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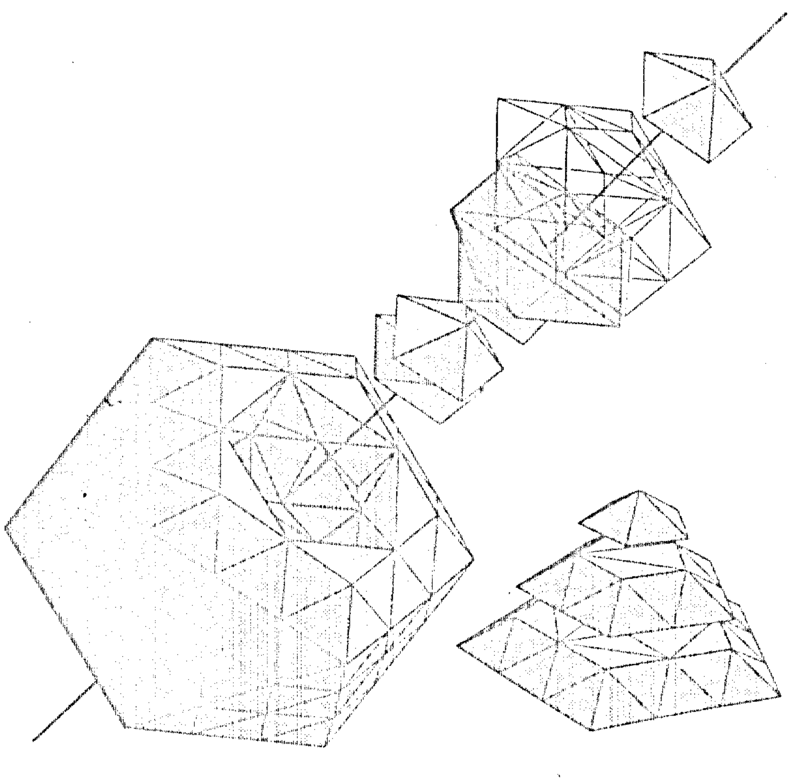

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GROWING ICOSAHEDRA

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It was discovered that by dividing an icosahedron into ten types of small modules, we can construct various 5-fold symmetrical polyhedra by recombining the modules again radially and symmetrically. This discovery suggests possible new paths of interaction between polyhedra, which are the subject of intense debate with respect to quasicrystals.

In crystallography, periodicity defines a structure that has both rotational and translational symmetry. Because of this restriction, 5-fold symmetry was ruled out in possible solutions to the question of the structure of crystals. However, 5-fold quasicrystals exist and have now been discovered. To cope with these inconsistencies, researchers are going to have to expand the framework of traditional crystallography.

In fact, all quasicrystals discovered so far have 5-fold symmetry. The Penrose lattice is a powerful means of coping with such structures, because it avoids too rigorous rotational and translational symmetry, and it provides us with an elegant order.

In geometry, there exist polyhedra with rigorous 5-fold symmetry. For example, if we project a dodecahedron or an icosahedron on a screen, we can see regular pentagons.

I first divided an icosahedron into ten types of small modules, then reconstructed various polyhedra by combining the modules radially and symmetrically. I found a beautiful hierarchical structure in the polyhedra, where in each layer in the reconstruction process there appear various 5-fold symmetrical polyhedra including the dodecahedron, icosahedron, and rhombic triacontahedron. Furthermore, it is possible to fill the entire space in these models while maintaining a hierarchical structure in the filling pattern.

Although this discovery was done outside the territory of translational symmetry in crystallography, it is quite suggestive of a solution to the structure of 5-fold symmetrical crystals that grow radially and symmetrically in the natural world.

Crystallography first began as a branch of geometry. From the 18th century until the early 20th century, crystallography researchers were developing an area of pure geometry of the structures of crystal lattices and the generalization of their patterns by investigating the symmetries which were revealed by measurements of the angles between adjacent faces of mineral crystals. They established the crystal lattice theory before the famous X-ray crystal structure analysis was invented by M. von Laue in 1912, which is deemed the monumental start of modern crystallography.

Crystal structure analysis is a method by which we can decide the atomic arrangement in a crystal by making a diffraction image of a crystal with an electron beam or X-ray. Note that the diffraction image does not show the edges and faces of a unit cell in the crystal directly. Instead, it only shows a transformed image which represents the relation between atoms in the crystal. Thus, in the image, the positions of the atoms are coded in an array of 2-dimensional dots.

Hence, information about distance between atoms (or bond length), their bond angles, and the space group is needed in advance in order to determine the shape and scale of the unit lattice from the diffraction image. In other words, the atomic arrangement cannot be determined without a minimal geometrical structure and the pattern of a 3-dimensional model.

Laue succeeded in adjusting the scale of independent geometrical concepts and the principles found in nature to allow them to coincide with each other and proved the correctness of the geometrical modeling method.

Now, we can visualize the atomic arrangement in a 3-dimensional form by using a geometrical approach with the three basic visual concepts of geometry (vertex, edge, and face) which allows us to reveal the relationship between dots in the diffraction image.

Technological advances in crystal structural analysis since Laue's invention have brought in a number of discoveries of quasicrystals with 5-fold symmetry which are crystals whose existence has never before been conjectured. The first quasicrystal Al-Li-Cu alloy was discovered by Hardy and Silcock in 1955, 30 years before the name 'quasicrystal' was first used for a newly discovered Schechtmanite Al-Mn alloy.

In 1986, a French researcher B. Dubost composed a very large quasicrystal of 1 millimeter in diameter, which contained 10^{20} atoms! It is an $Al_6Li_3Cu_1$ alloy whose shape is a complete rhombic triacontahedron. In 1988, A. P. Tsai and others of the Metal Material Laboratories of Tohoku University succeeded in composing a quasicrystal $Al_{65}Cu_{20}Fe_{15}$ of 2 millimeters in diameter whose shape is a complete dodecahedron.

GEOMETRICAL MODELING OF QUASICRYSTALS

A 3-dimensional geometrical model which can cope with 5-fold symmetry is necessary in order to investigate the structures of these new quasicrystals. There have been a number of proposed models in recent crystallography. Proposed models are, of course, decisively different from traditional ones. However, most of them are based on the traditional closest packing of spheres of the same diameter, and the all-space filling by two types of parallelepipeds called A_6 and O_6, respectively.

