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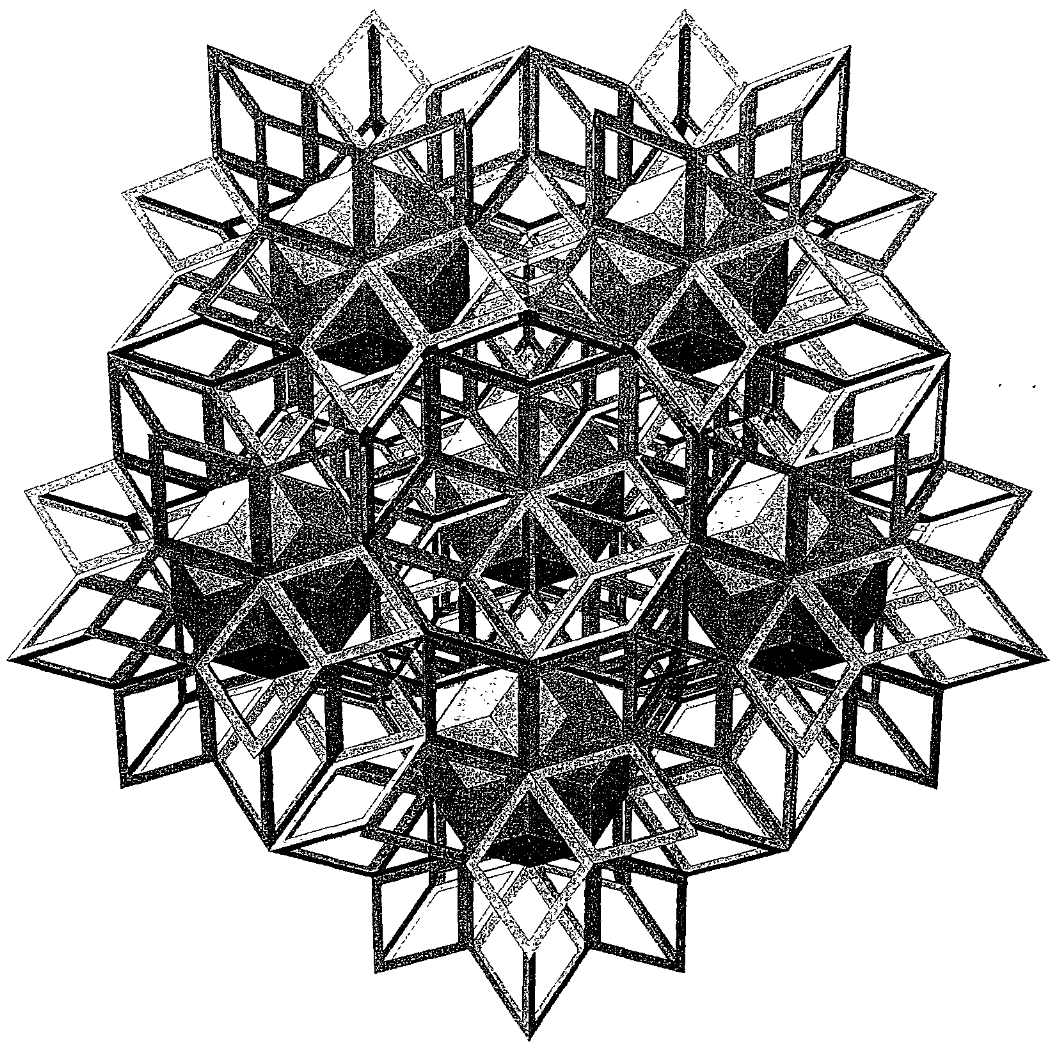

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MEAN, PROPORTION AND SYMMETRY IN GREEK AND RENAISSANCE ART

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'το γαρ ευ παρα μικρον δια πολλων αριθμων γιγνεσθαι'
'The beautiful is achieved little by little, by means of many numbers'[1]

This is one of the very few remaining fragments of the *Canon* of Polyclitus, who was probably the first artist to codify the rules to be followed in executing a work of art. There are other classical sources which speak of Polyclitus as creating a harmonic, well-structured 'system of proportions,' which he was to express directly in his statues – especially the *Doriforus* – and in his lost treatise, the *Canon*; and these sources indicate that it was his intention to supply a model of proportions which might create a relationship among the individual parts of the human body, based on numbers.[2] Never before had anyone given artists such a corpus of numerical rules which was to be of great influence not only on Greek but also on Roman and Renaissance art. His example was followed by a host of 'theoreticians of art' who, from Vitruvius on, found in Humanism and in the Renaissance the highest peak of the fusion between the mathematical doctrines of the mean and proportion, and the doctrines of art and of music. Erwin Panofsky and Rudolph Wittkower have already shown with masterly skill that the rediscovery of the classics on the one hand, and the conception of Man as expression of the microcosm and of the macrocosm on the other, were the factors that led to an appreciation of all those classical theories and conceptions of the beautiful, perfection and harmony as the synthesis and fruit of a systematic scientific study.[3] Thus the theoreticians of art and music in the periods of Humanism and the Renaissance, if they were to arrive at the scientific knowledge which enabled them to obtain the beautiful, harmony, symmetry, perfection, had to turn for their inspiration to the Pythagorean school, the canons of Polyclitus and the philosophical teachings of Plato, handed down by Vitruvius and subsequently by L. B. Alberti. It is my intention to add to Panofsky's and Wittkower's approaches the philological and historico-mathematical analysis of the classical Greek sources in which the terms 'mean' and 'proportion' appear (§. 1) in order to assess what classical traditions or sources were most influential for the art and music theoreticians of the 15th and 16th centuries (§. 2). Furthermore I take into consideration the occurrence of the golden section in handbook for artists (§. 3) to see the role played by irrational numbers in Renaissance treatises and finally I wish to single out the presence of two factors in the writings on the portrayal of the beauty of the human body (§. 4) and to suggest a possible link between the three main means (arithmetic, geometric and harmonic) and the human figure inscribed in square and circle (Table 6). I shall point out certain passages from 15th and 16th century writings on art, where it is clear that the application of mathematics to art is mediated by what Panofsky calls 'the visual experience of the artist'. In my view, two separate scientific components can be identified in the approaches of art theoreticians on the proportions of the human body: one theoretical, rooted in the Pythagorean and Platonian teaching of means and music; the other experimental, based on the observation of reality and on the meticulous gathering of numerical data, on which the mean is calculated. Both these components are linked to the concept of the mean, which has been given an exact mathematical definition by statistics scholars[4] only in this century, but which has been in practical use from the most ancient times.

1. MEANS AND PROPORTIONS IN THE HISTORY OF MATHEMATICS

In order to arrive at the origins of the concept of the mean and the use made of it in both mathematics and practical life, we must go back to pre-Hellenic civilisations. From surviving testimony it is clear that the Babylonians used the mean, though without defining it in their calculations. An Egyptian deed from 150 B.C. was found at Edfu: this shows a number of plane geometrical figures (triangles, rectangles, trapezoids and quadrilaterals) and the rule for finding their area. This rule - which, it has been proved, the Babylonians also knew - consisted of taking the product of the arithmetic means of the opposite sides.[5] In the same way the Babylonians calculated the volume of a frustrum of a cone or pyramid by taking the arithmetical mean of the upper and lower bases and multiplying it by the height.[6] Use of the arithmetical mean can also be seen in the Babylonians' procedure for extracting the square root.[7] But it was only with the Greeks that a rigorous, systematic treatment of the means and the proportions was achieved – a treatment which subsequently appeared in the surviving texts.[8] In Table 1 you can see the names of the principal figures who dealt with this topic and the titles of their works.

