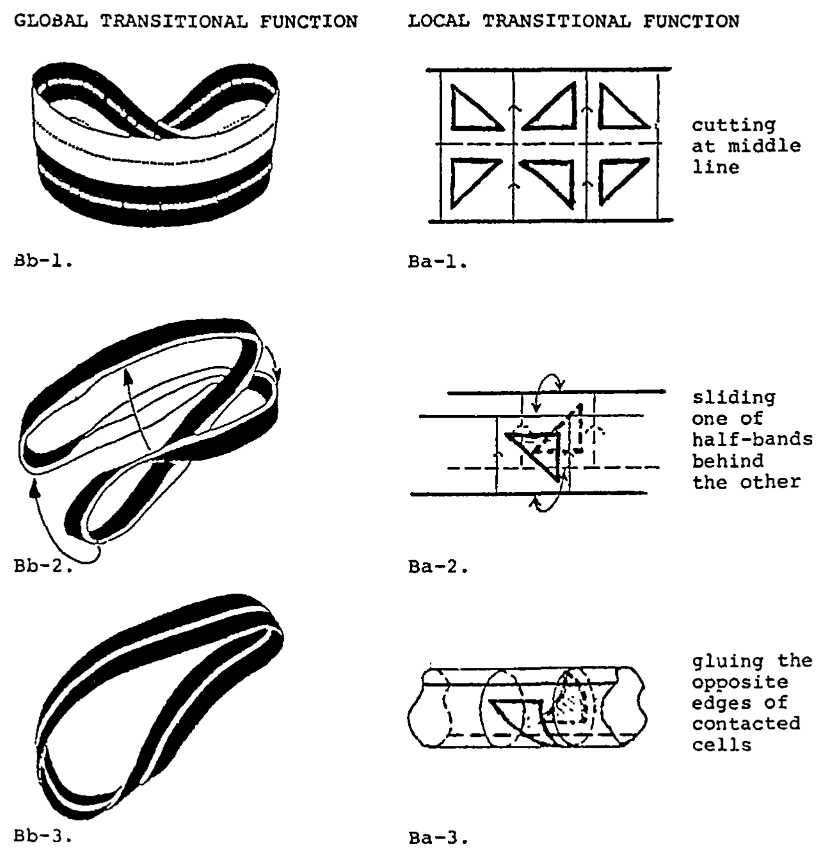


Figure 6: The global (left column) and the local (right column) transitional function in the cellular automatic formulation of the Möbius-band – to – torus transformation.

Figure 6 summarizes visually these steps both for the global and the local transitional functions. The program of our paper is to formulate the indirect problem: to transcript the global transitional function to the local transitional function. The steps in the local transitional function can be easily followed according to the movements given for *Bb-2* in comment. This is the *Ba* function:

(*Ba-1*) Contacts along band direction are preserved, separation in the perpendicular direction is executed between neighbouring cells.

(*Ba-2*) 'Sliding behind' one half band (cell-ribbon) to the other.

(*Ba-3*) Contact and glue the free, opposite cell edges. The twolayered – glued – structure should be blown up to form a torus. This blown up tube exhibits the plane-symmetry pattern according to the neighbourhood relations between the cells in it (Fig. 7). (Gluing results in a kind of Born-Karman boundary condition for the cellular bands regarding their patterns.)