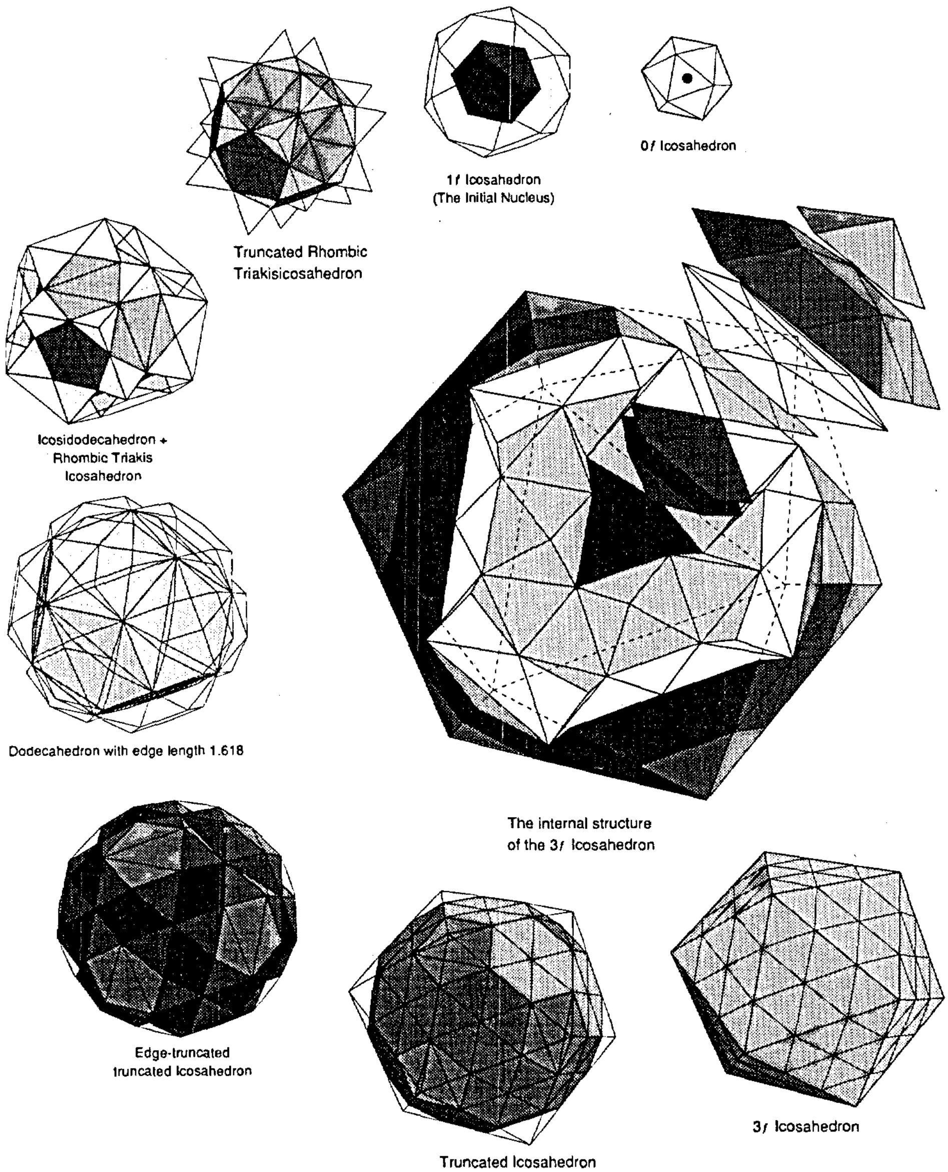


Figure 1: Hierarchical structure model

The model fills the space hierarchically starting from an icosahedron. The shape of each shell maintains the icosahedral 5-3-2 symmetry through the growing process. Some Platonic regular polyhedra and Archimedean semi-regular polyhedra which have 5-fold symmetry can be seen here. These polyhedra will fill space indefinitely, repeating a self-similar hierarchy every time their edge lengths become three times greater than before.

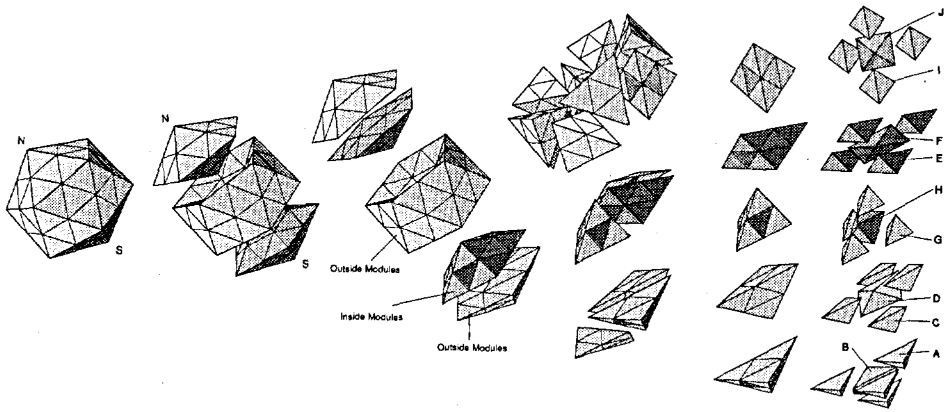


Figure 2: Synergetic modules

The model I discovered in which the space will be filled hierarchically on the basis of icosahedron consists of ten types of modules called the *synergetic modules*. These modules are obtained by dividing the icosahedron into two pentagonal bipyramids and one scooped out pentaprism and continuing to divide them at vertices, midpoints of edges, midpoints of diagonals and so on. The resulting ten types of modules are thought of as ultimate units in this division.

All of five tetrahedron modules *A, C, G, E, I* and five octahedron modules *B, D, H, F, J* are composed of only triangular faces. The volume of an octahedral module is four times as large as that of the corresponding tetrahedral module. (For example, the volume of *F* is four times as large as that of *E*.) The tetrahedron I is known to be able to constitute the whole icosahedron by itself. Let the volume of *I* be 1. Then, the volumes of the other tetrahedra *G, A*, and *C* are $1/\tau$, $2/\tau$, and $2/\tau^2$, respectively, where τ means the golden ratio.

The synergetic modules can be classified into the outside modules *A, B, C, D, I, J* whose face can be seen from outside, and the inside modules *E, F, G, H* which are hidden inside the icosahedron. Though icosahedra cannot fill the space without gaps, these ten types of modules can fill 5-3-2 symmetric space.

The closest packing of spheres of the same diameter has served as the traditional geometrical structural model to represent crystal structures. However, it cannot create polyhedra which have 5-fold symmetry, such as an icosahedron, a rhombic triacontahedron, or a dodecahedron composed of the same sized regular pentagons.

To construct an icosahedron by the closest packing of spheres, we have to reduce the size of the central sphere, around which 12 spheres of the same diameter can be arranged and if we want to enlarge the icosahedron by adding further spheres around it, it will soon turn out to be impossible because the bond angles and distances between spheres cannot be maintained exactly. Gaps in the outer shell will stop the growth.

This means that the closest packing sphere model cannot cope with even the simplest icosahedron and is thought to be inappropriate for the 3-dimensional geometrical model of quasicrystals which have 5-fold symmetry.

S. Baer is the first researcher who in 1970 discovered that A_6 and O_6 can yield 5-fold symmetry. He constructed a rhombic triacontahedron using ten parallelepipeds each of which is classified into either A_6 or O_6. The lengths of the diagonals of each parallelepiped equal the golden ratio. Then, for the first time, he developed a 3-dimensional space filling model, in which the internal construction of the units is non-periodic and the shape has 5-fold symmetry. He also succeeded in combining rhombic triacontahedra by overlapping adjacent ones without losing the 5-fold symmetry. He tried another all-space filling model using parallelepipeds whose diagonal ratios are different from those of A_6 and O_6, and found that 120 parallelepipeds of five types can fill a 5-fold symmetric rhombic enneacontahedron non-periodically.

