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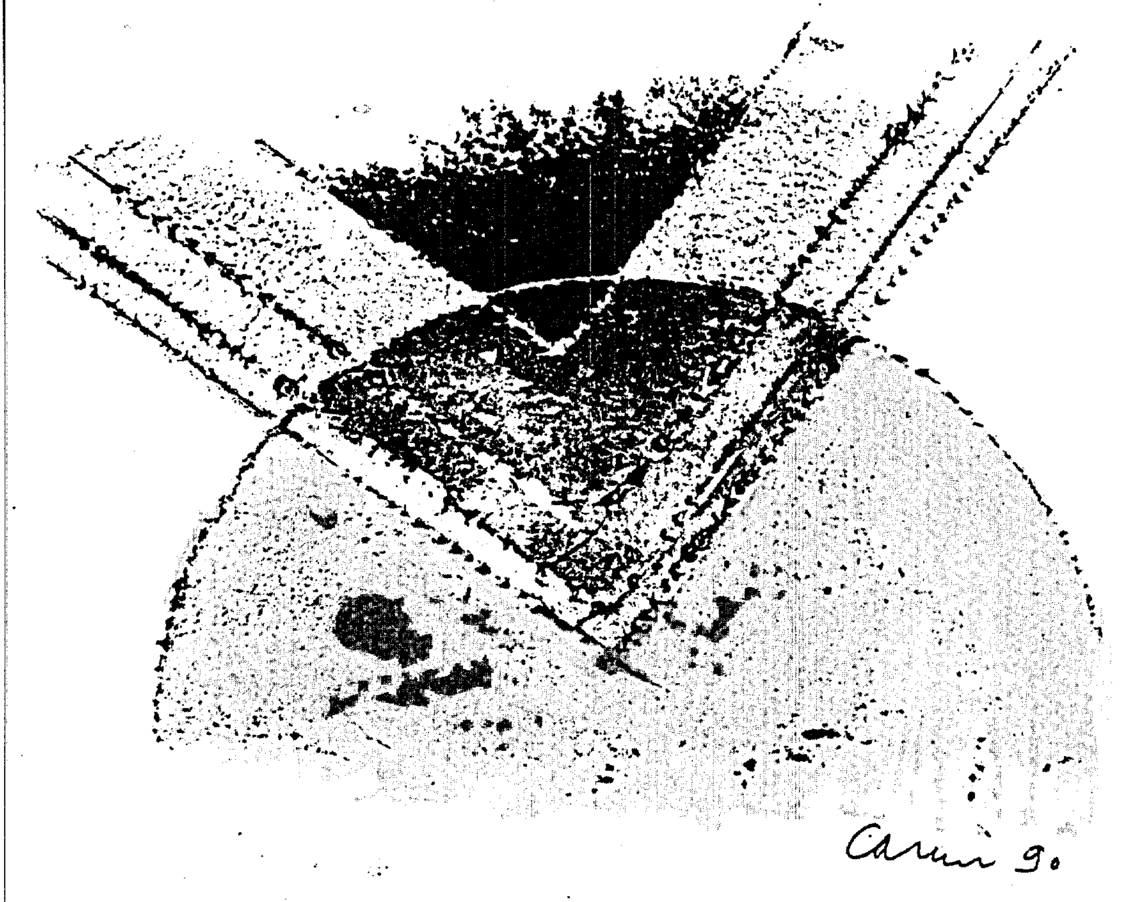

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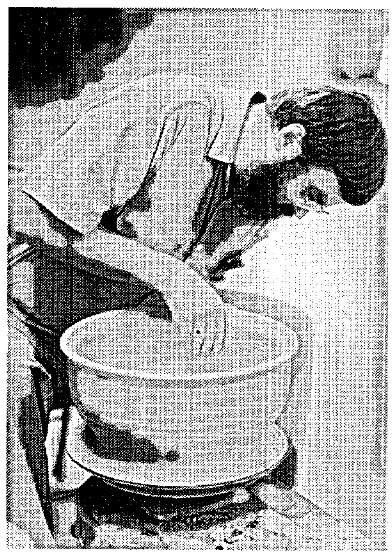
SYMMETRY: SCIENCE AND CULTURE