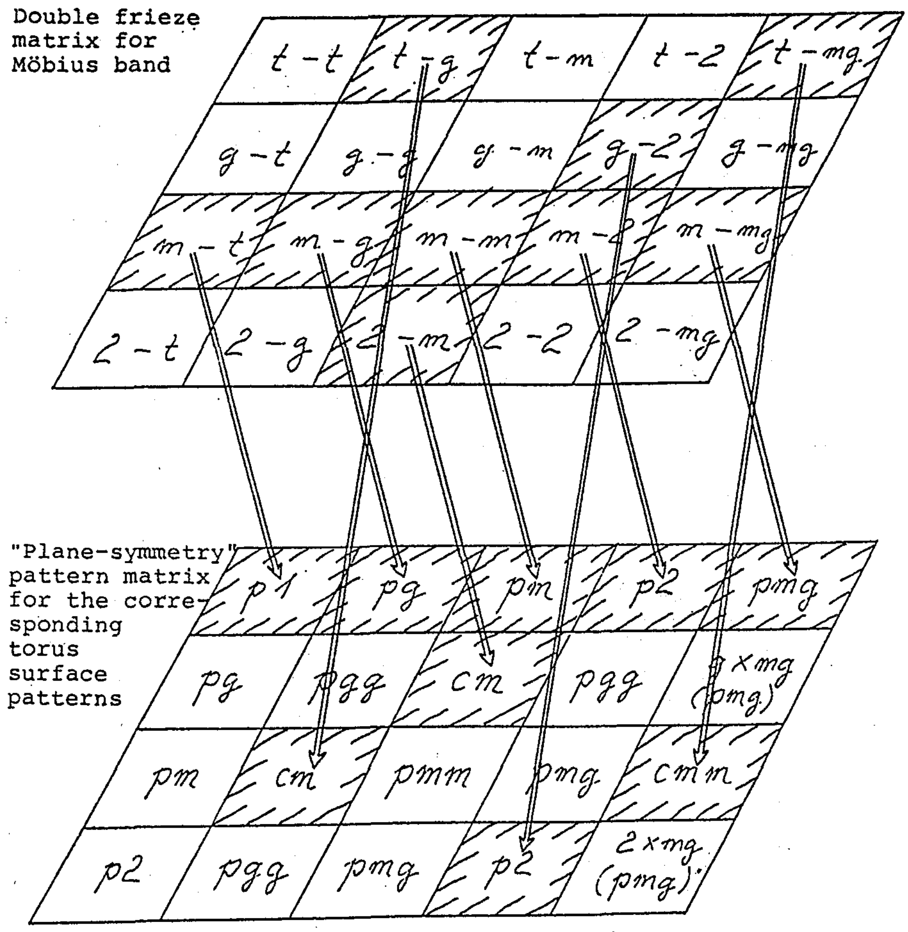
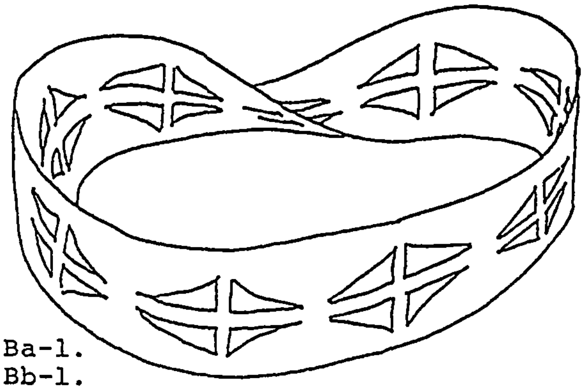
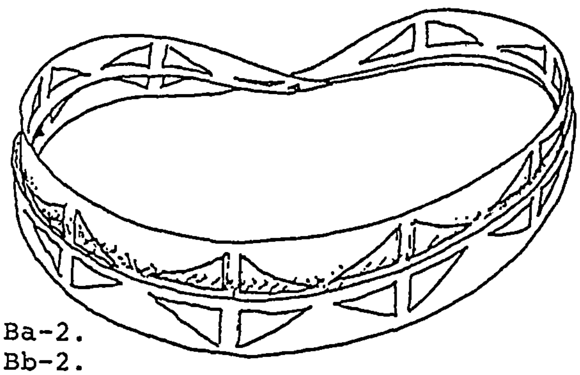


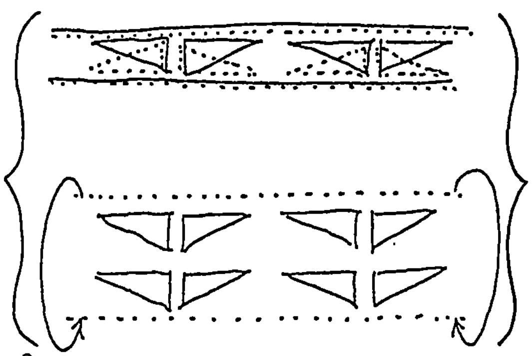
Figure 7: Transformations of Möbius-band — to — torus rearrange the pattern of the surface. These rearrangements are summarized in the double frieze matrix projected to the corresponding 'woven' plane-symmetry pattern matrix. The process how the transformation rearranges the pattern is shown for *m-m* double frieze pattern in Figure 8. The inverse rearrangements are referred in Figure 9.



Ba-1.
Bb-1.



Ba-2.
Bb-2.



Ba-3.
Bb-3.

Figure 8: Unification in a single sequence of sketches of the global and local transitional functions of the three steps (given earlier in Fig. 6) in the Möbius-band — to — torus transformation for the case of *m-m* double frieze on Möbius-band — towards the *pm* plane-symmetry structure on torus (or *t-m* double frieze pattern, if projected). Cells of the local model are represented by their abstract figure of triangles. The critical step of *Ba-2* and *Bb-2* shows how the sliding of one of the half bands behind the other forms *pm* structure after the third step from *m-m* frieze structure.

SURFACE PATTERNS /IN THEIR DOUBLE FRIEZE REPRESENTATIONS/ OF THE TORUS /OR TUBE-KNOTS/ SUITABLE TO THE INVERSE TRANSFORMATION