In 1981, A. L. Mackay discovered 3-dimensional Penrose tiling and it became known that the parallelepipeds A_6 and O_6 are the units which can also fill a rhombic icosahedron and a rhombic dodecahedron along with the rhombic triacontahedron. However, it was impossible to fill polyhedra with 5-fold symmetry like an icosahedron or a dodecahedron.

Since the discovery of quasicrystals, it seems that only these approaches have been tried in order to prove the possibility of 5-fold symmetry in the 3-dimensional geometrical model. Former space filling models in 3-dimensional geometry aimed at filling the entire space.

However, I thought that the fundamental problem of these space filling models with respect to quasicrystals was how to construct a pure geometrical space filling model in which an asymmetrical or non-periodic internal structure could comprise a symmetrical outer shape. I believed that pursuing the units by which 5-fold symmetric polyhedra can be composed would lead to a new geometry of symmetry. This was the motivation that started me to investigate a generalized geometrical all-space filling model which shows the way to fill any closed 5-fold symmetrical polyhedron without gap or inconsistency.

HIERARCHICAL MODEL OF ICOSAHEDRON AND SHELL FILLING

In October 1989, I found that an icosahedron can be divided into a number of triangularized modules of ten types and that they can comprise more than one shell of different geometrical structures which grow concentrically and hierarchically to make up an icosahedron with 5-fold symmetry. As a result of my particular asymmetrical modular divisions, the outermost shell of the icosahedron shows a regular triangularized pattern in which the length of each edge is an integer multiple of the original edge length. This means that an icosahedron can grow by this multiplication.

My ten new types of modules are called *synergetic modules*. These modules can all-space fill higher frequency icosahedrons each of whose faces is a lattice of regular triangles. We can make an icosahedron grow by dividing its edges into a number of equally long parts and connecting these divisions in a triangular manner. The number of these divisions is called the frequency (f) of the icosahedron.

My icosahedron grows symmetrically in the radial direction. *Synergetic modules* have both growth ability and interchangeability in this hierarchy. All-space filling models with such characteristics have never been reported in the past history of geometry, physics or crystallography.

In the growth hierarchy from a $1f$ (one frequency) icosahedron with edge length 1.0 to a $3f$ (three frequency) icosahedron, there appear recognizable symmetrical polyhedra, including the truncated icosahedron with edge length 1.0, the icosidodecahedron with edge length 1.0 and the dodecahedron with edge length 1.618. All polyhedra in this hierarchy have perfect 5-3-2 symmetry.

Note that the 5-3-2 symmetry is seen in the exterior shape of these polyhedra, not in the internal arrangement of modules nor in the triangularized pattern on their face. In fact, there is more than one combination or arrangement of the modules for each shell.

On the outside layer of the $3f$ icosahedron, the self-similar patterns of all the shell structures appearing in the hierarchy between the $1f$ and $3f$ icosahedron are replicated by a multiple of 3, because the number of divisions, or frequency, has to be a multiple of 3 in order to make each of the 20 vertices of the dodecahedron contact the center of the triangle lattice on the corresponding face of the outer icosahedron.

The $3f$ icosahedron can be considered as a minimum shell structure in the hierarchy in the sense that all the ten types of modules are used in it for the first time. In the $6f$ (six frequency) icosahedron, all the shell structures will appear again with their edge lengths doubled. However, in the $4f$ (four frequency) and $5f$ (five frequency) icosahedra, some of the layers are lost. That is, not all frequencies make icosahedra with complete hierarchical structures.

Super-high-frequency icosahedra are filled by iterating the hierarchy of the shell structure hierarchies. Thus, we can consider the concentrically expanding hierarchy of the polyhedra with 5-fold symmetry, a hierarchy which recurs periodically.

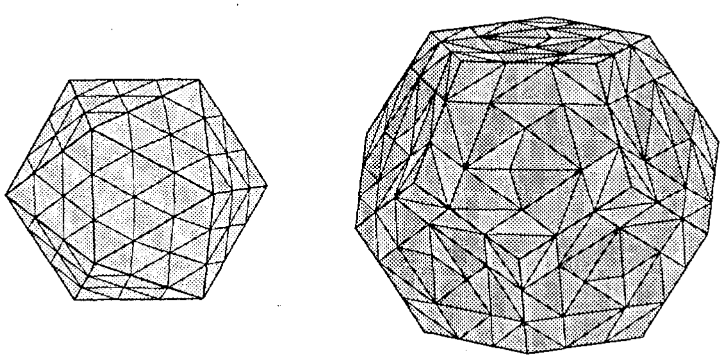


Figure 3: The $3f$ icosahedron and $3f$ dodecahedron
This figure illustrates the contrast between $3f$ (three frequency) icosahedra whose faces are composed of only outside modules and $3f$ (three frequency) dodecahedra whose faces are composed of only inside modules.