PERIOD	MATHEMATICS & PHILOSOPHY	MUSIC	ART & HISTORY
6-5th B.C.	PYTHAGORAS		
5-4th B.C.	HIPPOCRATES of Chios ARCHYTAS, HIPPASUS	PHILOLAUS	POLYCLITUS *Canon* THUCYDIDES *Hist. Belli Pelop.*
4th B.C.	EUDOXUS		PHILON *Mechan. Synt.*
	PLATO *Tim.*, *Resp.*, *Leges* ARISTOTLE *Eth. Nic.*		
3rd B.C.	EUCLID *Elements*		HIPPODAMOS of Mileto
	ERATOSTHENES	MYONIDES, EUPHRANOR	
1st B.C.			VITRUVIUS *De architec.*
1st A.D.			PLINIUS *Nat. Hist.*
1st-2nd A.D.	NICOMACHUS *Introductio arithmeticae*	NICOMACHUS *Introductio arithmeticae*	PLUTARCH *Moralia*
	THEON		GALEN *De temperam. Placita Hipp. et Plat.*
3rd-4th A.D.	IAMBLICHUS *Nicom. Arith. Introd.* PAPPUS *Collect. Math.*	PORPHYRY *In Ptol. Harm.*	
5th A.D.	PROCLUS *In Tim.*		
5th-6th A.D.	BOETHIUS *De Inst. Arithm.*	BOETHIUS *De Inst. Musica*	

Table 1: Ancient authors or writers on the popular theory of means and proportions

MEANS		NICOMACHUS		PAPPUS	
		Numerical examples	$a > b > c$	Numerical examples	$a > b > c$
First	*Arithmetic*	3 2 1	$\frac{a-b}{b-c}=\frac{a}{a}$	6 4 2	$\frac{a-b}{b-c}=\frac{a}{a}$
Second	*Geometric*	4 2 1	$\frac{a-b}{b-c}=\frac{b}{c}$	4 2 1	$\frac{a-b}{b-c}=\frac{a}{b}$
Third	*Harmonic*	6 4 3	$\frac{a-b}{b-c}=\frac{a}{c}$	6 3 2	$\frac{a-b}{b-c}=\frac{a}{c}$
Fourth	*Subcontrary to harmonic*	6 5 3	$\frac{b-c}{a-b}=\frac{a}{c}$	6 5 2	$\frac{a-b}{b-c}=\frac{c}{a}$
Fifth	*Subcontrary to geometric*	5 4 2	$\frac{b-c}{a-b}=\frac{b}{c}$	5 4 2	$\frac{a-b}{b-c}=\frac{c}{b}$
Sixth	*Subcontrary to geometric*	6 4 1	$\frac{b-c}{a-b}=\frac{a}{b}$	6 4 1	$\frac{a-b}{b-c}=\frac{b}{a}$
Seventh		9 8 6	$\frac{a-c}{b-c}=\frac{a}{c}$	3 2 1	$\frac{a-c}{a-b}=\frac{b}{c}$
Eighth		9 7 6	$\frac{a-c}{a-b}=\frac{a}{c}$	6 4 3	$\frac{a-c}{a-b}=\frac{a}{b}$
Ninth		7 6 4	$\frac{a-c}{b-c}=\frac{b}{c}$	4 3 2	$\frac{a-c}{a-b}=\frac{a}{c}$
Tenth		8 5 3	$\frac{a-c}{a-b}=\frac{b}{c}$	3 2 1	$\frac{a-c}{b-c}=\frac{b}{c}$

Table 2: The ten means in Nicomachus and Pappus' works

If tradition is to be trusted, Pythagoras learned in Mesopotamia of three means – the arithmetic, the geometric and the subcontrary, later called harmonic – and of the golden proportion relating two of these.[9] According to Iamblichus, the first three means were in use in the time of Pythagoras and his school, Archytas and Hippasus called the subcontrary harmonic mean, Eudoxus invented three new means: the antiharmonic or subcontrary, the fifth and the sixth and the other four were added later by Myonides and Euphranor, both Pythagoreans, in their musical studies.[10] So the means devised by Greek were ten in all (see Table 2), the number ten being "perfect" for the Pythagorean school. But it is certainly to Nicomachus and Pappus that we owe the fullest discussion on the means and on the history of the theory formulated by the Pythagoreans between the 6th and 4th centuries B.C. The importance of this theory is thus emphasised by Nicomachus:

> "After this it would be the proper time to incorporate the nature of proportions, a thing most essential for the speculation about the nature of the universe and for the propositions of music, astronomy, and geometry, and not least for the study of the works of the ancients, and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting. A proportion, then, is in the proper sense, the combination of two or more ratios [...] The first three proportions which are acknowledged by all the ancients, Pythagoras, Plato and Aristotle, are the arithmetic, geometric and harmonic; and there are three others subcontrary to them, which do not have names of their own, but are called in more general terms the fourth, fifth, and sixth forms of mean; after which the moderns discover four other as well, making up the number ten, which, according to the Pythagorean view, is the most perfect possible."[11]

He also noticed the role and importance of the means for the harmony of the cosmos:

> "Some however, agreeing with Philolaus, believe that the proportion/mean is called harmonic because it attends upon all geometric harmony, and they say that 'geometric harmony' is the cube because it is harmonised in all three dimensions, being the product of a number thrice multiplied together. For in every cube this proportion/mean is mirrored; there are in every cube 12 sides, 8 angles and 6 faces; hence 8, the mean between 6 and 12 is according the harmonic mean, for as the extremes are to each other, so is the difference between greatest and middle term to that between the middle and smallest terms, and, again, the middle term is greater than the smallest by one fraction of itself and by another is less than the greater term, but is greater and smaller by one and the same fraction of the extremes. And again, the sum of the extremes multiplied by the mean makes double the product of the extremes multiplied together. The diatessaron is found in the ratio 8:6, which is sesquitertian, the diapente in 12:8, which is sesquialter; the diapason, the combination of these two, in 12:6, the double ratio; the diapason and diapente combined, which is triple, in the ratio of the difference of the extremes to that of the smaller terms, and the diapason is the ratio of the middle term to the difference between itself and the lesser term. Most properly, then, has it been called harmonic."[12]

Porphyry in his *Commentary on Ptolemy's Harmonics* emphasised the use of the means in music by Archytas.[13] Basic musical intervals were found by the Pythagoreans studying the lengths of vibrating strings, which are expressible as ratios of integers:

1:2 for the octave, 2:3 for the fifth, 3:4 for the fourth. They would have observed that if they took three strings, of which the first gave out a note an octave below the second, while the second an octave below the third, the lengths would be proportional to 4, 2, 1, i.e., they are terms in geometric mean. When they took the strings sounding a given note, its major fourth and its upper octave, the lengths would be proportional to 12, 8, 6, i.e., they are terms in harmonic mean and when its major fifth and its upper octave, the lengths would be proportional to 12, 9, 6, i.e., they are terms in arithmetic mean.

The last five means can be traced back to the 'moderns', hence probably to Nicomachus himself, or to Theon, or to those music theoreticians who inherited Pythagorean teaching in their own field. The sum of this tradition reached Umanism and Renaissance mainly through Boethius, who translated and spread Nichomacus'theory. A contribution to the spreading of these concepts was supplied also by Marsilio Ficino in his comments to Plato's *Timaeus* and a significant influence, particularly on atrists, was produced by L. B. Alberti, who dedicated to proportions in buildings and to means a full chapter of his *De re aedificatoria*. But out of the ten means codified by the Greeks only the first three are mentioned in the art treatises of the XV and XVI centuries, and particularly the arithmetic, the geometric and the harmonic mean. A hint for the identification of the Greek sources, which had the most significant influence during Renaissance comes from the philologic examination of the terminology used to indicate means and proportions in Greece and thereafter.

The terms used in Greek to indicate the proportion and the mean, while are distinct from each other in a strictly mathematical context, often appear as synonyms in other contexts (e.g., physics, philosophy, ethics or ordinary language). This is another factor which may serve to differentiate the traditions followed, since those closest to scientific or mathematical language are careful not to confuse the two terms. From the most philologically accurate Greek dictionaries it emerges that the term $\alpha\nu\alpha\lambda o\gamma\iota\alpha$ indicated 'proportion, i.e. the equality between two ratios', expressed in Latin by the word *proportio*. This consists of a relation among four magnitudes *a, b, c, d* having the form

$$a:b = c:d,$$

and in this sense is used by mathematicians (Euclid, Archimedes, etc.). But philosophers, too, made use of it, though in different contexts, in connection with cosmology or physics. Plato, for example, uses it in the *Timaeus* 31c-32a in order to explain how the Demiurge began to create the universe with fire and earth: he says that these elements could not compound together without a link to unite them, and concludes by saying 'And it is proportion which naturally achieves this most beautifully.'

The term *μεσοτης* represented the 'mean', in Latin *medietas*, and sometimes central, medial or intermediate. In the mathematical context it is the abstract name given by the Pythagoreans to certain magnitudes which are intermediate between two given magnitudes, and are defined in function of these magnitudes. Prior to Euclid the term *μεσοτης* was used in Pythagorean teaching from the earliest times. The first Pythagoreans to write about means were Archytas, Hippocrates etc., to whom the historical sources attribute the ideation of the three means: the arithmetical, the geometrical and the harmonic. In the Renaissance art treatises it is often indicated with the term *mediocritas* o *mediocrità*.[14]

Since the geometrical mean is a continuous proportion, some Greek authors came to confuse the terms *analogia* and *mesotes*, treating them as synonymous – a misuse which became common in artists' writings too. Only Nicomachus of Gerasa (1st-2nd century), in his *Introductio arithmeticae*, condemned the extension of the term *analogia* to the arithmetical mean.