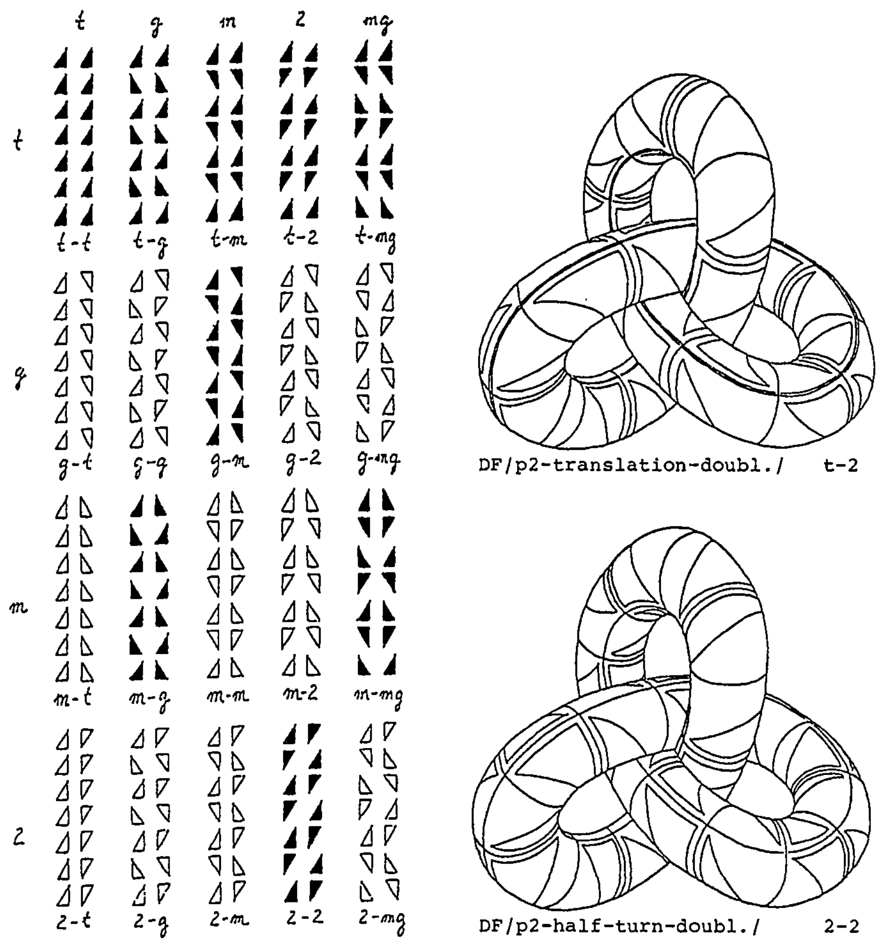


Figure 9: The initial surface patterns (given with their skinned and smoothed double frieze forms) on the torus (or on the tube-knots) suitable to the inverse Möbius-band — to — torus transformation. All these transformations and initial patterns are valid for tubeknots which fulfil the conditions given in the *Bb-3* point of global transitional function. Two cases for the simple knot are shown in the right column.

THE TORUS– (OR TUBE-KNOT–) –TO–MÖBIUS-BAND TRANSFORMATION

In the inverse (or reverse) of the Möbius-band – to – torus transformation (i.e., in the torus–to–Möbius-band transformation) the surface cell-mosaic patterns given in the lower matrix of Figure 7 are the initial conditions. These patterns are those which may form a correct double frieze pattern on the Möbius-band after the inverse Möbius-band–to–torus transformation.

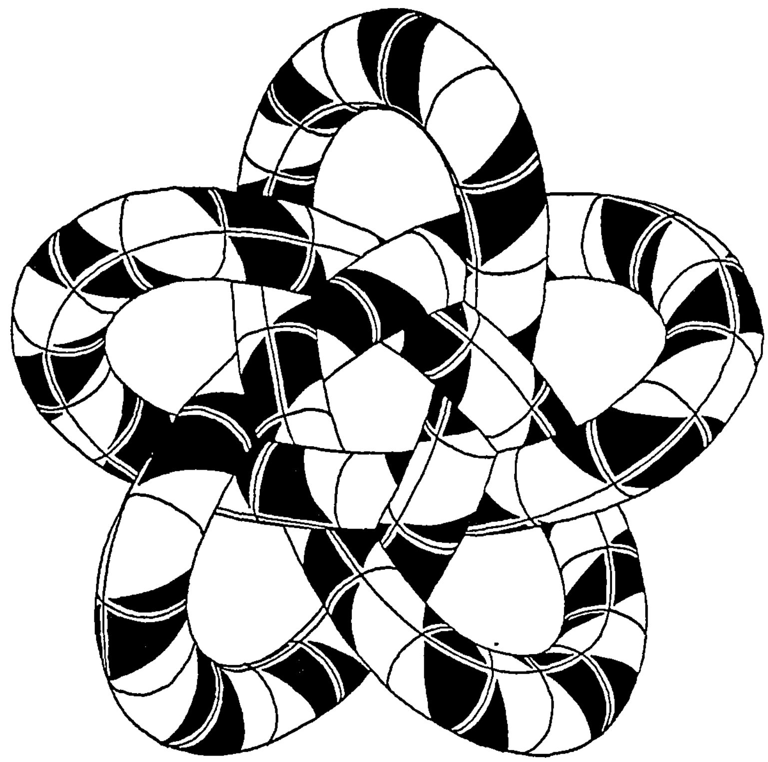


Figure 10: Idealized drawing of a tube-knot with C_5 rotational global symmetry with *p2* type surface pattern (or 2-2 corresponding double frieze pattern), on its surface which suitable to the inverse Möbius-band – to – torus transformation. This transformation forms *g*-2 double frieze structure on the surface of the Möbius-band-knot. Compare it to the simple tube knots of Figure 9.