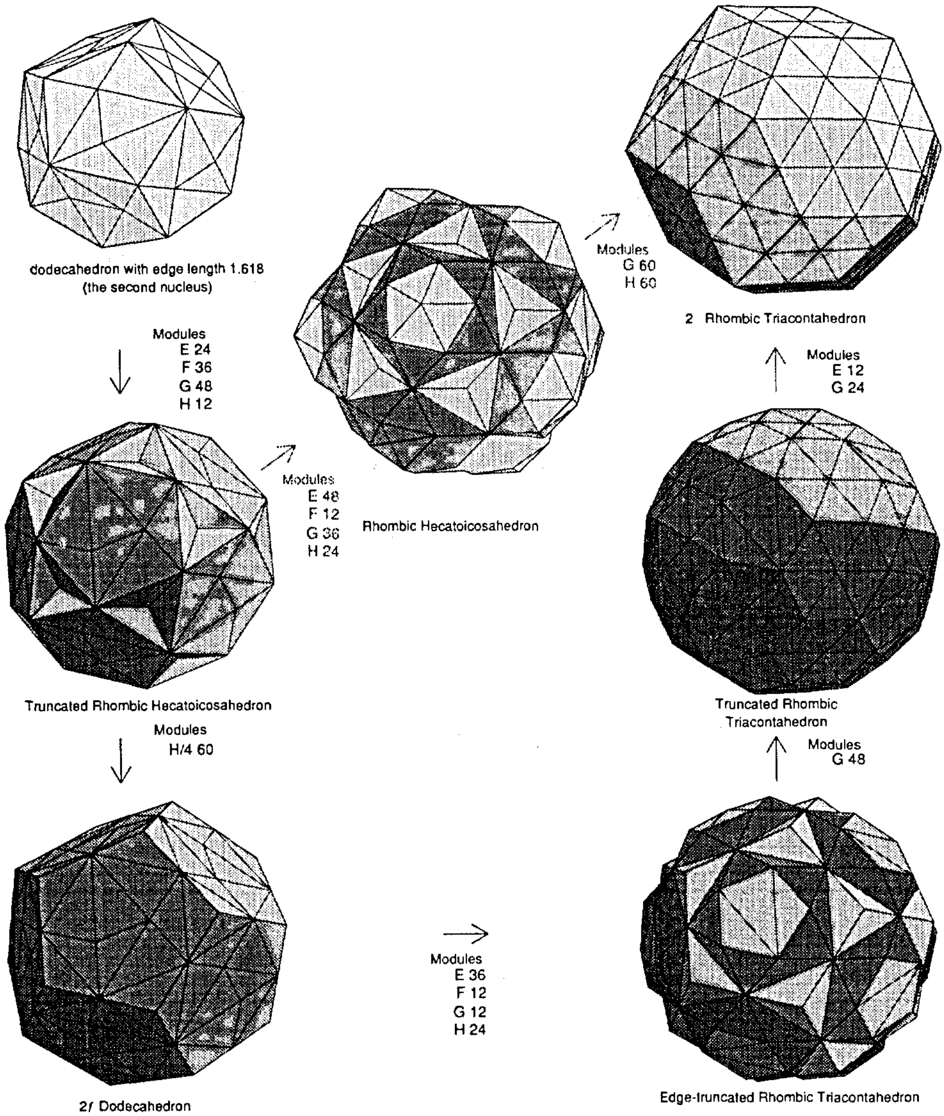


Figure 4: Rhombic triacontahedron

If we change the combination of modules from the first dodecahedron, we can obtain a rhombic triacontahedron which also has 5-3-2 symmetry. After that, the periodicity of the hierarchy will be 2. Beyond the first dodecahedron, only inside modules are used to make further shells. The face of the rhombic triacontahedron becomes a lattice of isosceles triangles. According to the model, two systems of the forms are possible. Quasicrystal alloys of millimeter size whose shapes are a rhombic triacontahedron and a dodecahedron have already been created. This choice from two may be explained by the hierarchical model for the super-high frequency rhombic triacontahedron. If this is the case, there may be an icosahedral quasicrystal alloy of three metal elements. The number immediately following a module name indicates the number of modules to be used in the growth step. Note that it is a multiple of 12.

So far, icosahedra with 5-fold symmetry have been thought impossible to all-space fill without inconsistencies or gaps. However, it has now been proved that growing icosahedra can be all-space filled by using the *synergetic modules.*

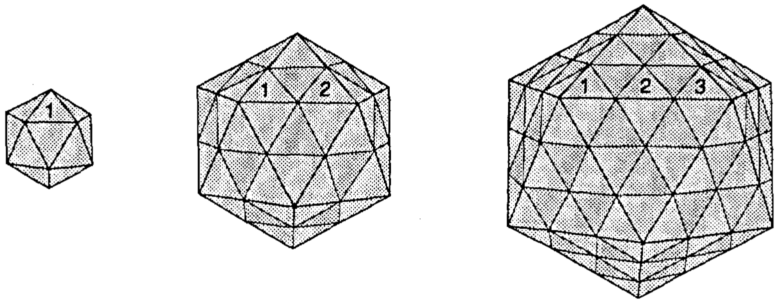


Figure 5: Icosahedral cluster structure with perfect 5-fold symmetry
The $1f$ icosahedron can be composed of only tetrahedral modules *I*. (Here, we use the same notation $1f$, $2f$, ...) The $2f$ and $3f$ icosahedra cannot be composed of *I* modules only. We have to use the corresponding octahedral module *J*. These icosahedra are formed by radially and symmetrically combining 20 tetrahedral subunits composed of modules *I* and *J*. There are no shell structures other than the icosahedron.

SYNERGETIC MODULES

The north pole and south pole among the 12 vertices of the icosahedron always function additionally to the system. In the formation process of the *synergetic modules*, the abstractness of the additive twoness is indeed replayed visually. These two poles are clearly distinguished from other vertices.

To obtain the *synergetic modules*, we have to divide an icosahedron into three basic parts first: the arctic part, the equator part, and the antarctic part, all of which are symmetrical with respect to an axis penetrating through the north and south poles. The arctic and antarctic parts are two identical pentagonal bi-pyramids and the equator part is a ring shape which is a pentaprism with both top and bottom scooped out. The volumes of the pentagonal bi-pyramid poles and the scooped out pentaprism are 5 and 10, respectively, if the volume of the icosahedron is assumed to be 20.

Next, we divide each pentagonal bi-pyramid into two (outer and inner) pentagonal pyramids by a plane perpendicular to its axis. This split is fundamentally related to the golden ratio. Further, we divide each pentagonal pyramid into three tetrahedra. We also divide the scooped out pentaprism into ten identical tetrahedra. Now, we have five types of tetrahedra, two from the outer pentagonal pyramids, two from the inner pentagonal pyramids and one from the equator scooped out pentaprism.

If we continue dividing each tetrahedron into four small similar tetrahedra and one small octahedron by splitting it at lines connecting its edge midpoints, we obtain five types of small tetrahedra and five types of small octahedra. Of course, the volume of each octahedron is four times as large as that of the corresponding tetrahedron.

All of the *synergetic modules* have triangular faces. However, there are only five types of triangular faces. Four of them are triangles with two equal sides, i.e, isosceles triangles with edge lengths 1.0 (the edge length of the original icosahedron), 0.95 (the distance between the center of the icosahedron and one of its vertices), or 1.618 (the length of a diagonal of the pentagonal face of the icosahedron). The other triangular face type is an equilateral triangle.

COMPLEMENTARINESS OF THE SYNERGETIC MODULES

The ten types of modules are classified into six outside modules and four inside modules, depending on where they are located in the original icosahedron. Modules can be joined to each other if they have the same mirror image face. However, the outermost face of a growing icosahedron can consist of only equilateral triangles with edge length 1.0, which is common to all outside modules.

The four types of inside modules can be arranged to form another shell inside the growing icosahedrons outer shell. Outside modules can in turn be arranged to form another inner shell to link with more inner shells. If faces of inside modules form some intermediate polyhedron, this polyhedron will not be an icosahedron. In other words, the formation of shells is intrinsically related to the complementariness of the outside and inside of the icosahedron.

Tetrahedral modules and octahedral modules are joined to each other in keeping with this complementary relationship. The first $1f$ icosahedron is composed by joining only tetrahedral modules with each other. However, in all other shell structures, each tetrahedral face must be joined to the face of an octahedron that can be joined to another tetrahedron on another face. That is, neither tetrahedral modules nor octahedral modules alone can fill these shells.

NUCLEUS FOR GROWTH

Historically, since they are dual to each other, we have not been able to determine which polyhedron is more fundamental, the icosahedron or the dodecahedron. However, in our growth system for the icosahedron, they differ in the hierarchy. We can distinguish them clearly by examining whether the pattern of the shell surface consists of only outside modules or only inside modules.

This leads to the important conclusion that the icosahedron belongs to a more fundamental hierarchy than the dodecahedron. The $1f$ icosahedron is a 5-fold symmetrical polyhedron which can be composed of the minimum number of modules. However, the very center of the $1f$ icosahedron, or its nucleus can be thought of as a $0f$ (zero frequency) icosahedron. In this sense, a single point is an initial polyhedron which has a hierarchy for the icosahedron already in it.