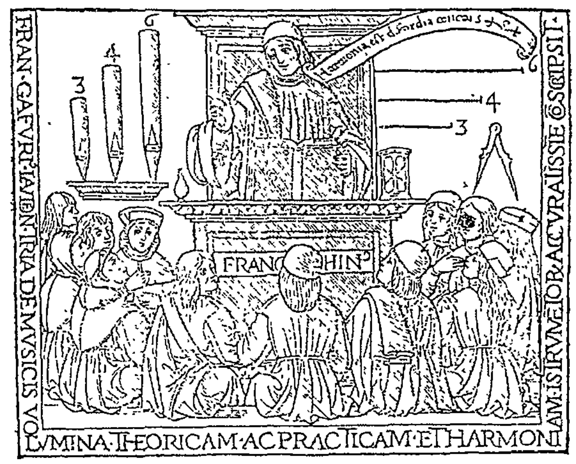
Mathematicians also made use of the terms *μεση αναλογον* for proportional mean or *ακρος και μεσος λογος* for extreme and mean ratio in the construction of that continuous proportion later named the "golden section", or "golden proportion", or "divine proportion."

In physics *mesotes* was used to reconcile opposing principles, to postulate the cosmic centre, and in ethics as the expression of moderation in two aspects: of *equilibrium* between two extremes or *moderation* exercised by the rational over the irrational.[15]

Also this aspect is transferred to art, where it is stated that beauty is balance between parts and the whole, and that for that purpose indeed the rules on proportions and means developed by mathematicians should be used.[16]

The sources of the theoreticians of art and music of the XVI century are mainly Vitruvius and Alberti. However it should be noted that there is no special care for a systematic and scientific use of the words "means" and "proportions", which are often confused. It should also be noted that, with the passing of time, the linguistic-scientific inaccuracy increases. [17] This signal indicates that, despite the fact that these theoreticians have attended abacus schools where arithmetic, geometry and the application to art of mathematical canons were taught, their interest was not in mathematical stringency, but rather in aesthetic adaptation.

1: F. Gaffurio (1480) *Opera teorica della disciplina musicale*



2: F. Gaffurio (1508) *Angelicum ac divinum opus musicum*

Table 3: Pythagorean means and proportions in musical treatises of XV-XVI centuries

The attitude of music theoreticians on relations between Pythagorean music and arithmetic during Humanism and Renaissance is not much different. Primary sources on the Pythagorean' musical theory have not survived, and all that remains is Boethius' account of them in his *De Inst. Musica.* The music theoretician Gafurio faithfully retells the story told by Boethius in his *Theorica musice* 1480 (published Posteriori 1492, 1508), whose frontispiece offers an effective illustration of the history of music theory in four scenes (Table 3). The characters portrayed are Tubalcain, biblical founder of music, shown in a forge where smiths are working iron; Pythagoras using a stick to touch bells of various sizes and vessels filled in varying degrees with liquid; and finally Philolaus and Pythagoras playing flutes of different lengths. In each figure appear the numbers 4, 6, 8, 9, 12 and 16, linked to the famous ratios between hammerheads (first scene), depth of liquid in vessels (second scene); weights suspended from a rope (third scene) and flutes of different lengths (fourth scene).

More precisely, in the second scene Pythagoras is shown demonstrating the octave 8:16 (i.e., 1/2) *diapason*, in the third scene Pythagoras demonstrates the fifth 8:12 (i.e., 2/3) *diapente*, and in the fourth he is holding in his hand two flutes which express the fourth 9:12 (i.e., 3/4) *diatesseron*, while Philolaus has two flutes which express the fifth 4:6. These last two figures are playing flutes one of which is twice the length of the other: Pythagoras 8, Philolaus 16, i.e. the octave 8:16 is represented once more, so that this last scene incorporates all the musical intervals. In the frontispiece as a whole, i.e. in the four scenes, two octaves thus appear, i.e., the compound chords which expressed the harmonic law of the universe, the 'Great perfect system of the Greeks': octave and fifth (1:2:3) and the two octaves (1:2:4). Raphael shows a similar scene on the blackboard in front of Pythagoras in his *School of Athens*. The fact that Plato appears in the same fresco, with the *Timaeus* in his hand, has led art historians to see in it 'Raphael's interpretation of the harmony of the universe, which Plato had described in the *Timaeus* on the basis of the Pythagorean discovery of the ratios of musical harmonies'. Indeed the inspiring sources of the artistic-musical theories were exactly Plato and Boethius.

Concerning the terminology used by art and music theoreticians to indicate fractions, it does not differ significantly from the one used by the mathematicians of the time. It is likely that in the abacus schools a similar table on proportions to the one reported by Luca Pacioli both in his *Summa* (1494) and in his *De divina proportione* (1509) (Table 4) was in use.

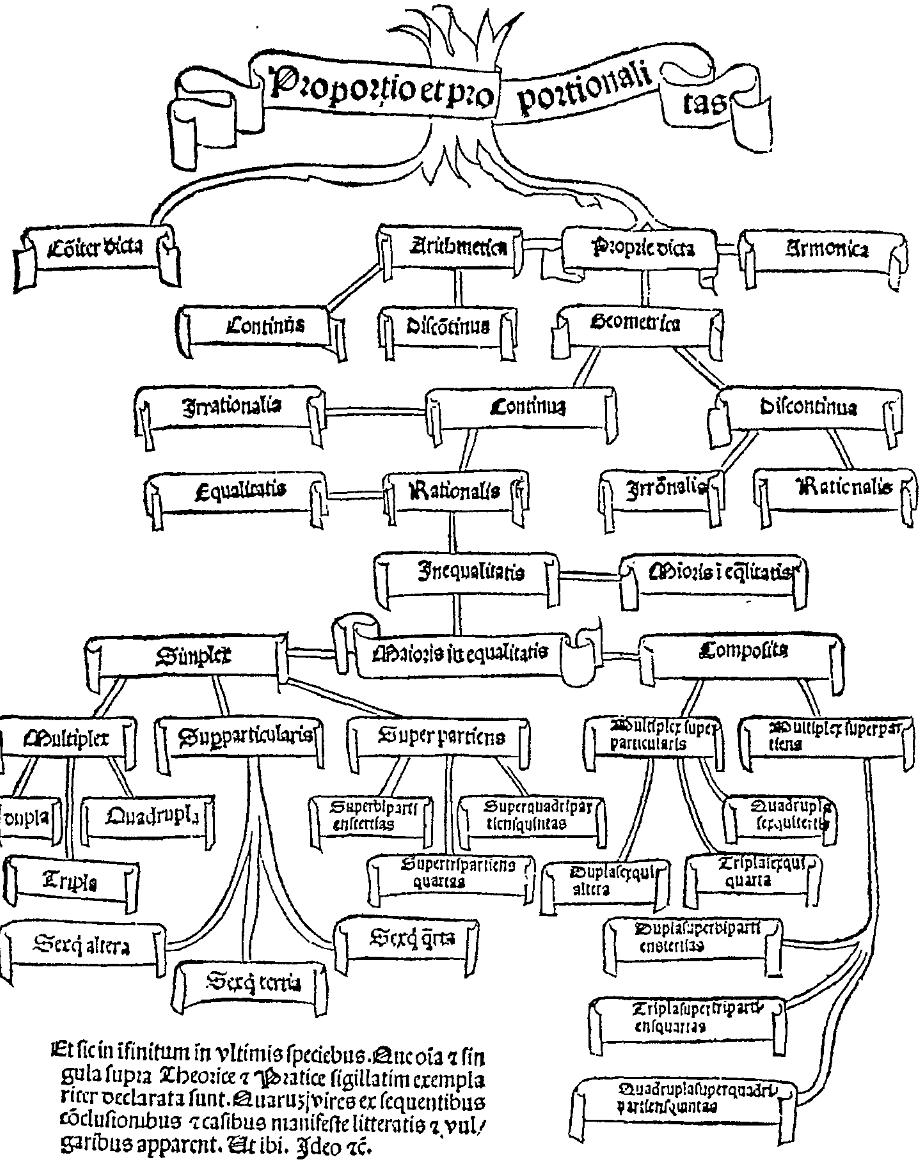


Table 4a: Diagram of proportions in L. Pacioli (1494) *Summa de arithmetica, geometria, proportioni et proportionalita*