The group structure of a knot can be understood easily, when we consider the knot as a representation of a frieze pattern woven from a strand, wound up around a central point. Here on Figure 11 the two most simple knots are shown with their corresponding frieze patterns.

A (top) The simple knot is the representation of the frieze pattern of 2 (half turn generated) wound up around a point: radius vector crosses two strands at once, but globally one strand runs around two times, according to the suitable alternation of the 2 (half turn) frieze pattern. The 12 (twelve) repeating elements (corresponded to sections between strand-crossings) of the corresponded frieze pattern prove that they may not be commensurable with odd number of cell units of the double-frieze pattern of the tube-knot with this knot-structure, if tube-knot is parcelled to cells according to a double frieze pattern suitable to be transformed by inverse Möbius-band – to – (tube-knot) – torus transformation.

B (bottom) The knot with C_4 rotational global symmetry (carrick band coaster) is the shortest frieze pattern representation of *g* (glide reflection) type structure wound up around a point: radius vector crosses three strands at once, but globally one strand runs three times, according to the alternation in the woven *g* frieze pattern (coloured with three tones). The 24 repeating elements (corresponded to sections between strand crossings) of the corresponded frieze pattern prove, that the frieze pattern of the knot structure may not be commensurable with the structure of the double frieze pattern of the tube-knot with such structure, if its parcelling to cells happened so that the double frieze pattern was suitable to have been transformed by inverse Möbius-band–to–(tube-knot)–torus transformation. Both examples shown on Figure 11 intuitively prove that structural hierarchy of tube knots contains hierarchy levels independent of each other.

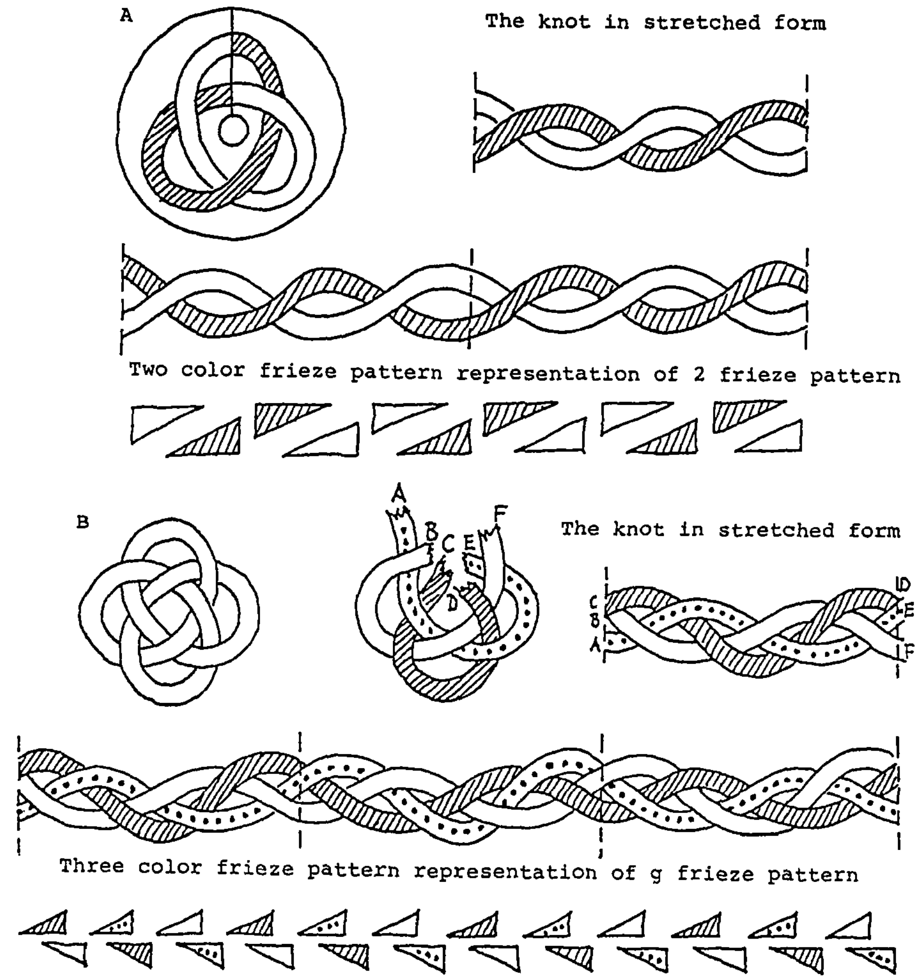


Figure 11: Wound up frieze pattern representation of the woven structure of the two most simple knot-structures. *A* Simple knot with C_3 global rotational symmetry and wound up 2 (half turn) frieze pattern structure around a point when considered its structure locally between strand-crossings. *B* Knot with C_4 rotational symmetry (global one) and wound up *g* frieze pattern structure around a point when considered its structure locally, between strand-crossings. Global and local structure is incommensurable in respect of the inverse Möbius-band — to — (tube-knot) — torus transformation if strand is a tube.