In our hierarchy, there are axes which radiate from the center toward the 12 vertices of the icosahedron, axes which radiate from the center toward the centers of the 20 faces of the icosahedron, and axes which radiate from the center toward the 30 edge midpoints of the icosahedron. Thus, there are 31 axes of 5-3-2 rotation symmetry in total. In the hierarchy of growth, any point (including the kernel) where the vertices of modules meet contains the 31 rotation axes. The bond angle of the edges of modules at each point is a central angle made by some combination of these 31 axes. At the maximum in our hierarchy, 18 directions are selected among the 62 radial directions.

Both the local non-periodicity and the synergetic hierarchy emerge from the angular divisions made by these axes at the nucleus. The asymmetrical combinations of modules are caused primarily by the symmetry of the kernel where three types of rotational axes can co-exist.

The concentric polyhedral shell structures grow symmetrically with respect to the 5-3-2 rotational axes from the nucleus. The growth of the arrangement by the *synergetic modules* is governed by the rule that the bond angles must match each other at any point no matter how non-periodic and asymmetrical these angles are.

A point must contain the system in order to make sure the combination of *synergetic modules* will grow radially. This leads to the idea that the point represents a complex of realistic substances with some properties, rather than the idea in traditional Euclidean geometry that a point has no parts in it.

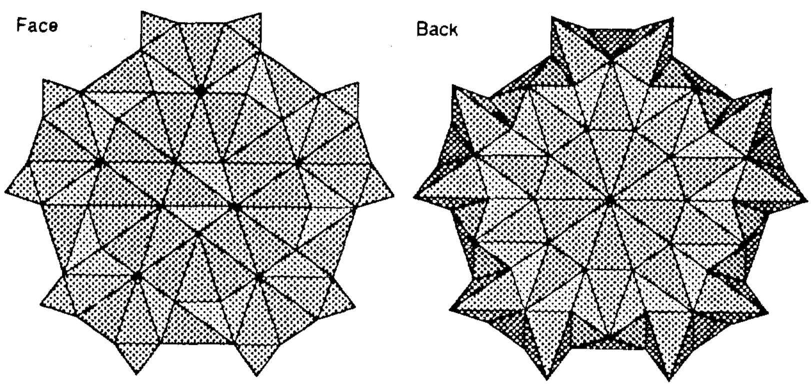


Figure 6: Tiling
In the concentric polyhedral hierarchy, the *synergetic modules* make possible such generalizations as 5-fold symmetry, periodicity of the hierarchy and radial growth. However, if we limit them to construct only a plane (not a mathematically rigorous plane since it has some thickness), we obtain non-periodic arrangements. This figure illustrates an arrangement composed of four kinds of inside modules E, F, G and H, which have a global orientation order and a global translation order. The patterns of the face (left) and back (right) are never identical. Any Penrose tiling pattern can be realized in this model. If we pile up symmetric layer on layer, and so on, we obtain a periodic structure with respect to the vertical axis. On the other hand, if we use the different modules A, B, C, and D, we obtain a layer with the same pattern but with a volume $2/\tau$ times as large as before.

RADIAL GROWTH ON ROTATION AXIS

In the growing icosahedron, there are parallel layers expanding successively which are perpendicular to any 5-fold axis. The cross section perpendicular to the axis is always a regular pentagon. It is composed of outside modules and inside modules joined to each other. There are two kinds of thicknesses of the pentagonal layers which can be expressed in terms of the golden ratio τ if we assume the distance between the center of a $1f$ icosahedron and its surface equilateral triangles to be 1. Since the boundary between layers is also a boundary between shells, we can remove the layer from the hierarchy.

Parallel surfaces of regular pentagonal layers have intrinsically non-periodic patterns. These patterns are the result of the 3-dimensional combination of modules. Adjoining patterns are mirror images of each other. The combination of modules has 3-2 rotation symmetry. Furthermore, a pattern has either right-handedness or left-handedness. Each shell shows right-handedness or left-handedness alternately on its surface pattern, which is integrated into the inside surface of the next shell.

Perpendicular to the 3-fold axes, equilateral triangles expand out successively. These triangular layers cannot be removed since modules which compose the layers between two equilateral triangles are overlapping modules which compose the pentagonal layers perpendicular to the 5-fold axes. The thickness of a triangular layer is an integer multiple of the distance between the center of the $1f$ icosahedron and its face equilateral triangles. The patterns on these layer surfaces are periodic.

With respect to 2-fold axes, there is no module which has a surface perpendicular to these axes.

The ten types of *synergetic modules* can be classified into three classes by their height. Three kinds of layers appearing on the 5-fold and the 3-fold axes are truss structures of three thicknesses, each of which is composed of tetrahedral and octahedral modules of the same height. This is an economic and dynamically stable combination of modules.

SYNERGETICS TILING

I found a generalization which abstracts the method of module arrangement in the layers on the 5-fold axis in October 1989, and named it *Synergetics Tiling*. We can make tilings of two different thicknesses, which expand horizontally the layer of either the inside modules or the outside modules on the 5-fold axes. These tilings are non-periodic and do not have 5-fold symmetry.

The patterns appearing on both sides of these tilings can reproduce all of the non-periodic Penrose tilings. The patterns on both sides are never identical to each other. Moreover, one can arrange modules so that the pattern on the face is not periodic, while the pattern on the back has perfect 10-fold symmetry.

If we pile up the layers of the *Synergetics Tiling* so as to make the adjoining patterns match each other, we have a periodic structure in the vertical direction which can fill the entire space. The reason why these modules can spread to infinity is the degree of freedom that exists when there is such a high level of possible combinations, 20 directions at maximum out of 62 radial total directions.

QUANTIZATION BY MODULES

Let the volume of the module I and E be 1. I found that the formula which tells the volume V of a growing icosahedron with frequency f is represented as:

$$V = 2 \times (2 \times 5)\, f^3$$

Although modules other than I and E have irrational volumes, they add up to an integer value when they compose an icosahedron, eventually canceling the irrationality of each other. Moreover, the volume of any shell structure is also an integer. Hence, the volumes of the shell structures are quantized by the *synergetic modules* and increase proportionally to the cube of the frequency.

Next, let f be the number of divisions or frequency of an edge, X be 1 for a regular tetrahedron, 2 for a regular octahedron, 5 for a cuboctahedron or an icosahedron. In 1960, R. Buckminster Fuller found that when a regular polyhedron is filled closely by spheres, the number of spheres on the faces, say N, can be represented as:

$$N = 2 \times f^2 + 2$$

He also found a similar formula for the number of points on the faces of an icosahedron which is divided into equilateral triangles.

Fuller's general formula also holds true for our hierarchical structural model of the icosahedron as follows:

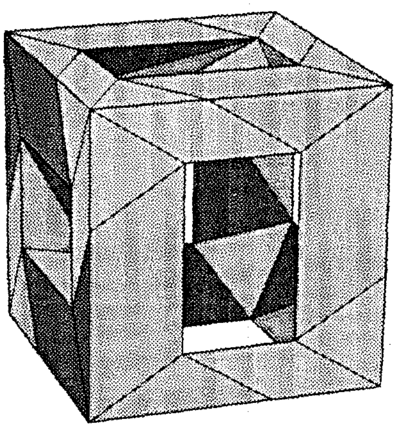
$$N = 2 \times 5\, f^2 + 2$$

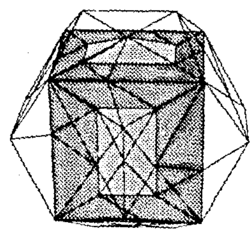
where N means the number of points on the outer shell faces (points where edges of modules meet). That is, the number of points on the outer shell faces of an icosahedron is the square of the frequency times the particular prime number 5, multiplied by 2, and finally plus 2. Here we can notice another appearance of additive twoness.