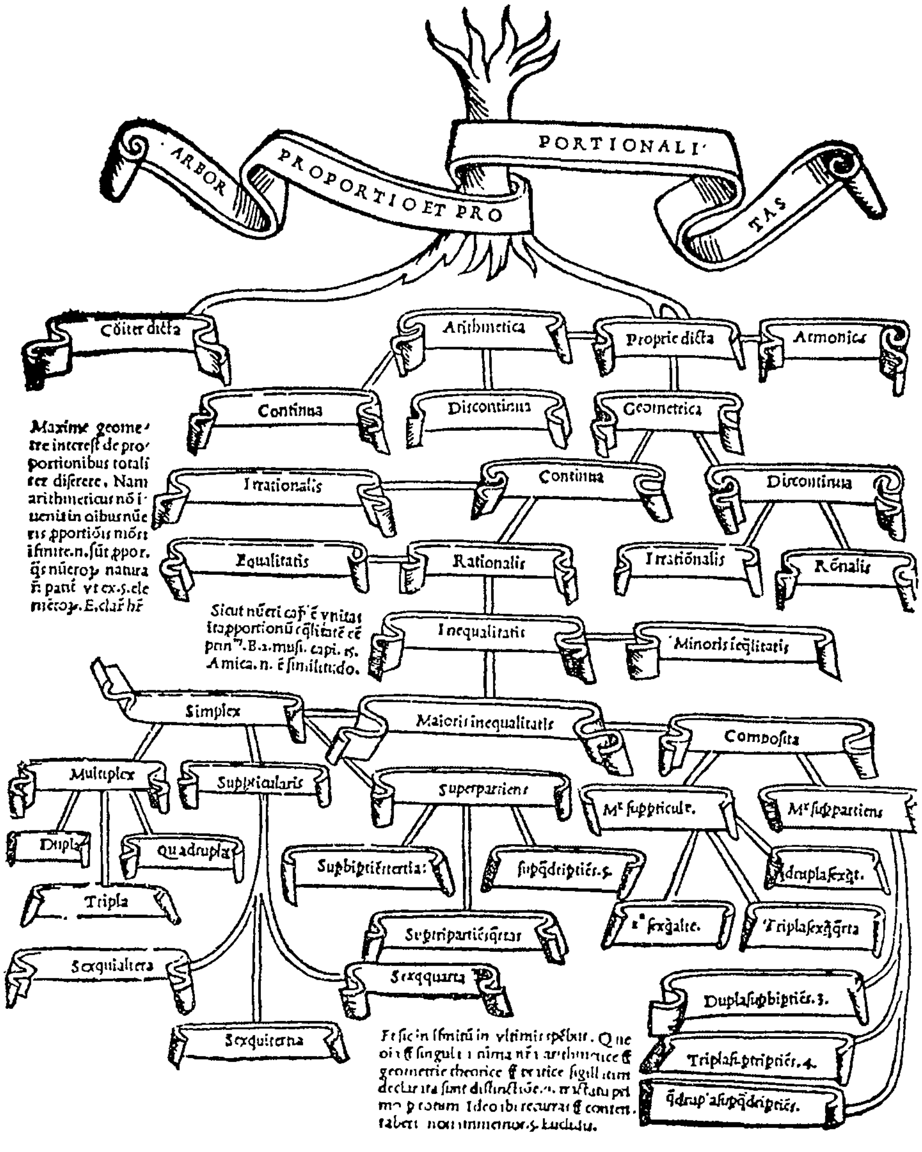


Table 4b: Diagram of proportions in L. Pacioli (1509) *De divina proportione*

2. THE LINK BETWEEN SYMMETRY, PROPORTION AND MEAN IN ART TREATISES

No doubt Nicomachus had absorbed Plato's theories (see also the initial section of the *Introductio* reproduced above) and his treatise, filtered by Boethius and flanked by Ficino and Vitruvius, marked the way which led to the birth of the eurithmy and symmetry concepts, concepts which are fundamental for research and codification of beauty in the art treatises of the XVI century. These could have been the roots of said concepts:

> Plato, *Timaeus,* "... in beginning to construct the body of the All, God was making it [...] The fairest of bonds is that which most perfectly unites into one both itself and the things which it binds together; and to effect this in the fairest manner is the natural property of *proportion.* [...] the body of the Cosmos was harmonised by *proportion* and brought into existence. [...] Whereas, because of this reasoning, He fashioned it to be One single Whole, compounded of all wholes, perfect and ageless and unailing."
>
> Vitruvius, *De Architectura, "Aedium compositio constat ex symmetria, cuius rationem diligentissimi tenere debent. Ea autem paritur a proportione, quae graece analogia dicitur. Proportio est ratae partis membrorum in omni opere totiusque commodulatio, ex qua ratio efficitur symmetriarum. Namque non potest aedis ulla sine symmetria atque proportione rationem habere compositionis, nisi uti ad hominis bene figurati membrorum habuerit exactam rationem."*
>
> "The planning of temples depends upon symmetry: and the method of this architect must diligently apprehend. It arises from proportion (which in Greek is called *analogia*). Proportion consists in taking a fixed module, in each case, both for the parts of a building and for the whole, by which the method of symmetry is put into practice. For without symmetry and proportion no temple can have a regular plan; that is, it must have an exact proportion worked out after the fashion of the members of a finely-shaped human body."

And here is their evolution in Humanism and Renaissance:

> *"Sed pulchritudo atque ornamentum per se quid sit, quidve inter se differant, fortassis animo apertius intelligemus, quam verbis explicari a me possit. Nos tamen brevitatis gratia sic deffiniemus: ut sit pulchritudo quidem certa cum ratione concinnitas universarum partium in eo, cuius sint, ita ut addi aut diminui aut immutari possit nihil, quin improbabilius reddatur."*[18]

> *"Ex ea re suas esse partes instituit Grecia suscepto in opere id conari, ut quos fortunae opibus aequare non possent, hos, quoad in se esset, ingenii dotibus superaret: coepitque uti caeteras artes sic et hanc aedificatoriam ipso ex naturae gremio petere atque educere, totamque tractare, totamque pernoscere, sagaci solertia prospiciens perpendesque. Quid inter ea, quae probentur, aedificia intersit atque ea quae minus probentur, ista in disquisitione nihil praetermisit. Omnia tentavit, naturae vestigia lustrans et repetens. ...neque destitit etiam in minutissimis iterum atque iterum considerasse partes singulas, qui dextra sinistris, stantia iacentibus, proxima distantibus convenirent. Adiunxit, detraxit, aequavit maiora minoribus, similia dissimilibus, prima ultimis; quoad compertum fecit, laudari aliud in his, quae ad vetustatem perferendam quasi stataria constituerentur, aliud in his, quae nullam aeque ad rem atque ad venustatem fabricarentur."*[19]

> *"Itaque superiorum exemplis et peritorum monitis et frequenti usu, admirabilium operum efficiendorum absolutissima cognitio, ex cognitione praecepta probatissima deprompta sunt. ... Istarum rerum praecepta alia universam omnis aedificii pulchritudinem et ornamenta complectuntur, alia singulas partes membratim prosequuntur."*[20]

> *"Quae in rebus pulcherrimis et ornatissimis placeant, ea quidem aut ex ingenii commento et rationibus aut ex artificis manu deveniunt, aut a natura rebus ipsis immissa sunt. Ingenii erit electio distributio collocatio et eiusmodi, quae operi afferant dignitatem."*[21]

> *"Quae si satis constant, statuisse sic possumus: pulchritudinem esse quendam consensum et conspirationem partium in eo, cuius sunt, ad certum numerum finitionem collocationemque habitam, ita uti concinnitas, hoc est absoluta primariaque ratio naturae, postularit."*[22]

Not longer before Alberti, trying to find which is the cause which by its own nature originates beauty *("quidnam sit quod natura sui pulchritudinem efficiat")*[23] listed the criteria used by the ancients and concluded:

> *"Ex quo statuisse possumus, ne catera istiusmodi prolixius prosequar, praecipua esse tria haec, in quibus omnis, quam quaerimus, ratio consumetur: numerus, et quam nos finitionem nuncupabimus, et collocatio. Sed est amplius quippiam ex his omnibus compactis atque nexis, quo tota pulchritudinis facies mirifice collucescat: id apud nos concinnitas nuncupabitur, quam eandem profecto omnis esse gratiae atque decoris alumnam dicimus. Atque est quidem concinnitatis munus et paratio partes, quae alioquin inter se natura distinctae sunt, perfecta quadam ratione constituere, ita ut mutuo ad speciem correspondeant."*[24]

Concerning numbers,[25] after having made the distinction between even and odd and listed notions of numerology Alberti went on to define the perfect number 6=1+2+3=1x2x3. He then states that according to Aristotle the perfect one is 10 because its square 100 is the sum of the cubes of 1, 2, 3 and 4, and that in nature even numbers go up to 10 and the odd ones up to 9. Proportions appear for instance in the paragraph on the forms preferred by the ancients for the plans of rectangular buildings. Sides follow the rules of musical harmony, that is they are in 3:2 (fifth), 4:3 (fourth) , 2:1 (eight). Areas were divided in three groups: short ones, long ones and medium ones. The first ones were square (1:1) or rectangular with a length equal to $1+\frac{1}{2}=3{:}2$, or $1+\frac{1}{3}=4{:}3$. The long ones with the sides for instance in proportion 3:1 or 8:3, while the medium ones were in double 2:1 or triple 3:1 proportion.

> *"In quadrangulis ferme omnibus templis maiores observaverunt aream producere, ut esset ea quidem longior amplius ex dimidia quam lata; alii posuere, ut latitudo parte sui tertia a longitudine superaretur; alii voluere longitudinem duas capere integras latitudines."*[26]

In book IX, going back to the subject of proportions between areas[27] Alberti calls short areas those in the ratios 1:1, 2:3, 3:4, medium areas those in the ratios 1:2, 4:9, 9:16 and long areas those in the ratios 1:3, 1:4, 3:8. He then goes on to discuss buildings, that is the three dimensions, and on ratios between base areas and height he states that there also they can be in harmonic ratios, i. e. as 1:2:3:4, but we can also occur irrational numbers, for instance square roots, when the diagonal of the base quadrangle is taken as height.

> *"Ternatim autem universos corporis diametros, ut sic loquar, coadiugabimus numeris his, qui aut cum ipsis armoniis innati sunt aut sumpti aliunde certa et recta ratione sunt. In armoniis insunt numeri, ex quorum correspondentiis proportiones earum complentur, uti in dupla, tripla, quadrupla. [...] His numeris, quales recensuimus, utuntur architecti non confuse et promiscue, sed correspondentibus utrinque ad armoniam. [...] Diametris etiam finiundis innatae sunt quaedam correspondentiae, quae numeris necquicquam terminari possunt, sed captantur radicibus et potentiis. [...]Tales igitur, quales recensuimus, diametris finiendis et numerorum et quantitatum correspondentiae innatae sunt. Istorum omnium usus est, ut minima linea detur areae latitudini, maxima vero huic correspondens longitudini, mediae vero dentur altitudini. Sed interdum pro aedificiorum commoditate commutabuntur."*[28]

Alberti goes on to discuss other devices used in art to "group dimensions in three" and introduces on this subject the means, mentioning only the first three and supplying for each numerical examples. The arithmetic mean between 4 and 8 is $6=\frac{8+4}{2}$, the geometric mean between 4 and 9 is $6=\sqrt{4\cdot 90}=\sqrt{36}$, and the harmonic mean between 30 and 60 is 40. These means, Alberti states, are mainly used by architects to determine the heights of buildings:

> *"Quae autem diffinitionis ratio non innata armoniis et corporibus sed sumpta aliunde ad diametros ternatim iungendos subserviat, nunc dicendum est. Etenim sunt quidem trium diametrorum in opus coaptandorum annotationes quaedam valde commode ductae cum a musicis tum a geometris tum etiam ab aritmetricis, quas iuvabit recognovisse. Has philosophantes appellarunt mediocritates. Earum ratio et varia et multiplex, sed in primis apud sapientes captandarum mediocritatum modi sunt tres, quorum comparetur numerus ambobus illis positis correspondens certa cum ratione, hoc est, ut ita loquar, affinitatis quadam adiunctione.*
>
> *[...] Ex tribus, quas in primis probarint philosophantes, facillima inventu mediocritas est, quam aritmetricam dicunt. Positis enim extremis numerorum terminis, hoc est hinc maximo, puta octavo, atque hinc e regione minimo, puta quattuor, hos ambos in una iungis summam: fient igitur duodecim; qua summa compositarum in duas partes divisa accipiam ex eis alteram: ea erit unitatum sex. Hunc numerum senarium istic esse aritmetrici mediocritatem statuunt, quae quidem inter illos positos extremos quattuor atque octo aequo ab utrisque distet intervallo.*

Altera vero mediocritas geometrica est; ea captatur sic. Nam minimus quidem terminus, puta quattuor in maximum, puta novem, ducitur. Ex his ita multiplicatis fit summa unitatum sex et triginta; cuius summae, ut loquuntur, radix, id est lateris totiens sumptus quotiens in eo adsit unitas, ipsam compleat aream numerorum triginta sex. Erit igitur radix istaec sex: nam sexties sumpta aream dabunt triginta sex. Hanc aritmetricam [geometricam] mediocritatem perdifficile est ubivis adinvenisse numeris, sed lineis eadem bellissime explicatur; de quibus hic non est ut referam.

Tertia mediocritas, quae musica dicitur, paulo est quam aritmetrica laboriosior; numeris tamen bellissime definitur. In hac proportio quae minimi est terminorum positorum ad maximum, ista eadem proportione se habeant oportet distantiae hinc a minimo ad medium, istinc a medio ad maximum terminorum...

Huiusmodi mediocritatibus architecti et totum circa aedificium et circa partes operis perquam plurima dignissima adinvenere, quae longum esset prosequi. Atqui mediocritatibus quidem istiusmodi ad altitudinis diametrum extollendam apprime usi sunt."[29]

In the following chapter Alberti shows in which way the arithmetic mean had been used for instance to determine the dimensions of columns in the different stylesi: ionic, doric and corynthian. After having made the analogy between the human body and the column, where the base is the unit, he states that height was evaluated by the ancients sometimes 6 and some other 10, so that, in order to correct these values the arithmetic mean was used. The ionic column has a height which is the arithmetic mean between 6 and 10, that is 8. The doric one the arithmetic mean between 6 and 8, i.e. 7, and the corynthian one the arithmetic mean between 10 and 8, i.e. 9:

"Ad tales dimensiones fortassis columnas posuerunt, ut essent aliae ad basin sexcuplae, aliae vero decuplae. Sed naturae sensu animis innato, quo sentiri diximus concinnitates, tanquam istic crassitudinem et contra hic tantam gracilitatem non decere moniti, abdicarunt utranque. Denique hos inter excessus esse quod quaererent existimarunt. Ea re aritmetricos in primis imitati, ambo illa extrema in unum coegere et summam iunctorum per mediam divisere."[30]

Alberti's interest for the measurements to use in art and for mutual relations between the elements of buildings are the result of observations on works of art of antiquity as well as of the rules established by ancient artists. On this subject following statements, characterizing his concept of beauty and harmony, are particularly significant:

> *"Quare in primis observabimus ut ad libellam et lineam et numeros et formam et faciem etiam minutissima quaeque disponantur, ita ut mutuo dextera sinistris, summa infimis, proxima proximis, aequalia aequalibus aequatissime conveniant ad istius corporis ornamentum, cuius futurae partes sunt."*[31]

> *"Ergo recte asseverant, qui dicunt reperiri vitium nullum deformitatis obscenius atque detestabilius, quam ut angulos aut lineas aut superficies numero magnitudine ac situ non diligenter examinateque inter se comparatas coaequatas atque compactas intermiscere."*[32]

> *"Omnium ratio et ordo ita comparentur, ut non modo ad opus honestandum certatim conveniant, sed ne altera quidem sine alteris per se constare aut satis suam servare dignitatem posse videantur."*[33]

> *"Sic istic, quotquot ubique aderunt opinione et consensu hominum probata opera, perquam diligentissime spectabit, mandabit lineis, notabit numeris, volet apud se diducta esse modulis atque exemplaribus; cognoscet repetet ordinem locos genera numerosque rerum singularum, quibus illi quidem usi sunt praesertim, qui maxima et dignissima effecerint, quos fuisse viros egregios coniectura est, quandoquidem tantarum impensarum moderatores fuerint."*[34]

The word used by Alberti to indicate the mean is *mediocritas* and among the classical sources, where he looked for inspiration might have been Nicomachus'arithmetic. Indeed this author is mentioned as arithmetician[35] and among the other classical authors who wrote on means Plato, Aristotle, Pythagoras, Tucidides, Vitruvius, Galen and the historians Plinius, Plutarch are mentioned in Alberti's treatise. It should be noted that Alberti does not simply report the statements of the ancients, but often adds his own deductions resulting from direct observation.[36] The terminology that he uses for fractions is the one introduced by Nicomachus.