CONSTRUCTION OF KLEIN-BOTTLE BY DOUBLING MÖBIUS-BANDS

Doubling of Möbius-band double friezes and gluing them at their edges one by the other in mirror symmetric position result in known topological surfaces: the Klein-bottle (see Fig. 12). Doubling consists of two parts. First one is the doubling of Möbius-band cellular background by a mirror reflection. Second one is the doubling of the suitable patterns: this may be carried out according to four operations shown in Fig. 2 (by *t*, *g*, *m*, and 2 by-generators). In this operation pairs of cells work as generators.

On the other hand, doubling of the pattern may be accomplished in two variants. In the case of normal doubling pattern of the doubled pair is in phase with the initial one. A modification of this normal case is that doubling may not be not-in-phase. In the case of doubled *t*, *g*, *m*, and 2 lines there is one (for *mg* three) possibility to slide doubled pattern one cell unit. This variation results in new doubled Möbius patterns on the Klein-bottle, (see Fig. 15).

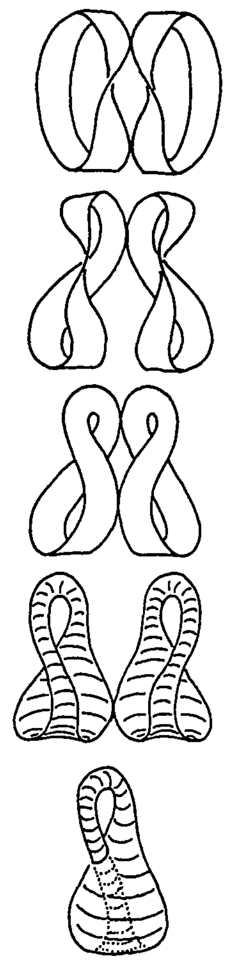


Figure 12: Construction of Klein-bottle by mirror reflection of a Möbius-band.

SUMMARY AND CONCLUSIONS

Different ways of pattern generations from a single, constructed, basic frieze pattern were constructed and analysed in this paper. After the constructive definition (in combinatorical way) of basic frieze patterns, first the double frieze patterns were generated by a neighbour-state generator local operation. Then those double friezes were selected from the double frieze matrix, which are suitable to fit them onto Möbius-band. The most important and interesting results shown in this paper were the cellular automatic description of the Möbius-band–to–torus transformation and the implicated pattern rearrangements in the double frieze patterns constructing them. These results may have importance in description and modelling of transformations and reproductions of molecular double-threads (especially that of viruses).

Different developments of the construction by local doubling operations were shown. It was shown for tube-knots, that their knot-hierarchy level is independent of the surface cell-mosaic structure of the tube, and the group structure of the knot is incommensurable with the numerosity of the cell-mosaic pattern of the tube, when it is suitable to the inverse Möbius-band–to–(tube-knot)–torus transformation. Finally, further constructions from Möbius-bands with double frieze cell-mosaic patterns were shown to build Klein-bottle from them. There were used also a local doubling operation to repeat the structure of the initial Möbius-band. It was sketched how can be constructed new Klein-bottle surface patterns from Möbius-band double friezes by *t*, *g*, *m*, and 2 doubling, and in-phase and not-in-phase cases were also shown to enrich the number of pattern variations.