CONSTRUCTING TESSELLATIONS AND CREATING HYPERBOLIC ART

Raymond F. Tennant

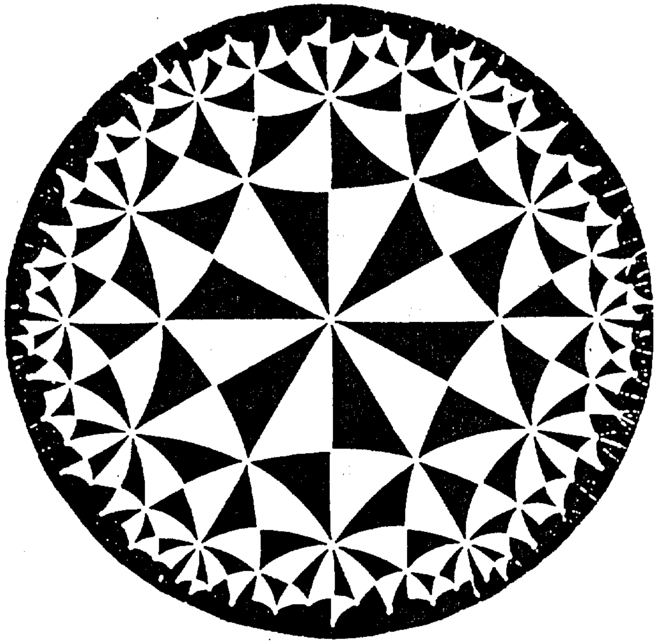
Mathematician, potter (b. North Adams, Massachusetts, 1950).
Address: Department of Mathematics, Statistics and Computer Science, Eastern Kentucky University, Richmond, KY 40475-3133, U.S.A.; E-mail: tennant@acs.eku.edu .
Fields of interest: Combinatorial group theory and topology, Euclidean and non-Euclidean geometry, mathematical symmetry, mathematical art, creating mathematical pottery.
Publications: The swap conjecture, *The Rocky Mountain Journal of Mathematics*, 23, 3, Summer, 1992 (with E. C. Turner); Tessellations with hyperbolic squares, The Journal of Recreational Mathematics, 1993; Small study groups in the classroom, The Caninus College Teaching Quarterly, 1, 1, September, 1992.



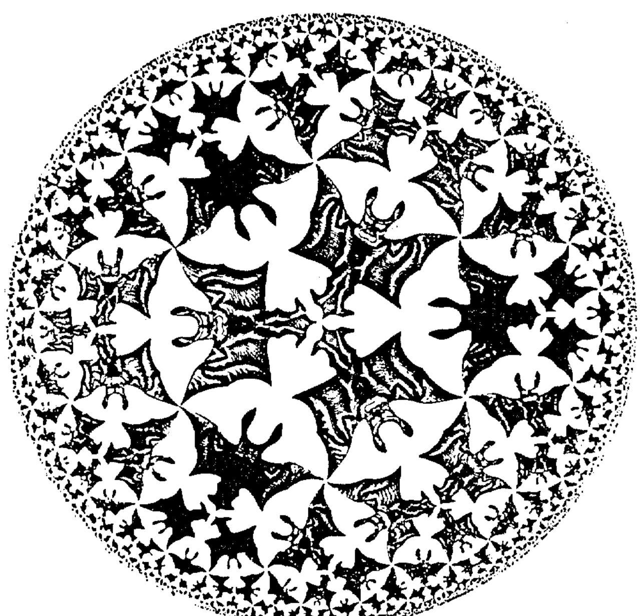
Abstract: *The article describes a method for geometrically constructing hyperbolic tessellations in the Poincaré disk with the aid of ruler, compasses, and protractor followed by techniques for transforming these tessellations into "Escher-type" patterns. It begins with a brief discussion of hyperbolic geometry and regular tessellations of the Euclidean plane. The concept of tessellation is then extended to Poincaré's disk and a detailed description of the tessellation by hyperbolic squares meeting six at a vertex is given.*

INTRODUCTION

The Dutch artist M. C. Escher (1898-1972) was perhaps the first to use hyperbolic geometry to create art. His inspiration came from a tessellation that was illustrated in a paper written by H. S. M. Coxeter in 1957. Below are Coxeter's *Illustration* and Escher's woodcut, *Circle Limit IV*, (Bool et al. 1981, p. 322).



(a) The Coxeter Illustration



(b) Escher's Circle Limit IV
Figure 1

These figures raise two questions:
1. How does one geometrically construct such a tessellation with the aid of ruler, compasses, and protractor?
2. How does one utilize the tessellation to create an artistic design?
Both of these questions are investigated in this paper.

THE POINCARE DISK

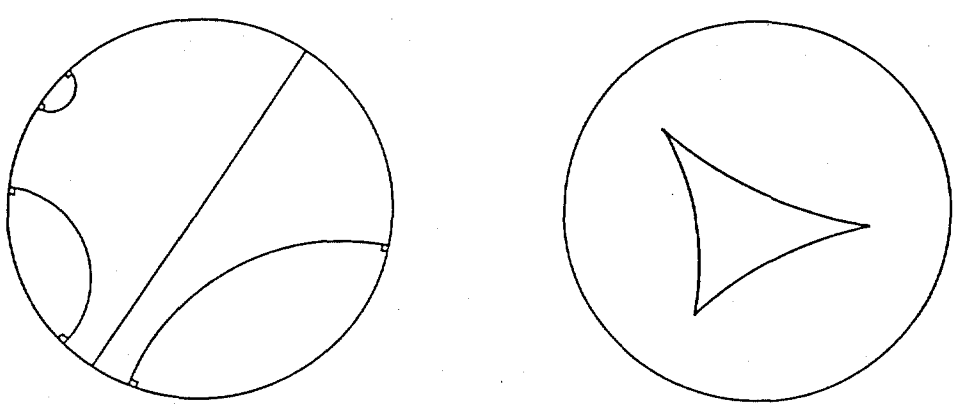
In the late 19th century, a model for hyperbolic geometry was developed by Henri Poincaré (1854-1912). Now known as the Poincaré disk, the model can be defined as consisting of all the points in the plane that lie inside a bounding circle C, with geometric concepts defined in the following manner:

Hiperbolic point. Any point interior to C.

Hyperbolic line. Any diameter of C or the portion of any circle that lies inside C and is perpendicular to it.

Hyperbolic angle. An angle between two hyperbolic lines is the Euclidean angle between the circles that represent them.