4. THE PROPORTIONS IN THE PORTRAYAL OF HUMAN BODY

Particularly concerning the rules to be applied in representing the human figure the two aspects, touched at the beginning, the theoretic and the experimental one, can be seen to surface. As regards the theoretic one the main source is Vitruvius, who uses exclusively rational ratios. The canons are modified in the course of the centuries, but the basic theory implying tht beauty is determined by the mutual relations of parts, remains.

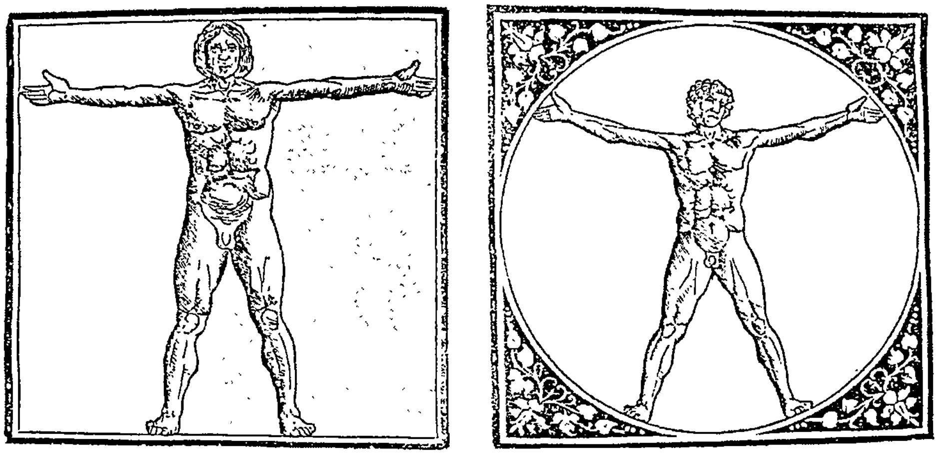
Alberti deviates somewhat from this tradition and, possibly having red Thucidides, absorbs the idea of the mean of observations, and in his *De statua* he imitates that method. Polyclitus Canon is thus recovered![37]

If at this point we go back to Polyclitus and the period in which he was writing his Canon, it is clear that he based his purpose on the philosophical, mathematical and technical knowledge of his time. His aims thus fit into Greek philosophical thinking, which aimed to find harmony and beauty of the universe; and since, according to Pythagoras, this harmony lay in musical relations, based on numbers, Polyclitus turned to these philosophico-mathematical theories in considering proportion and mean. To some extent his treatise echoed the Pythagorean theories on music, mathematics and art, as expressions of a universe ruled by number. The direct view of nature, which was certainly the inspiration for the Greek *mimesis*, i.e., the attempt to instill the naturalistic vision into art (and not the fantastic or richly imaginative vision of the Egyptians or the Babylonians or other populations bordering on the Greek world) is a further piece in this construction ruled by number and by arithmetic.

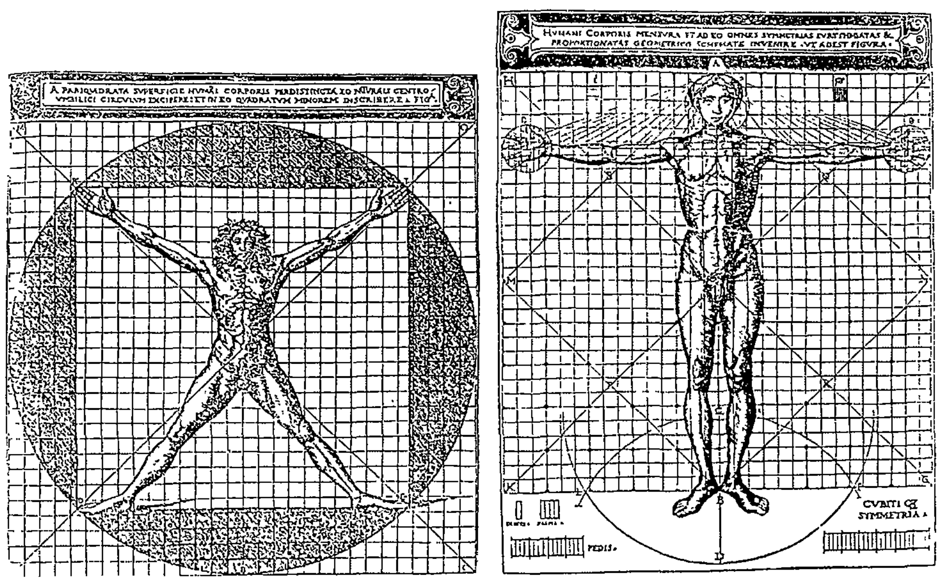
It is in this fusion of *mimesis* and mathematics/arithmetic that Polyclitus' originality lies, and this is why he had such a following not only in ancient times, but in Humanism and the Renaissance. Harmony expressed by means of numbers. Creating, on the basis of mathematical ratios or relationships, a form which, in addition to being naturalistic, might express an ideal essence. Just as music translates mathematical rules, proportions and means into perceptible form, thus becoming the expression/manifestation of theory through the senses, so the visual arts, architecture, the portrayal of the beauty of the human body require exact mathematical rules. Mathematics, the expression of exactness, of the possibility of understanding the universe regulated by number, as the Pythagoreans thought, is sublimated by Plato in the *Timaeus* and subsequently, in the Neoplatonic tradition (1st BC-5th AD)) is rediscovered and reappraised in Humanism. Plato, then, is the go-between for the transfer of the precepts of harmony and music, linked to mathematics.

Although so little survives of the 'rules' or 'canons' given by Polyclitus in his treatise, there can be no doubt that it was from him that a line of thought originated which tried to 'codify' the principles to be followed by artists, in schemes which were either mathematical (harmony linked to arithmetic and to music, following the Pythagorean tradition) or statistical (harmony which is introduced into the universe and into reality through the forms of greatest beauty is 'observed' and translated into mathematical rule with the arithmetical mean of the data observed). This is what, perhaps somewhat inappropriately, I have called experimental, linking it to some degree to the not dissimilar observation in Alberti's work *Della statua*:[38]