Clusters of such rare gas atoms as argon and xenon are stable since they form icosahedral packing structures. Their magic numbers are said to be 13, 55, and 147. These values can be obtained by adding successively the numbers of the points of the $0f$ icosahedron, $1f$ icosahedron, $2f$ icosahedron, and $3f$ icosahedron, 1, 12, 42, 92, respectively. That is, 13=1+12, 55=13+42, and 147=55+92. The *synergetic modules* used here are only the tetrahedron I and the octahedron J. The filling has no internal dodecahedron structure but has perfect 5-fold symmetry.

This means that the icosahedron in a cluster form has no non-periodic combination and has a self-similar structure with perfect 5-fold symmetry. This geometrical model is a good example to explain non-formative quantum leaps in physics. The fact that the magic numbers can be calculated this way, even though the edges of modules *I* and *J* are not equilateral, suggests that the closest packing by identical spheres is not an appropriate geometrical model for these clusters.

Synergetic modules with growth ability and interchangeability repeated in the hierarchical combination quantize the volume of 5-fold symmetric polyhedra. There exists a system which cannot be guessed only from the individual modules. Each *synergetic module* does not appear as a macroscopic shape in the hierarchy. In other words, *synergetic modules* form the structures and patterns as a whole by some system unpredictable from its parts.





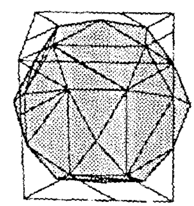
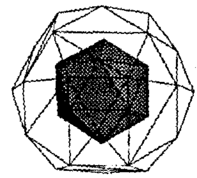


Figure 7: A regular hexahedron which inscribes a dodecahedron
We will ultimately obtain only tetrahedral modules if we divide the octahedral modules in the *synergetic modules* and decrease the degree of symmetry. If we use such 'minimum' tetrahedral modules, we can fill a regular hexahedron which inscribes a dodecahedron. In this case, modules are arranged symmetrically with respect to the kernel of the icosahedron which takes the regular hexahedral arrangement. Therefore, the 5-3-2 symmetry and the 4-3-2 symmetry can coexist in this space filling. In other words, if we use these tetrahedral modules, we can obtain both hierarchical systems of icosahedra and regular hexahedra. Note that even in this case, the golden ratio between volumes of modules is kept.



HIERARCHICAL MODEL OF RHOMBIC TRIACONTAHEDRON

My discovery of the hierarchical model for the rhombic triacontahedron was motivated by the discovery of a dodecahedron whose edge length is an integer multiple of 1.0 (the edge length of the 1*f* icosahedron) in the course of the combining of *synergetic modules*. Dodecahedra whose edge length is an integer multiple of 1.618 are necessary to grow icosahedra whose edge length is an integer multiple of 1.0, while dodecahedra whose edge length is an integer multiple of 1.0 are not needed at all.

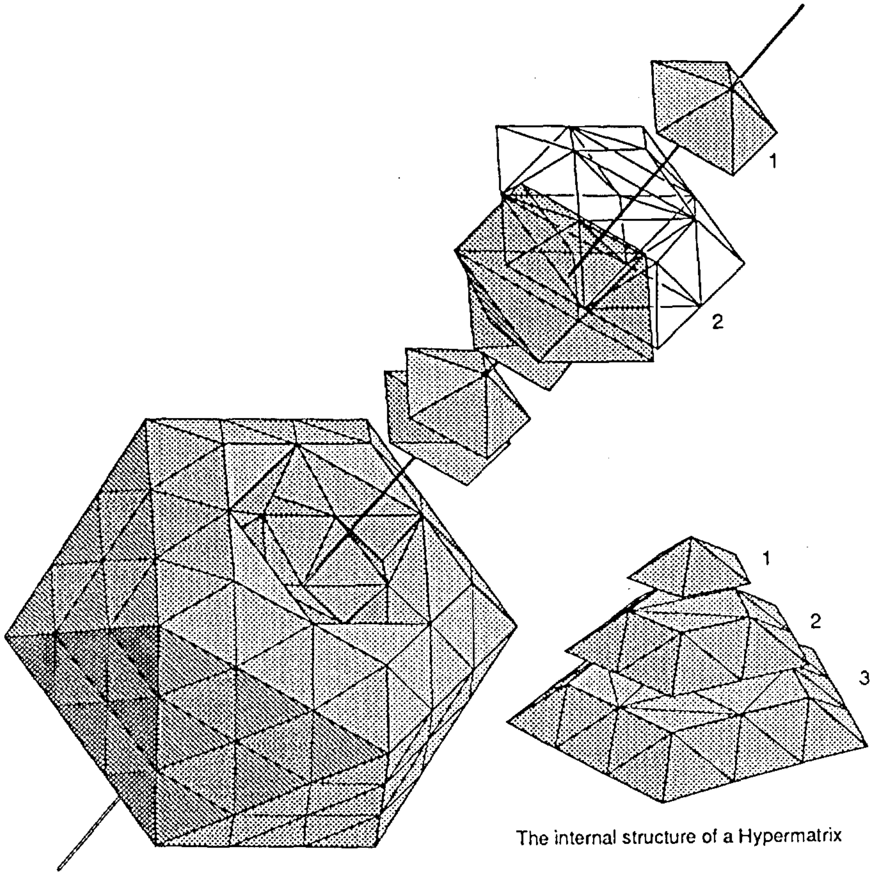


Figure 8: Hypermatrix

I call the wire-frame model of the hierarchy made by the *synergetic modules* the *hypermatrix*. The three kinds of vectors whose lengths are in the ratios 0.95, 1.0, & 1.618 and the angles of combinations of the 31 rotation axes make up a complicated shell structure system. On the 5-fold axes (one of them is shown in the figure) and the 3-fold axes, there are parallel and consecutive layers perpendicular to the axes. Each layer is a non-equilateral and non-periodic truss composed of tetrahedra and octahedra, which can be obtained at a small cost of energy and is dynamically stable. There must be a regular pentagon on the layers perpendicular to a 5-fold axis. The number 3 in the figure illustrates the layer which appears in the 4*f* icosahedron.

I thought that a dodecahedron whose edge length is an integer multiple of 1.0 belonged to another hierarchy and tried to find a rhombic triacontahedron which circumscribes this dodecahedron. I eventually succeeded in filling a rhombic triacontahedron that has 5-3-2 symmetry hierarchically with a 3-dimensional combination of irrational diagonals and non-integer angles as in the case of the icosahedron.