Hyperbolic triangle. A closed three sided figure formed by three hyperbolic line segments.



(a) Hyperbolic lines (b) A hyperbolic triangle

Figure 2

Hyperbolic geometry is a logical and self-consistent world but its geometry is different from the Euclidean geometry of our everyday experience so we can only hope to illustrate the hyperbolic plane and its geometry in a distorted fashion.

Imagine that the Poincaré disk is a shallow circular pool of radius ten meters that is filled with water and contains one lone inhabitant, a hyperbolic goldfish. As the goldfish swims around the pool, he remains the same hyperbolic size but because the Poincaré model distorts length to our 'Euclidean eyes', the goldfish appears to us to change size.

Specifically, if the goldfish has a Euclidean length of one meter when it is in the center of the pool then it will have Euclidean length equal to $(1\text{-}r^2/100)$ meters when it is r meters from the center. Hence, the distortion in the model makes the goldfish appear to shrink as it swims towards the boundary. So, hyperbolic length is distorted but equal hyperbolic angles are surprisingly represented by equal angles in the model so that for instance an angle of 45° in the model represents an angle of 45° in hyperbolic reality.

TESSELLATIONS

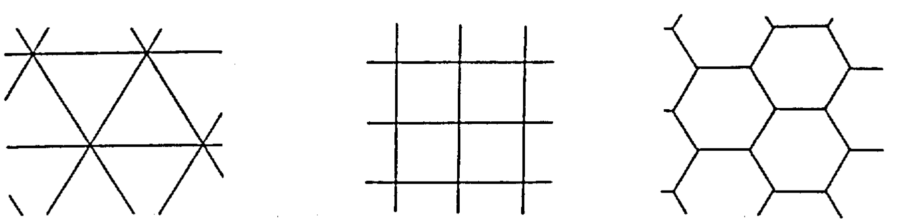
A regular tessellation or tiling of the plane is a covering, without gaps or overlaps, of the plane by congruent copies of a regular polygon. A regular tiling is descibed by two positive integers, $[p, q]$, where p is the number of sides on the tiling regular polygon and q is the number of these polygons that meet at a vertex. In the Euclidean plane, since the vertex angle of a regular p-sided polygon is equal to 180° - 360°/p and q polygons meet at a vertex, we have

$$q(180° - 360°/p) = 360°$$

or equivalently

$$(p - 2)(q - 2) = 4.$$

This leads us to the three familiar regular tessellations of the Euclidean plane.

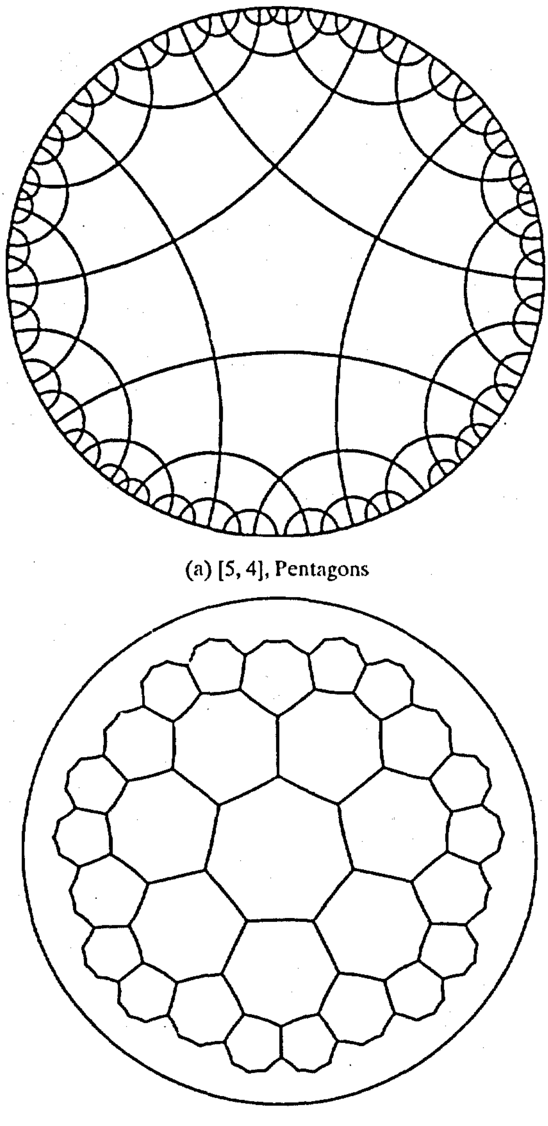


(a) [3, 6] (b) [4, 4] (c) [6, 3]

Figure 3

Now, we consider regular tessellations of the Poincaré disk in an analogous manner. Since the angles of a hyperbolic triangle add up to something less than 180°, the vertex angle of a regular p-sided polygon is equal to less than $180° - 360°/p$ and since q polygons meet at a vertex, we have