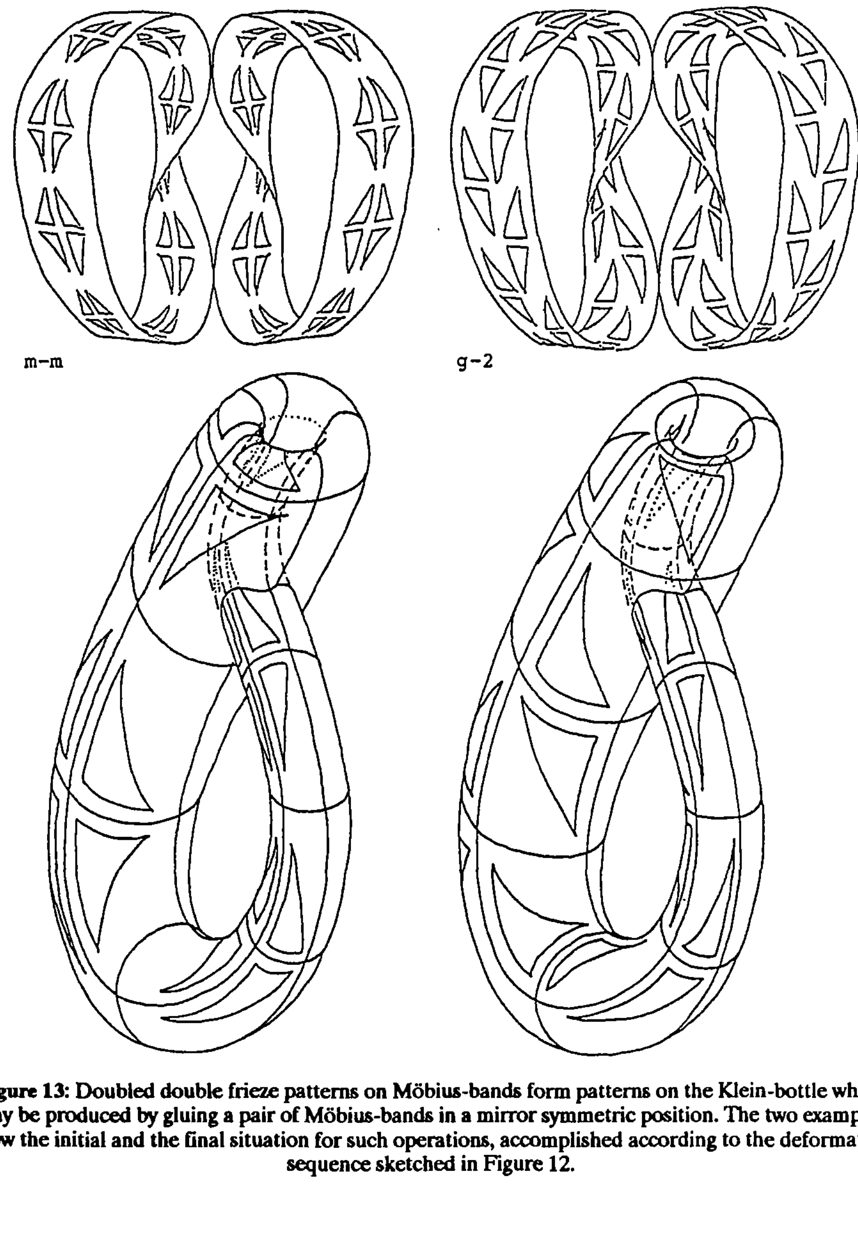


Figure 13: Doubled double frieze patterns on Möbius-bands form patterns on the Klein-bottle which may be produced by gluing a pair of Möbius-bands in a mirror symmetric position. The two examples show the initial and the final situation for such operations, accomplished according to the deformation sequence sketched in Figure 12.