In projective geometry, a rhombic triacontahedron can be constructed from the duality of the golden ratio between an icosahedron and a dodecahedron. However, it cannot be constructed in our hierarchical system although the icosahedron, dodecahedron, and rhombic triacontahedron with the same center have the same 5-3-2 symmetry. The reason is that a dodecahedron whose edge length is a multiple of 1.618 can inscribe an icosahedron whose edge length is a multiple of 1.0 while a dodecahedron whose edge length is a multiple of 1.0 can inscribe a rhombic triacontahedron whose edge length is a multiple of 0.95. Therefore dodecahedra, icosahedra and rhombic triacontahedra cannot constitute a single hierarchical structure. That is, icosahedra and rhombic triacontahedra constitute their own hierarchical structures, respectively.

On the other hand, these two hierarchical structures do share some first shells from the first $1f$ icosahedron with edge length 1.0 to the first dodecahedron with edge length 1.618. In the hierarchy of rhombic triacontahedra, however, only inside modules are used beyond the first dodecahedron.

Thus, if we call the $1f$ icosahedron the initial nucleus, the first dodecahedron common to the both hierarchies can be thought of as the second nucleus. Here emerges a hierarchy of nucleuses. The initial nucleus and the second nucleus are dual to each other. The duality in the hierarchy is based on the time axis of the frequency growth.

As the frequency of rhombic triacontahedra whose rhombic faces make a lattice of isosceles triangles increases, *synergetic modules* are able to completely fill them . Rhombic triacontahedra grow symmetrically, forming non-periodic parallel layers perpendicular to the radial direction from the center just as in icosahedra. The *synergetic modules* also maintain growth ability and interchangeability in this case.

The first rhombic triacontahedron appearing in this hierarchy is the $2f$ rhombic triacontahedron with edge lengths of 0.95. Before it appears, there appear a $2f$ dodecahedron with edge length 1.0, a $1f$ truncated rhombic hecatoicosahedron with edge length 1.618 and a $1f$ dodecahedron with edge length 1.618 which is common to the hierarchy of icosahedra. They all have 5-3-2 symmetry.

Beyond the $2f$ rhombic triacontahedron, similar shell patterns to those which have appeared so far are replicated every second time. That is, the period of the hierarchy is 2.

A rhombic triakisicosahedron inscribes the $1f$ dodecahedron with edge length 1.618. A rhombic icosahedron adjoins to each of its 12 cavities in the hierarchy of rhombic triacontahedra. A $1f$ rhombic triacontahedron can be formed by sharing structural parts of these two kinds of polyhedra.

Choose one 5-fold axis, then we can find a $1f$ rhombic triacontahedron whose center is on the axis and which contacts both the center and one of the vertices of the $2f$ rhombic triacontahedron. This $1f$ rhombic triacontahedron does not share the center with other shell structures, and its diameter is just half that of the $2f$ rhombic triacontahedron.

Hence we have now two $1f$ rhombic triacontahedra which contact each other at the center of the hierarchy.

Rhombic triakisicosahedra, rhombic triacontahedra, and rhombic icosahedra can be composed of A_6 and O_6. However, there appear parallel layers with non-periodic patterns successively in our hierarchical system of *synergetic modules* unlike the construction by A_6 and O_6. There are 7 layers in the $1f$ rhombic triacontahedron and 14 layers in the $2f$ rhombic triacontahedron.

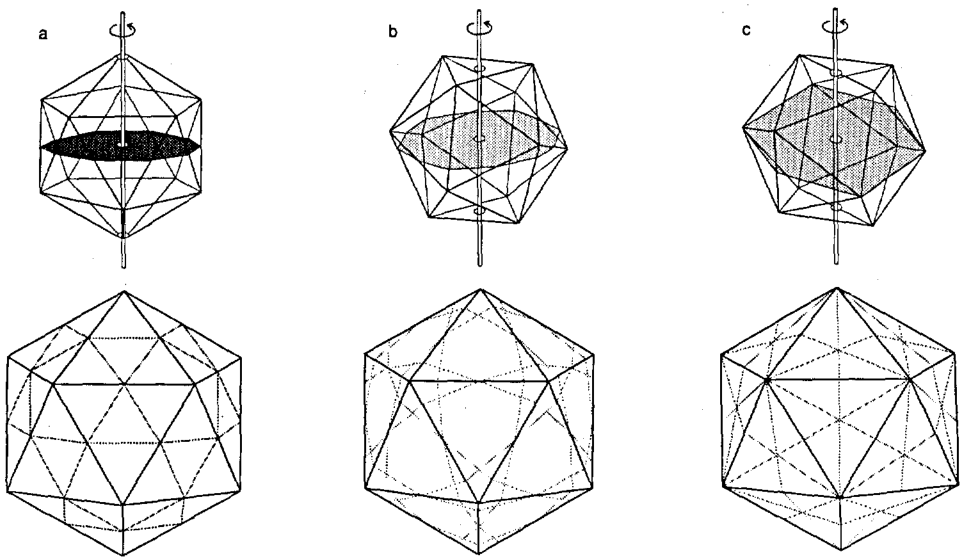


Figure 9: Rotation symmetry axes of the icosahedron
Axes which link two antipodal vertices are 5-fold axes (a). Since there are 12 vertices, there are six 5-fold axes. There are fifteen 2-fold axes, each of which links two antipodal midpoints of the edges (c). There are ten 3-fold axes, each of which links two antipodal centers of the faces (b). Hence we have thirty-one 5-3-2 symmetry axes in total. We call the cross section which cuts the icosahedron into two identical pieces at a plane perpendicular to a rotational axis a great circle. There are 31 great circles since there are 6 great circles (a) on 5-fold axes, 10 (b) on 3-fold axes, and 15 (c) on 2-fold axes. The 8 angles which appear in the *synergetic modules* are the central angles between these great circles: 31.717 degrees, 36 degrees, 58.283 degrees, 60 degrees, 63.435 degrees, 72 degrees, 108 degrees, and 116.565 degrees.

I found two new polyhedra in the growth process from a $1f$ to a $2f$ rhombic triacontahedron. One is the rhombic hecatoicosahedron composed of 120 identical rhombuses. This polyhedron can be obtained by joining 12 rhombic icosahedra around a rhombic triakisicosahedron. The other is a truncated rhombic triacontahedron composed of 12 regular pentagons and 30 regular hexagons.

There is another hierarchy in which a Kepler's small stellated dodecahedron can be obtained by joining 12 pentagonal pyramids composed of modules F and E to a $2f$ rhombic triacontahedron. In the course of filling the vacant space among these pyramids, there appears Kepler's great dodecahedron. If we go further to fill the triangular pyramidal cavities in the great dodecahedron, we get another big rhombic triacontahedron.