$$q(180° - 360°/p) > 360°$$



(a) [5, 4], Pentagons

(b) [7, 3], Septagons
Figure 4

Thus the regular hyperbolic tessellation, $[p, q]$, must satisfy

$$(p - 2)(q - 2) > 4$$

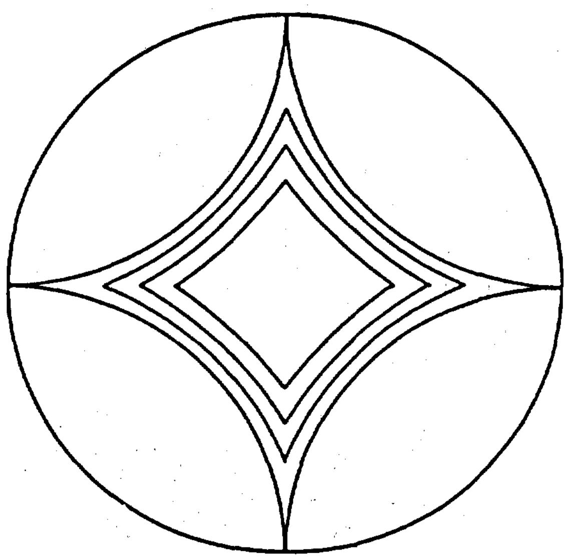
First, we notice that there are infinitely many possibilities for p and q. Two examples of hyperbolic tessellations are shown below. One tessellation is by pentagons meeting four at a vertex and the other is by septagons meeting three at a vertex.

In fact, for a given p, there are infinitely many potential values for q. In order to make our eventual construction of tessellations simpler, all of the regular hyperbolic tessellations that we consider will contain a centered regular p-sided polygon, that is, one polygon will lie directly in the Euclidean center of the Poincaré disk.

Consider the case where $p = 4$. These are tessellations by hyperbolic squares. There is one tessellation for each q where $4 < q \leq \infty$. As q increases, the centered regular square for the tessellation $[4, q]$ grows out towards the bounding circle C, and at the same time, the angles of the centered square decrease toward zero.

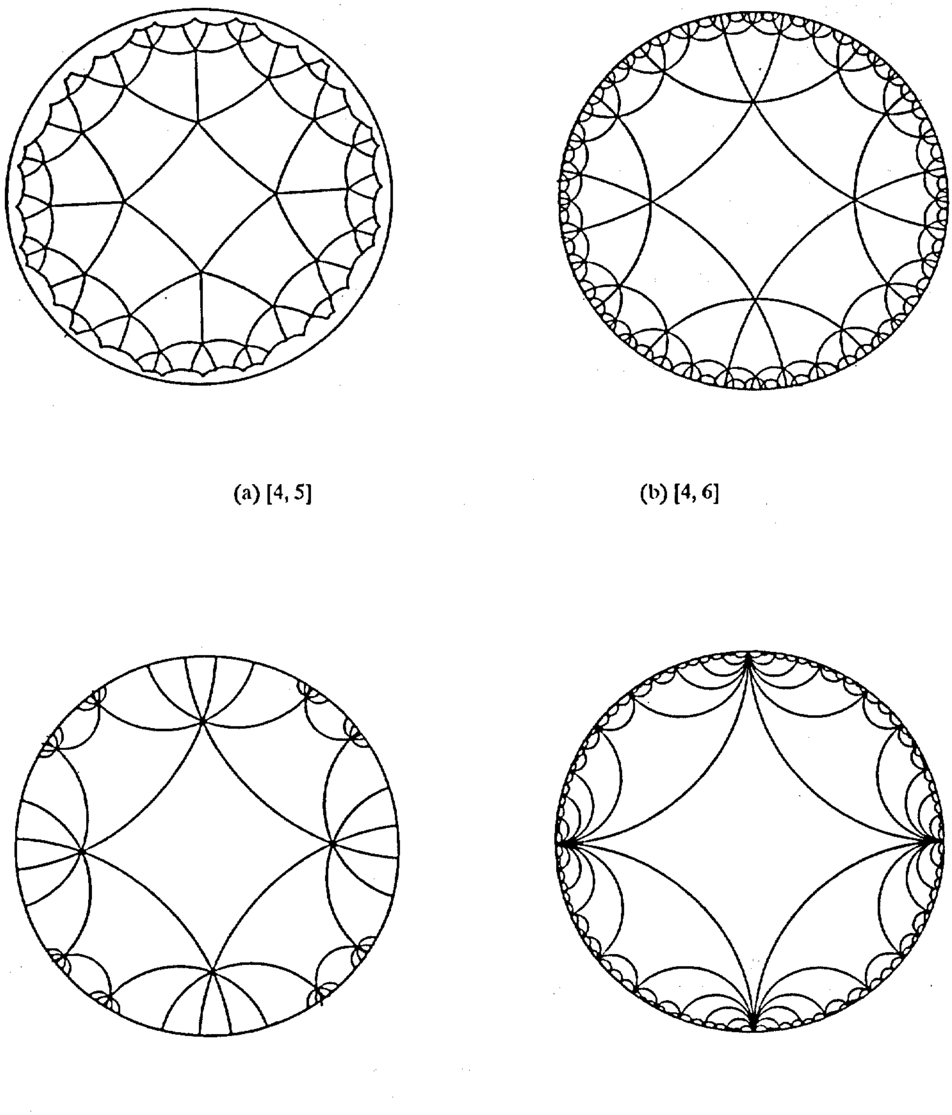
The case where $q = \infty$ is a special case because the vertices of the centered square as well as the vertices of all the squares in the tessellation lie on the bounding circle. It is included because it is an appealing tessellation and relatively easy to construct.

Some examples of centered squares are shown below.



Inside to out, $q = 5, 6, 8, \infty$ with angle 72°, 60°, 45°, 0° respectively
Figure 5: Concentric regular squares

These centered squares are pivotal to the four tessallations shown below.