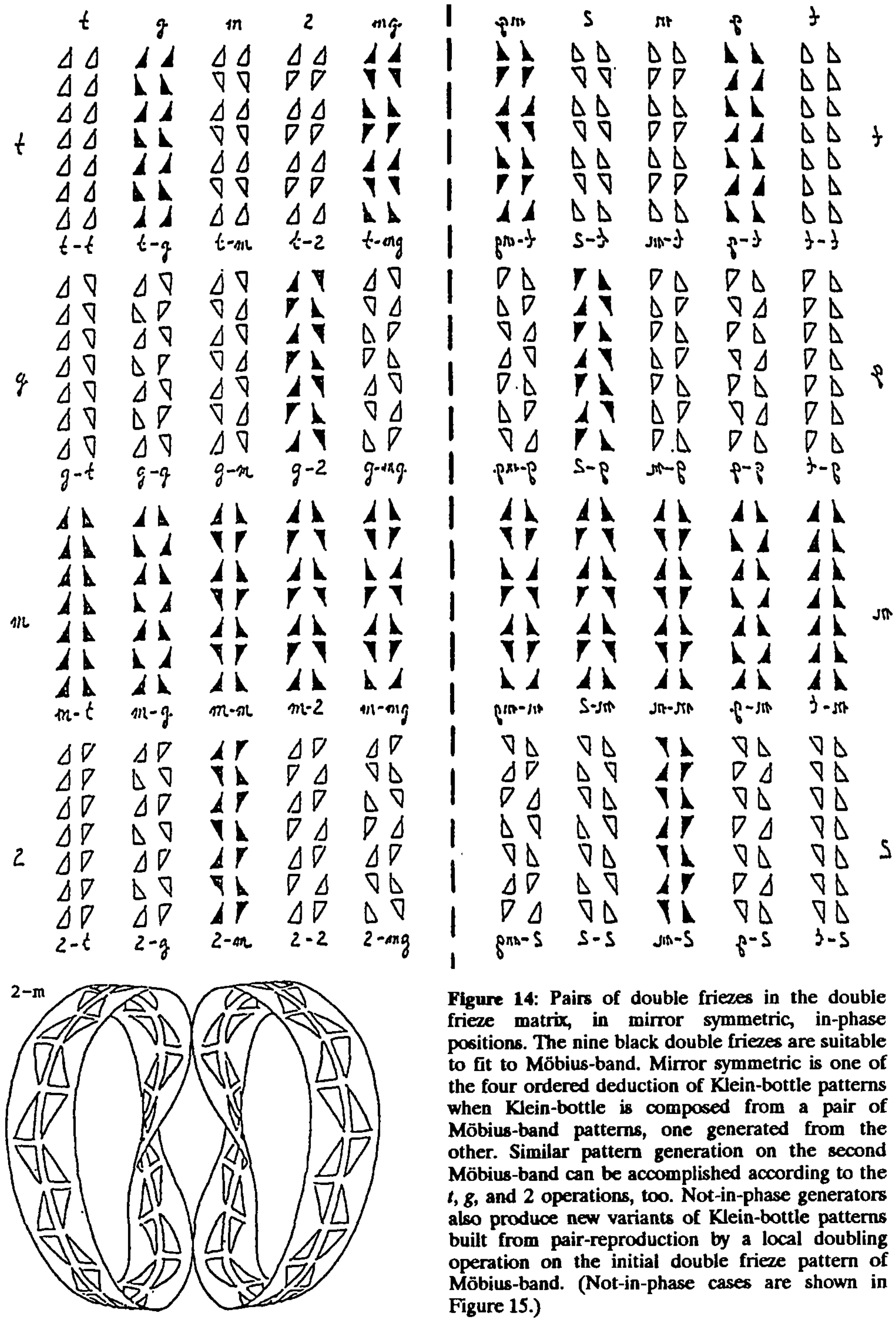


Figure 14: Pairs of double friezes in the double frieze matrix, in mirror symmetric, in-phase positions. The nine black double friezes are suitable to fit to Möbius-band. Mirror symmetric is one of the four ordered deduction of Klein-bottle patterns when Klein-bottle is composed from a pair of Möbius-band patterns, one generated from the other. Similar pattern generation on the second Möbius-band can be accomplished according to the *t*, *g*, and 2 operations, too. Not-in-phase generators also produce new variants of Klein-bottle patterns built from pair-reproduction by a local doubling operation on the initial double frieze pattern of Möbius-band. (Not-in-phase cases are shown in Figure 15.)

All the constructions shown in our paper are independent of structural hierarchy level in the evolution of matter. Therefore phenomena which can be modelled using up such constructional or transformation processes may be found on different levels of structural hierarchy. But mainly molecular level of structural formation is the hierarchy level in focus, when these transformations are discussed. On this level it is an important conclusion from our paper, that Möbius-band structures are very rarely observed, because of the strong criterion of *g* symmetry operation which is necessary condition for the existence of Möbius-band double thread structure. But *g* operation contains reflection, which is not a motion, so molecular enantiomorphy should be used up in such constructions. Alternating selection of enantiomorphous molecules seems almost impossible process in nature. Only small number of elements long Möbius-band structures may be expected or may be hoped to be found in natural circumstances. But the possibility of constructing them is open. Construction of molecular double thread structures with Möbius-band global structure is a challenge to chemists and may open a new field in chemistry. On the other hand it may be interesting field for theoretical calculations (with small number of elements) for quantum chemistry, too. All these 'predictions' are valid for structures with both Möbius-band structure and with a knot superstructure, too, but it seems a far future program.

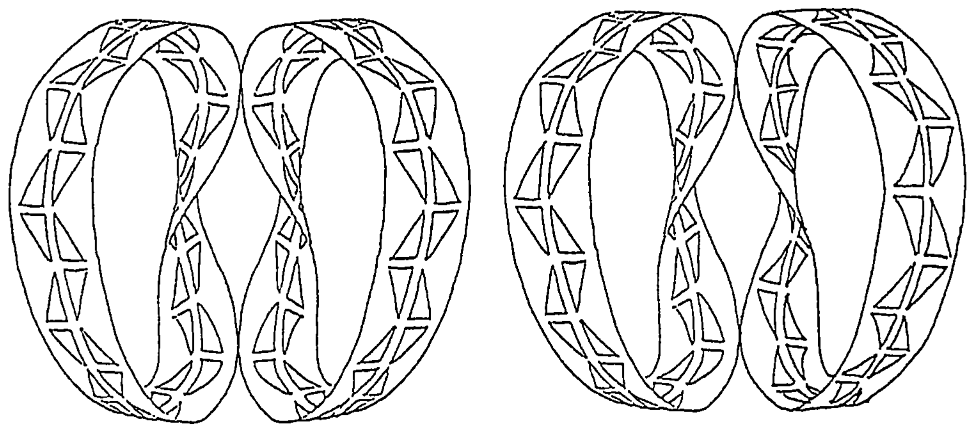


Figure 15: In-phase and not-in-phase doubling of the 2-*m* pattern of the initial Möbius-band (left part of each pairs in the figure). In-phase doubling results in *cmm* pattern (left double-Möbius-band), not-in-phase doubling results in *pmg* pattern (right double-Möbius-band) according to local neighbourhood relations of cells (with the triangle pattern element) in the final pattern of the Klein-bottle. The local structure can be observed when the patterns were stretched onto the plane; the global structure can be expressed, when the earlier pattern were given with the so called Born-Karman boundary conditions, which represent the feedback of the structure on itself, when it were closed.

REFERENCES

Bérczi, Sz. (1976) Növényi szimmetriák, [Plant symmetries, in Hungarian], *Fizikai Szemle*, 26 (2), 59-62.

Bérczi, Sz. (1979) A szabályos és féligszabályos (platoni és archimedészi) testek és mozaikok periódusos rendszere, [The periodic system of regular and semi-regular solids and mosaics, in Hungarian], *Középiskolai Matematikai Lapok*, 59 (5), 193-199.

Bérczi, Sz. (1980) The periodic system of Platonic and Archimedean solids and tessellations, *Acta Geologica Academia Sciencia Hungarica*, 23 (1-4), 184-200.

Bérczi, Sz. (1985) Symmetries in the plant surface lattice systems: Development of Fibonacci numbered structure in a cellular automaton model, Lecture at Intuitive Geometry Conference, Balatonszéplak, May, 1985.