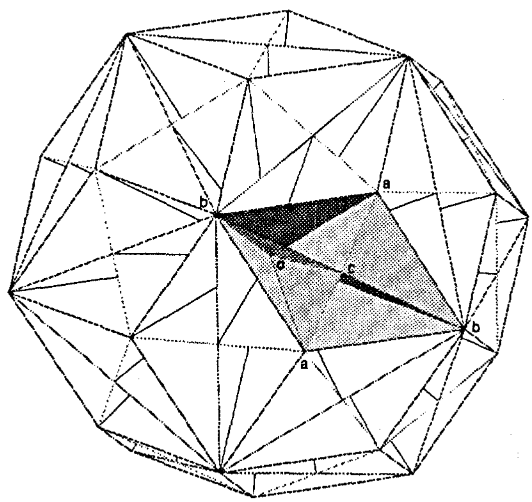


Figure 10: 5-3-2 Rotational symmetry axes
A rhombic triacontahedron can be divided into 120 identical tetrahedra (a-b-c-o in the figure). Once this division is done, we can integrate polyhedra with 5-3-2 rotation symmetry simultaneously. If we focus our attention on the rhombuses on the surface, we find a rhombic triacontahedron. If we focus our attention on the longer diagonals in the rhombuses, we find an icosahedron. If we focus our attention on the shorter diagonals in the rhombuses, we find a dodecahedron. In other words, 62 radial lines from the center amount to 31 rotational symmetry axes which are common to the rhombic triacontahedron, icosahedron and dodecahedron. All the combinations of edges at lattice points in the *hypermatrix* can be represented by the combination of the central angles made by these 31 rotational symmetry axes.

The possibility of radial symmetrical combinations of modules increases and 5-fold symmetry can be reproduced more abundantly as the frequency increases. Since only inside modules are used in the growth process, the volumes of polyhedra appearing in the hierarchy of rhombic triacontahedra are not integer valued except for the first few. Instead, they are intrinsically related to the golden ratio.

The formulas I discovered with respect to the hierarchy of rhombic triacontahedra are as follows :

$$N = (2 \times 3 \times 5)\, f^2 + 2$$

$$V = (1+ \sqrt{5}) \times (2 \times 3 \times 5)\, f^3$$

where N, V, & f mean the number of vertices, the volume and the frequency (which has to be even), respectively.

From another viewpoint, a rhombic triacontahedron is obtained by adding 12 pentagonal pyramids to a dodecahedron. Hence we can calculate the volume of a dodecahedron whose edge length is an integer multiple of 1.0 by the formula:

$$V = (1+ \sqrt{5}) \times (2 \times 3 \times 5)\, f^3 - 12\sqrt{5}\, f^3$$

The volume of each pentagonal pyramid is $\sqrt{5}f^3$.

HYPERMATRIX

Synergetic modules can fill the two Platonic regular polyhedra, the dodecahedron, and the icosahedron, which were not filled by the unit cells in traditional crystallography, but they can also fill the rhombic triacontahedron, which was a basic element in traditional crystallography. This suggests that the hierarchies of icosahedra and rhombic triacontahedra bring in new lattices with different characteristics from those space lattices in traditional crystallography.

The space lattices which explain crystal structures in traditional crystallography are composed of identical unit lattices which are arranged periodically, each of which has the same peripheral arrangement. Each point can be obtained by translation and all points are equivalent. The form of the basic unit cell which determines the crystal can vary indefinitely.

The lattices formed by the hierarchy made from *synergetic modules* form a kind of wire frame of the corresponding shell structures. Unlike the lattices so far, they do not spread infinitely, but form a closed space within each shell structure. This growth limit results from 5-fold symmetry.

The lattices growing along their rotational symmetrical axes in the hierarchies of icosahedra and rhombic triacontahedra are called the *hypermatrix*.

The ratios of lengths for the three vectors in the *hypermatrix* are 0.95, 1.0, & 1.618, lengths which exist inherently in the icosahedron. The space group *hypermatrix* for concentric polyhedra is formed by these ratios and the 31 rotational axes; again they are the 12 radial directions toward the icosahedron vertices, 20 radial directions toward the icosahedron face centers, and 30 radial directions toward the icosahedron edge centers.

Any point has 5-3-2 symmetry and thereby has the possibility of being a nucleus. However, one of the points is selected as the nucleus and lattice points whose peripheral arrangement are mutually different are obtained by symmetrical radiation of the ten types of *synergetic modules*.

As described above, there are two hierarchies beyond the $1f$ dodecahedron with edge length 1.618. However, the concepts of frequency, 5-3-2 symmetry, shell filling, quantization, 31 rotation axes, complementariness of tetrahedral and octahedral modules, non-periodicity, periodicity of the hierarchy, face and back, polarity, right-handedness and left-handedness, and radial growth are independent on the scale of the *hypermatrix*.

Five-fold symmetry is one of the generalized characteristics of the *hypermatrix* integrated radially in the concentric hierarchy.

FORMS AND MODELS

In 1985, L. Pauling pointed out that the quasicrystal alloy $Al_{86}Mn_{14}$ with 5-fold symmetry does not match any Bravais lattice and has a twin structure of more than one regular hexahedra. In 1986, T. Rajasekharan reported that the 5-3-2 symmetric units of the quasicrystal alloy $Mg_3(Al, Zn)_{49}$ are filled in the body-centered cubic lattice and that quasicrystals of Al Cu Fe family form the face-centered cubic lattice.

A. P. Tsai and others confirmed in their experiments that if three metal elements can be fused into a metal alloy by the liquid quenching method, the ratio of the radii of the solvent atom and the solute atom is 1 to between 0.85 and 0.95. The other ratio 1.618 can be detected as the distance between atoms by electron diffraction image.

This suggests an important analogy between the forms of quasicrystals and the formation and growth of the *synergetic modules*.

Each octahedron in the *synergetic modules* can be further divided symmetrically into two tetrahedra. Thus, we have now only tetrahedral modules. Therefore, these tetrahedral modules can be thought of as the minimum modules with the highest interchangeability which can be derived from the icosahedron. By using these minimum tetrahedral modules, we can obtain a shell in the shape of a regular hexahedron which has no 5-fold symmetry in the hierarchy of icosahedra. This is a regular hexahedron with edge length 2.618 which inscribes the $1f$ dodecahedron with edge length 1.618. Here, the 4-fold axis of the hexahedron and 2-fold axis of the icosahedron coincide with each other, as well as the 3-fold axis of the hexahedron and the 3-fold axis of the icosahedron.

The fact that some of the 31 rotational axes of the icosahedron and the rotational axes of the hexahedron are common in the *hypermatrix* suggests that the 4-3-2 symmetry of the hexahedron and the 5-3-2 symmetry of the icosahedron are fused physically.

Moreover, the fact that there are 5-fold symmetric shells inside the hexahedron (in fact, there is an icosahedron and an icosidodecahedron inside) suggests that the discrimination between crystals and quasicrystals by the concept of periodicity is not essential. This fact seems to fade out the contrast between quasicrystals and crystals smoothly and naturally.

Form is the spatial arrangement of constituents in a material. Positions of individual atoms have come to be detectable by the rapid development of electron diffraction and X-ray diffraction technology. However, the structural patterns behind the form appear to be more important than the information of individual constituents.

In other words, the arrangement of the modules is far more important than the forms of the modules themselves.

The *hypermatrix* displays the internal interaction between real substances and concepts, but is not an illustrative reproduction of the spatial arrangement of the constituents of real materials. Hierarchical models made with the *synergetic modules* do not require that there really exist ten new types of unit cells corresponding to the modules. Instead, it is a visualization of a closed abstract system of relations between atoms.

The *hypermatrix* woven by the angles of the three kinds of rotational axes of the icosahedron and the three kinds of vectors of the *synergetic modules* represents a symmetry with very high structural stability, as in nature, where all things are structured in triangles.

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