(a) [4, 5] (b) [4, 6]

(c) [4, 8] (d) [4, ∞]

Figure 6

Note that the edges for the tessellation [4, 5] (Figure 6a) do not extend to the boundary of C because $q = 5$ is odd.

A GEOMETRIC METHOD FOR CONSTRUCTING A TESSELLATION

The first step in constructing the hyperbolic tessellation $[p, q]$ is to construct the centered p-sided polygon inside the bounding circle C. Assume that the bounding circle C is the circle of radius 1 centered at the origin of the Euclidean plane.

Let $(s, 0)$ be the vertex of the centered polygon which lies on the positive x-axis.

To construct the hyperbolic line segment in the 1st quadrant which forms a side of the centered polygon joined to $(s, 0)$, we must find the center and radius of the circle which represents this hyperbolic line segment.

Since the center of this circle lies on the line $y = x\tan(\pi/p)$, let the center equal $(h, h\tan(\pi/p)$ and let r equal the radius of the circle (Figure 7).

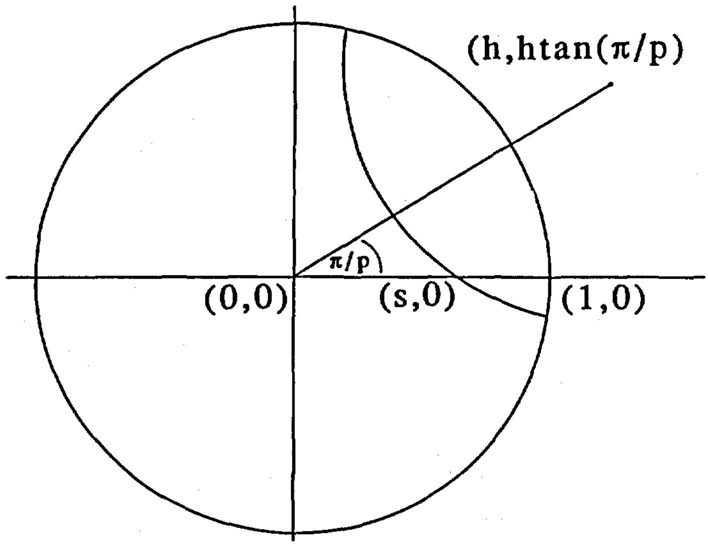


Figure 7

Since a vertex angle of the centered polygon is $2\pi/q$, calculus and algebra yield the following equations involving s, h, and r.

$$s = \sqrt{\frac{1 - \tan(\pi/p)\tan(\pi/q)}{1 + \tan(\pi/p)\tan(\pi/q)}}$$

$$h = \frac{s}{1 - \tan(\pi/p)\tan(\pi/q)}$$

$$r = \sqrt{h^2\sec^2(\pi/p) - 1}$$

To illustate this geometric method, the tessellation [4, 6] is constructed. This is a tessellation by squares where six squares meet at a vertex. The first step involves constructing the centered hyperbolic square. Substituting $p = 4$ and $q = 6$ into the equations above shows,

$$s=\sqrt{\frac{3-\sqrt{3}}{3+\sqrt{3}}} \qquad h=\frac{3s}{3-\sqrt{3}} \qquad r=\sqrt{2h^2-1}$$

Simplifying, $s \cong 0.518$ $\quad h \cong 1.225$ $\quad r = 2$

The three remaining centers involving the centered hyperbolic square are found using symmetry. Inside the bounding circle C, the four hyperbolic lines are constructed to form the centered hyperbolic square. Outside the bounding circle C, the four centers are connected to form a Euclidean square. The diagonals and perpendicular bisectors for the outer square are constructed. This divides the centered square into eight congruent right triangles. This triangulation will be carried out on all the squares in the tiling to aid in the construction. These additional hyperbolic lines will be represented by dashed arcs while the edges of the squares are represented by solid arcs. (Figure 8)

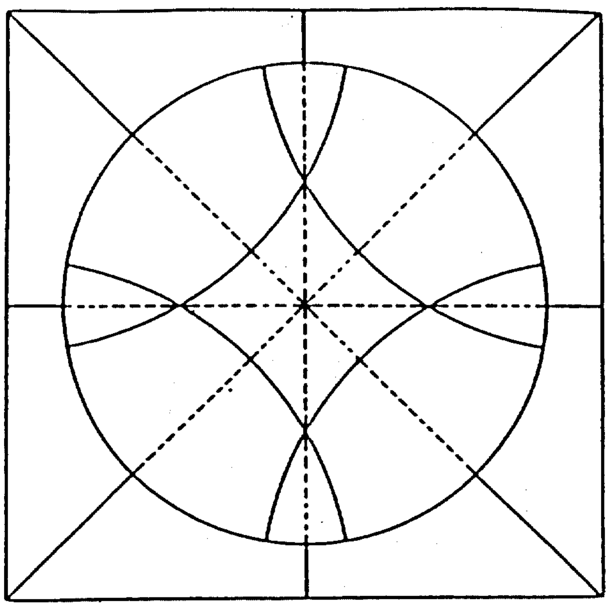


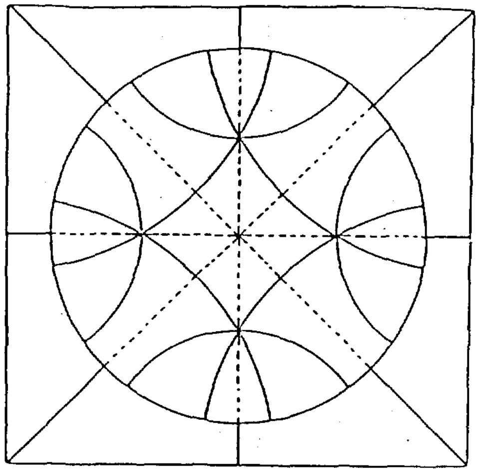
Figure 8: 1st construction step, [4, 6]

We prepare for the remaining steps in the construction by listing three theorems that will play a key role.