Bérczi, Sz. (1986) Escherian and Non-Escherian developments of new frieze types in Hanti and old Hungarian communal art, In: Coxeter, H. S. M. *et al.*, eds., *M. C. Escher: Art and Science*, Amsterdam: North-Holland, pp. 349-358.

Bérczi, Sz. (1986) New double-frieze types in the Avarian communal art, lecture at the Symmetry Symposium and Exhibition, Darmstadt Technical University, Darmstadt, June 13-17, 1986.

Bérczi, Sz. (1987) Symmetry constraints in development and evolution of Fibonacci-plants, paper presented at the Symposium on Organizational Constraints on the Dynamics of Evolution, Budapest, June 29 - July 3, 1987.

Bérczi, Sz. (1989) Symmetry and technology in ornamental art of old Hungarians and Avar-Onogurians from the archaeological finds of the Carpathian basin, seventh to tenth century A.D., In: Hargittai, I., ed., *Symmetry 2, Computers and Mathematics with Applications*, Oxford: Pergamon Press, 17 (4-6), 715-730.

Bérczi, Sz. (1990) Symmetry constraints in development and evolution of Fibonacci plants, In: Maynard-Smith, J., Vida, G., eds., *Proceedings in Non Linear Science (IV), Organizational Constraints on the Dynamics of Evolution*, Manchester: University Press.

Bérczi, Sz. (1990) *Szimmetria és Strukturaépités*, [Symmetry and Structure Construction, in Hungarian], Lecture Note Series, J3-1441, Budapest: Tankönyvkiadó, 260 p.

Bérczi Sz.(1990) Local and Global Model of Fibonacci Plant Symmetries, In: Gruber, B. and Yopp, J.H., eds., *Symmetries in Science IV, Biological and Biophysical Symmetries*, New York: Plenum Press.

Bérczi Sz. (1990) Szimmetria és topológia: Rácsátrendeződések a Möbius-szalag — tórusz transzformáció során, Természet Világa, 10, 464-466.

Bérczi Sz., Lukács B., and Molnár I. (1991) On Symmetry and topology of the organisms in macroevolution, In: Lukács, B., Bérczi, Sz., Molnár, I., and Paál, Gy. eds., *Symmetry and Topology in Evolution*, Budapest: MTA-KFKI-1991-32/C, pp. 73-79.

Bérczi, Sz. (1991) Symmetry and topology in a 'cell-mosaic-automata' model of the Fibonacci-plant structures, In: Lukács, B., Bérczi, Sz., Molnár, I., and Paál, Gy. eds., *Symmetry and Topology in Evolution*, Budapest: MTA-KFKI-1991-32/C, pp. 80-90.

Bérczi, Sz. (1991) Platonic-Archimedean spherical cellular automata in the solution of the indirect Von-Neumann problem on sphere for transformations of regular tessellations, In: Lukács, B., Bérczi, Sz., Molnár, I., and Paál, Gy. eds., *Symmetry and Topology in Evolution*, Budapest: MTA-KFKI-1991-32/C, pp. 111-116.

Bérczi, Sz. (1991) Symmetry by cellular automata, Lecture on the Intuitive Geometry Conference, Szeged, September 1991.

Bilinski, S. (1952) Homogene Netze geschlossener orientierbarer Flachen, *RAD Bulletin International*, 277, Zagreb: Academie Yugoslave des Sciences et des Beaux-Arts, 129-164.

Codd, E.F. (1968) *Cellular automata*, New York: Academic Press.

Coxeter, H.S.M. (1969) *Introduction to Geometry*, 2nd ed., New York: John Wiley and Sons.

Grunbaum, B. (1987) *Tilings and Patterns*, New York: W. H. Freeman and Co.

Hargittai, I. ed., (1989) *Symmetry 2*, Oxford: Pergamon Press.

Koch, S. A. (1990) The evolution of viruses, In: Lukács, B. *et al.*, eds., *Evolution: Cosmogenesis to Biogenesis*, Budapest: MTA-KFKI-1990-50\C, pp. 69-76.

Lábos, E. (1991) *Attractors and knots*, In: Lukács, B. *et al.*, eds., *Symmetry and Topology in Evolution*, Budapest: MTA-KFKI-1991-32\C.

Lukács, B. (1993) On geometric symmetries and topologies of forms, *Symmetry: Culture and Science*, 4, 1, 93-103.

von Neumann, J. (1966) *Theory of Self-Reproducing Automata*, (ed. and completed by A. W. Burks), Urbana, Illinois.

Schattschneider, D., and Walker, W. (1977) *M. C. Escher Kaleidocycles*, New York: Ballantine Books.

Senechal, M. (1986) Escher designs on surfaces, In: Coxeter, H. S. M., *et al*, eds., *M. C. Escher: Art and Science*, Amsterdam: North Holland, pp. 97-109.

Vollmar, R. (1978) *Algorithmen in Zellularautomaten*, Stuttgart: Verlag B. G. Teubner.

Weyl, H. (1952) *Symmetry*, Princeton: Princeton University Press.