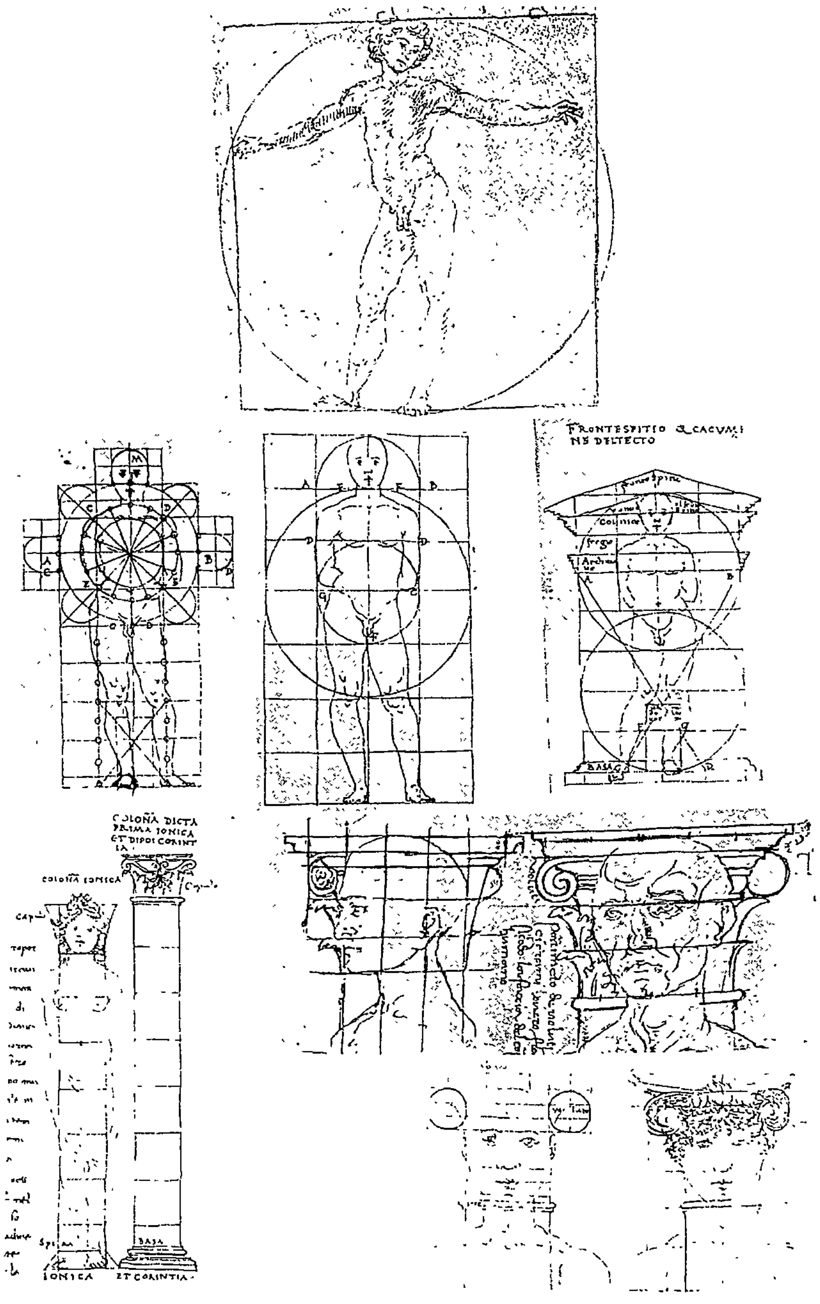
> *"Finalmente mediante tutte quelle cose che insino a qui si son dette, si vede assai manifesto, che si posson pigliare le misure, e i determinamenti da un modello, o dal vivo comodissimamente, per fare un lavoro o un'opera che sia, mediante la ragione, e l'arte perfetta. Io desidero che questo modo sia familiare a' miei Pittori e Scultori, i quali se mi crederanno se ne rallegreranno. E perché la cosa sia mediante gli esempi più manifesta ... ho presa questa fatica di descrivere cioè le misure principali che sono nell'uomo. E non le particolari solo di questo o di quell'altro uomo; ma, per quanto mi è stato possibile, voglio porre quell'esatta bellezza, concessa in dono dalla natura, e quasi con certe determinate porzioni donata a molti corpi, e voglio metterla ancora in scritto, imitando colui che avendo a fare appresso a' Crotoniati la statua della Dea, andò scegliendo da diverse Vergini, e più di tutte l'altre belle, le più eccellenti, e più rare, e più onorate parti di bellezze che egli in quelle giovani vedesse, e le messe poi nella sua statua. In questo medesimo modo ho io scelto molti corpi, tenuti da coloro che più sanno, bellissimi, e da tutti ho cavate le loro misure e proporzioni; delle quali avendo poi insieme fatto comparazione, e lasciati da parte gli eccessi degli estremi, se alcuni ve ne fossero che superassino, o fossero superati dagli altri, ho prese da diversi corpi e modelli, quelle mediocrità, che mi son parse le più lodate. Misurate adunque le lunghezze, e le larghezze, e le grossezze principali e più notabili, le ho trovate che son così fatte."*[39]



1. M. Vitruvius (1511) *per Iocundum .. cum figures et tabula ...*



2. C. Caesarirano (1521) *Di Lucio Vitruvio Pollione; De Architectura libri...*



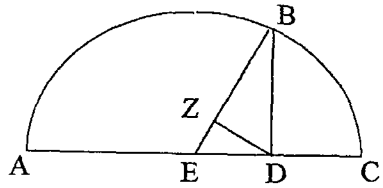
3. F. di Giorgio Martini (1487-1492) *Architettura Ingegneria, Arte militare*

Table 5: Vitruvian figures *(Homo ad circulum et quadratum)*

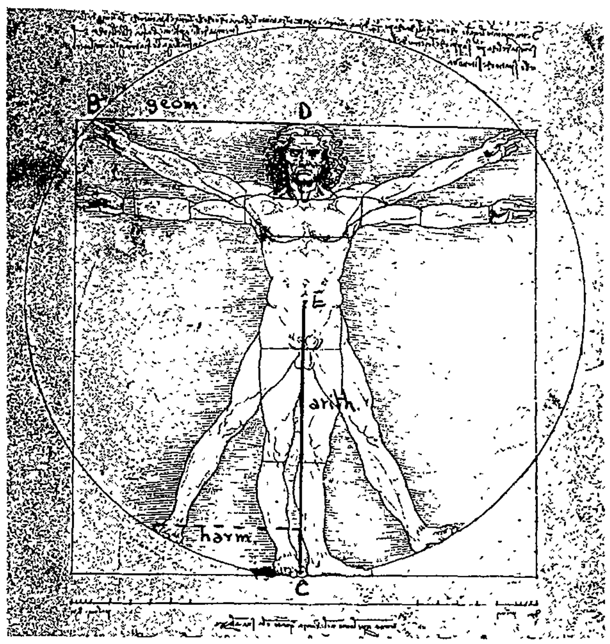
This road is followed by Leonardo[40] and finally by Dürer, with whom the most complete codification is reached. Canons are defined not only for adult men, but also for children, women and old men (Table 5). Leonardo codifies not only the proportions of the various arts of the human body, but also those resulting from the mouvement of the limbs. Beauty is enhenced through the statistical aspect. And even if, following a theorem reported by Pappus,[41] the three means can be found/seen even in the portrayal of the human body inside a circle and a square, artists do not mention it. This mathematical property was either unknown or unimportant to them! In my opinion we find here the same attitude found above on the golden mean. The rule exists and is important for the mathematician, but it is not codified by the artist and the art theoretician. It remains inside the work of art without reaching consciousness.

Things are different for the statistical mean. In this case the artist perceives something which has not yet been codified by mathematics. The law of the large numbers which surfaces in Polyclitus, in Thucidides and in Alberti will find its final formulation only in Jacob Bernoulli's *Ars Conjectandi* (1713). This passing on of 'experience' has meant that the mathematical concept of the mean, which has an important place in the calculation of probabilities and in statistics, two disciplines which date back only to the 17th century, was anticipated by the artists and art theoreticians of Classical Greece (Polyclitus) and of Humanism and the Renaissance (Alberti). If we accept the anthropological point of view according to which mathematics, as an officially recognised activity, is simply the outlet, into the light of awareness, of an activity rooted in the human subconscious, or even in the unconscious, just as the tip of an iceberg is only the surface appearance of a hidden, vastly larger mass, we may see here the example of an artistic manifestation in which mathematical properties are applied even before the relevant theory has been developed. The figurative arts, in this case, may be significant for the mathematics historian in showing the sensitivity to and gift for geometry or mathematics in a particular cultural context, although they cannot in themselves provide documentation of the effective acquisition of the mathematical concept. No concept can emerge in the consciousness unless it has been made the object of research and study.