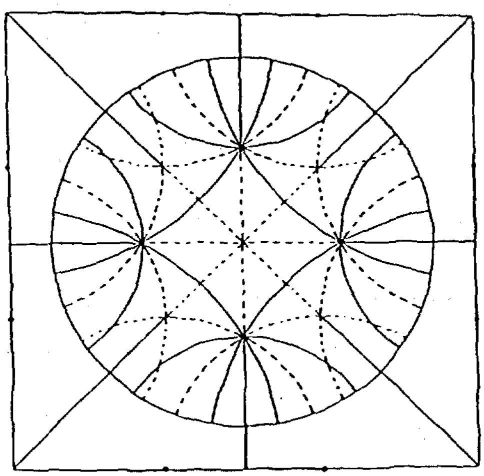
The Tangent Theorem. A tangent to a circle is perpendicular to the radius at the point of tangency.

Corollary to the Tangent Theorem. The center of a circle, whose arc forms a hyperbolic line, lies outside the bounding circle, C.

The Collinear Center Theorem. If hyperbolic lines pass through the same point inside the bounding circle C, then the centers of their respective circles in the Euclidean plane are collinear.



(a) 2nd construction step, [4, 6]



(b) 3rd construction step, [4, 6]
Figure 9

At each vertex of the centered square, there are three remaining circular arcs that we need to construct. By the Collinear Center Theorem, we know that the centers of these circles must lie on the line connecting the other centers associated to this vertex. Since the angles between neighboring arcs are 30°, we can find the centers of the arcs. Whenever two circular arcs intersect at a point where another circular arc is known to pass, connect their centers with a straight line segment (Figure 9).

We continue the construction outward from the center by repeating this step (Figure 10). It is noted that care should be taken when making calculations of distances in the construction since an error on an early step may be magnified into a larger error on a subsequent step.

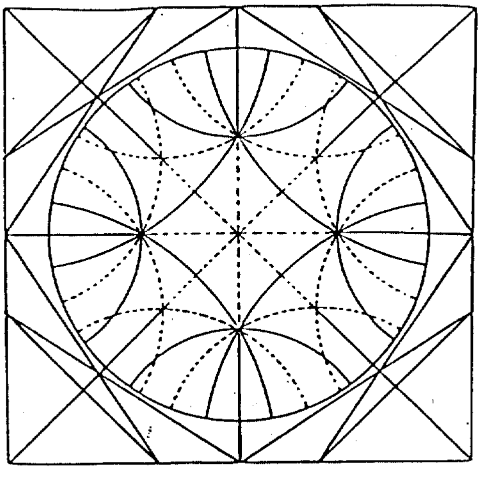


Figure 10: 4th construction step, [4, 6]

On subsequent steps the tessellation moves out towards the bounding circle C and the Euclidean frame containing the centers of the circles moves in closer to the bounding circle. In Figure 11, most arcs have not been extended to the boundary to make it easier to see the tessellating squares.

In general, the construction begins by choosing integers p and q for the tessellation $[p, q]$ that we wish to construct. After drawing the bounding circle C, use p and q to determine the centers and radii of the circular arcs that form the sides of the regular centered p-sided polygon. This is accomplished by using the following facts:

(a) p determines lines through the origin on which the centers lie.

(b) q determines the vertex angles of the centered p-sided polygon.

(c) the centers of the circular arcs form a regular Euclidean p-sided polygon outside of C.

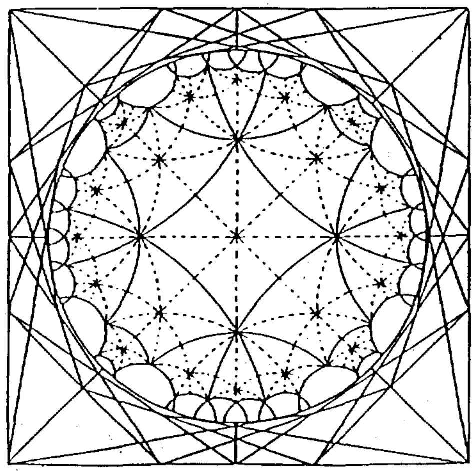


Figure 11: 5th construction step, [4, 6]

Note, the equations involving s, h, and r for finding the centered polygon, that were stated earlier, will work for any p and q.

Then the tessellation is constructed outward in the same manner as the tessellation by hyperbolic squares described previously.

CREATING HYPERBOLIC ART

Once a tesselation has been constructed, it can be used as a starting point for creating a hyperbolic design. As an example, we will use the hyperbolic tessellation, [4, 6], that was constructed above. One technique for transforming a tessellation into a pattern involves first changing the tessellation into a black and white tiling. This is a coloring of the tessellation in black and white so that some rotational symmetry is preserved at the centers of the hyperbolic polygons and at the vertices where the hyperbolic polygons come together. The black and white tiling, (Figure 12), has 180° rotational symmetry at the centers of the hyperbolic squares and 60° rotational symmetry at the vertices where the squares come together.

The next step is to form the black or white regions into a new shape. Since each white region is bounded by black regions and vice versa, by altering all the black regions, the white regions will automatically change. In the tessellation above, if we transform the black triangles into devils (Figure 13), we arrive at a ariation of Escher's *Circle Limit IV* (Figure 14). Actually, the tessellation, [4, 6] used in this pattern is dual to the tessellation, [6, 4], used in Escher's *Circle Limit IV*. The choice in center for each tessellation is the major difference.

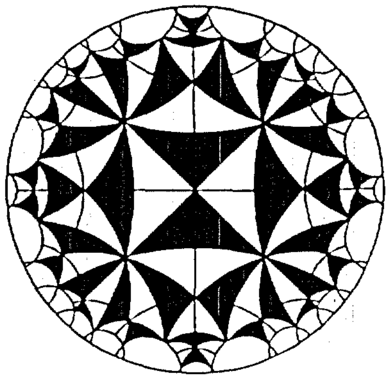


Figure 12: Black and white tiling

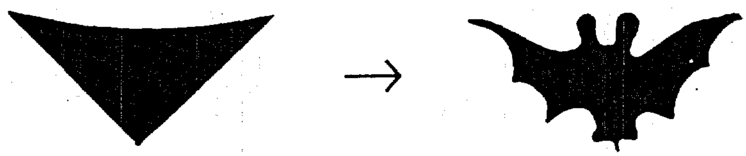


Figure 13: Triangles to devils

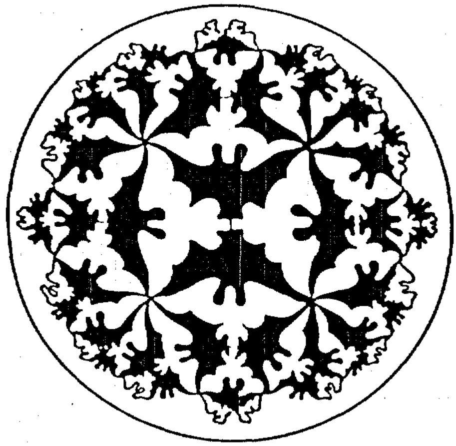


Figure 14: Angels and Devils

The distortion of the size and shape of the tiles in the Poincaré' model masks imperfections of sketching and this allows the aesthetic power of symmetry to shine through. To create other hyperbolic patterns, six black and white tilings for the tessellation [4, 6], containing rotational symmetry, are shown below (Figure 15).

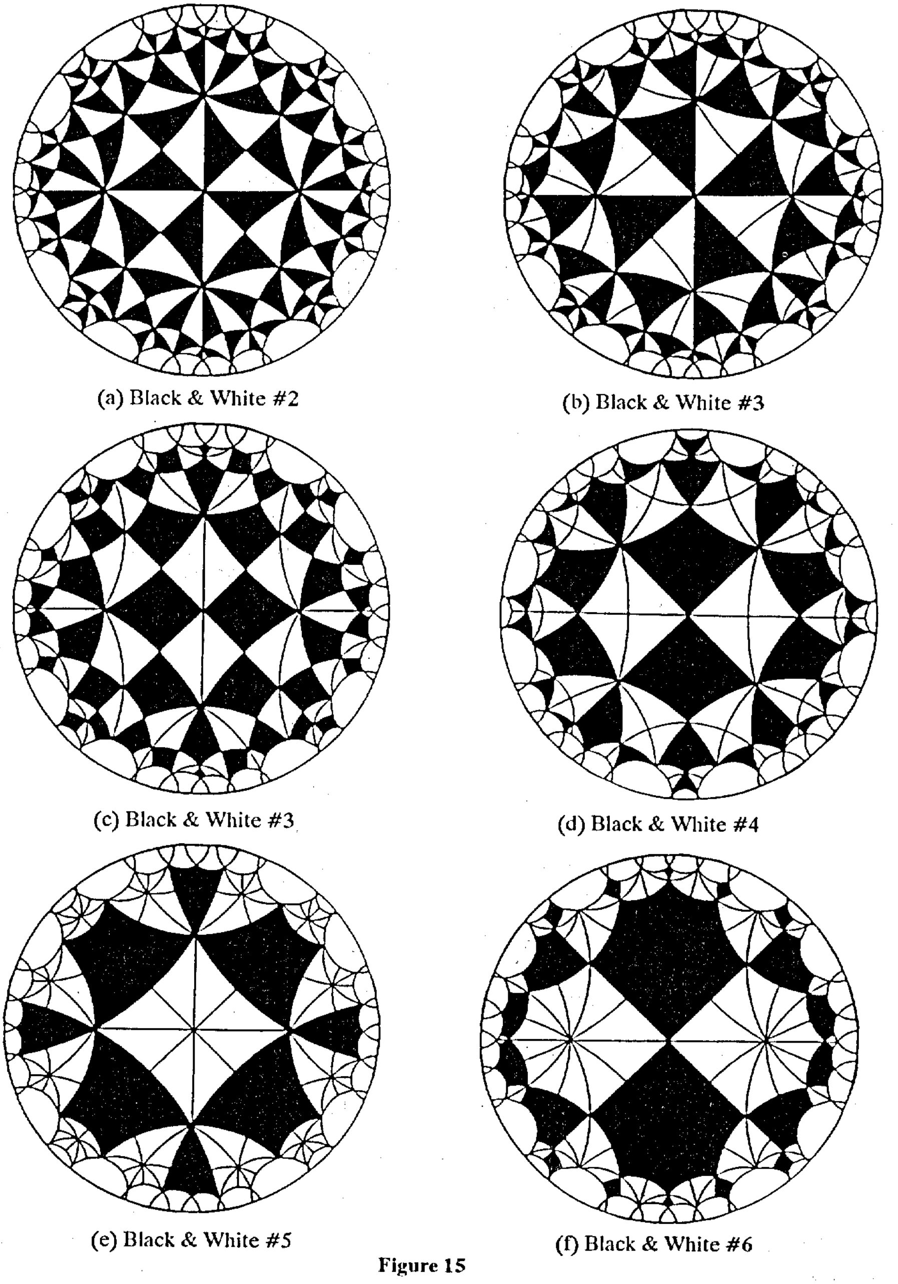


Figure 15

Another method for creating hyperbolic patterns involves using the tesellation as an outline and sketching designs inside the tiles. As an example, we start with the tessellation, [4, 6] (Figure 16).

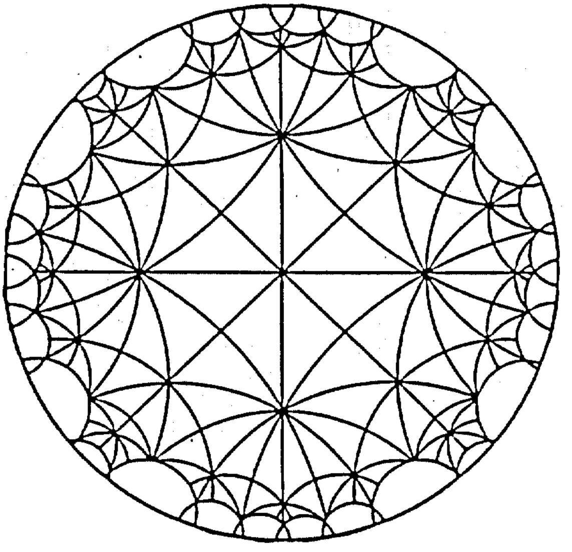


Figure 16: Tessellation [4, 6]

The word 'Zoo' is sketched in a tile (2 triangles) from the tessellation, [4, 6] (Figure 17).

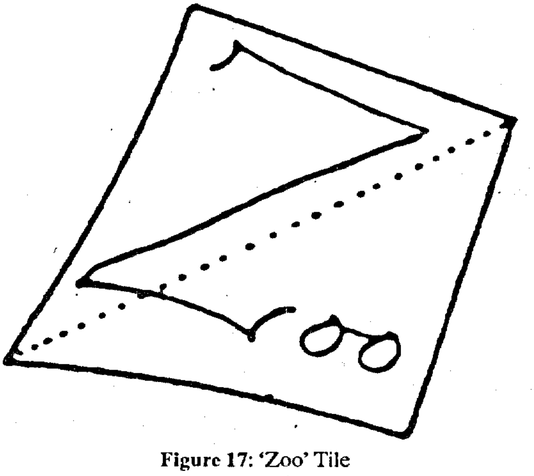


Figure 17: 'Zoo' Tile

This tile is then reflected around the tessellation, [4, 6], to create the hyperbolic pattern, 'Butterflies and Flowers' (Figure 18), with the six-petalled flowers appearing in between the butterflies.

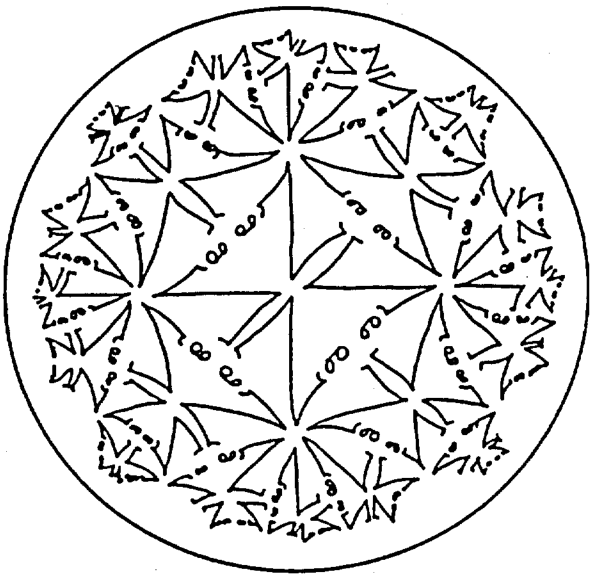


Figure 18: 'Butterflies & Flowers'

COMPUTER DESIGNS

In recent years, computer programs have been written to produce hyperbolic designs. A program created by Dunham, Lindgren, and Witte (1981) generates the designs by describing the symmetry transformations in terms of 3 by 3 real matrices.

Dedication. The author wishes to dedicate this article to the memory of his sister, Thelma Marie Dubrieul, who was one of M. C. Escher's biggest fans.

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