1. L. B Albertı (1435) *Della statua*

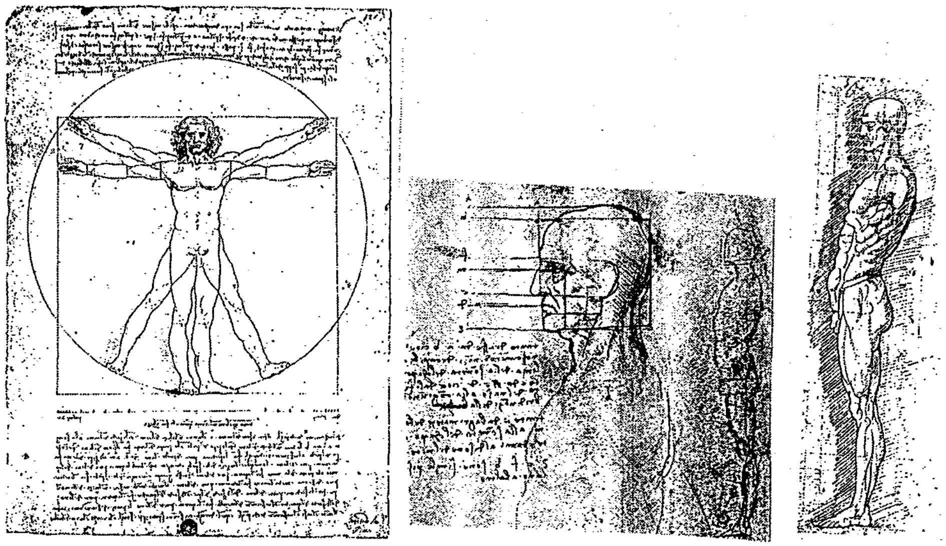


2. Pappus, *Collectio Mathematica* III.,11, 28
EC arithmetic mean
BD geometric mean
ZB harmonic mean

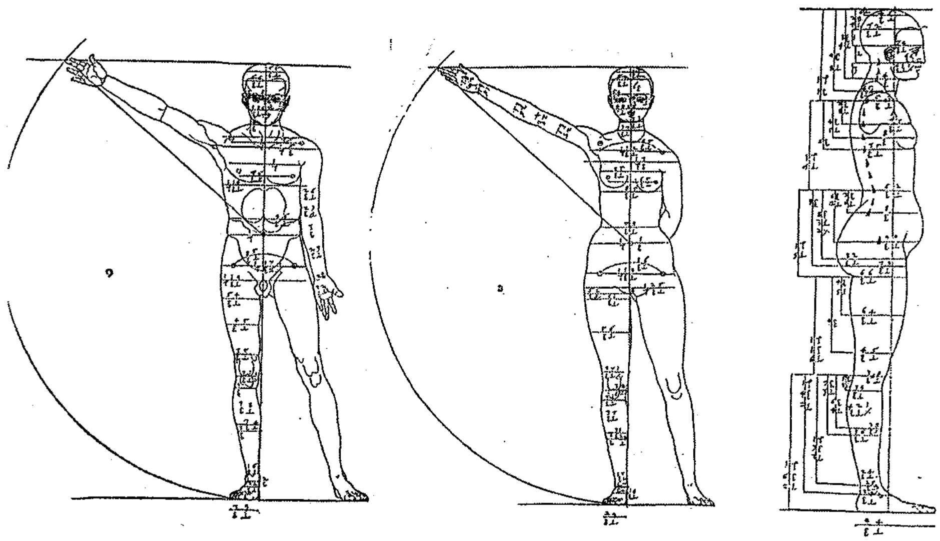


3. Leonardo (1492-1510) *Codex Urbınax Latinus 1270*

Table 6: The means and the portrayal of the beauty of the human body



1. Leonardo (1492-1510) *Codex Urbinax Latinus 1270*



2. A. Dürer (1528) *Bücher von menschlicher Proportion*

Table 7: Studies of the proportions of the human body

ENDNOTES

[1] Polyclitus' *Canon* [Philon *Mechan. synt*, IV, I, 49, 20 - Diels (1949-54, 40B2)] Cf also Diels (1889, note I, p 10); Panofski 1962, p. 73 and La Rocca (1979, pp 524-526.)

[2] Cf. Plutarch, *Moralia*, I, 91 and Galen, *Placita Hippocratis et Platonis*, V, 3

[3] Cf Panofski (1921) and Wittkower (1952, 1960), pp. 199-201 and (1962), pp. 1-29.

[4] The general definitions of the mean which are closest to the true essence of the concept are those which Oscar Chisini gave (Chisini 1929, p 106): *"Data una funzione* $y = f(x_1, x_2, \ldots, x_n)$ *di un certo numero n di variabili indipendenti* $x_1, x_2, \ldots, x_n$, *rappresentanti grandezze omogenee, dicesi media delle* $x_1, x_2, \ldots, x_n$ *rispetto alla funzione f quel numero M che sostituito alle* $x_1, x_2 \ldots x_n$ *dà il medesimo valore per la f che le* $x_1, x_2, \ldots, x_n$ *stesse, cioè quel numero M tale che* $f(M, M, \ldots, M) = f(x_1, x_2, \ldots, x_n)$. *È facile trasformare questa definizione in modo da dare l'espressione analitica esplicita della media A tale scopo si osservi che se, nella f, al posto delle* $x_1, x_2, \ldots, x_n$ *mettiamo un unico valore x, la f stessa diviene una funzione di una sola variabile; chiamiamo* f_1 *questa funzione. E' dunque* $f_1(x) = f(x, x, \ldots, x)$*."* ("Given a function $y = f(x_1, x_2, \ldots, x_n)$ of a certain number *n* of independent variables $x_1, x_2, \ldots, x_n$, representing homogeneous magnitudes, the mean of $x_1, x_2, \ldots, x_n$, with respect to the function *f* is that number *M* which, when it replaces $x_1, x_2, \ldots, x_n$, gives the same value for *f* as those same $x_1, x_2, \ldots, x_n$, i.e that number *M* such that $f = (M, M, \ldots, M) = f(x_1, x_2, \ldots, x_n)$. This definition can easily be transformed in such a way as to give the explicit analytical expression of the mean. To this end, note that if, in *f*, in place of $x_1, x_2, \ldots, x_n$, we put a single value *x*, *f* itself then becomes a function of a single variable; we call this function f_1. Thus, $f_1(x) = f(x, x, \ldots, x)$.") and Corrado Gini's in his essay *Le medie* (1958, pp. 57-58): *"media fra più quantità è il risultato di una operazione eseguita con una data norma sopra le quantità considerate, il quale rappresenta o una delle quantità considerate che non sia superiore né inferiore a tutte le altre (media reale o effettiva) oppure una quantità nuova intermedia fra la più piccola e la più grande delle quantità considerate (media di conto)"* ("Mean among several quantities is the result of an operation carried out with a given rule on the quantities being considered, which represents either one of those quantities which should be neither greater nor smaller than all the others (real or effective mean) or a new quantity intermediate between the smallest and the greatest of the quantities considered (count mean)"). Some statistics scholars (De Finetti 1931, pp. 369-370; Jacklin 1949, pp. 3-11; Pizzetti 1950, p 428) have written that these are the first acceptable formulations to appear in the history of mathematics in that they show the intrinsic significance of the concept of the mean and the purposes it must answer They also disagree with those who regard Cauchy as the first founder of the formulation of this concept. In his *Cours d'analyse* [*1re partie, Analyse algebrique*, Imprimerie Royale, Paris (1821), pp. 26-29] Cauchy defined 'mean of several given quantities [as] a new quantity lying between the smallest and the greatest of those considered' Similarly vague, inexact definitions had already appeared in the history of mathematics. As Chisini himself maintains (Chisini 1929, p. 106), supported by De Finetti (1931, p. 370), Cauchy's definition 'is virtually meaningless, and defining the single kinds of mean habitually encountered (arithmetical, geometrical, harmonic, etc) is admittedly an exact operation, but merely formal and antiphilosophical, which may be used - badly - only for empirical purposes.'

[5] Boyer 1968, p. 18.

[6] Boyer 1968, p. 42.

[7] Boyer 1968, p. 31.

[8] Cf Tannery 1912, pp. 80-105, Michel 1950, pp. 365-411; Klein 1968

[9] Boyer 1968, p. 61. In modern terms if *a*, *b*, *c* are natural numbers and $a>b>c$, the arithmetical mean can be written by the proportion

$$(a-b):(b-c) = a:a = b \cdot b = c:c,$$

$$a-b = b-c \text{ i.e., } b = \frac{a+c}{2}, \ \frac{a}{b} < \frac{b}{c}$$

the geometrical mean by

$$(a-b):(b-c)=a:b$$

$$a:b=b\cdot c \quad \text{i.e ,} \quad b=\sqrt{ac}$$

and the harmonic mean by

$$(a-b)\ (b-c)=a:c$$

$$ac-bc=ab-ac \text{ i.e , } b=\frac{2ac}{a+c},\ \frac{a}{b}>\frac{b}{c}$$

[10] Iamblichus, *In Nicom. Arith Introd.* 100, 19 ff: "In ancient days in the time of Pythagoras and the mathematicians of his school there were only three means, the arithmetic and the geometric and a third in order which was then called subcontrary, but which was renamed harmonic by the schools of Archytas and Hippasus, because it seemed to furnish harmonious and tuneful ratios. And it was formerly called subcontrary because its character was somehow subcontrary to the arithmetic. [. .] After this name had been changed, those who came later, Eudoxus and his school, invented three more means, and called the fourth properly subcontrary because its properties were subcontrary to the harmonic [..] and the other two they named simply from their order, fifth and sixth."

[11] Nichomacus Chap. XXI, p 264, Chap. XXII, pp. 266-267.

[12] Nichomacus Chap. XXVI. In chap. XXIX he again wrote on the harmonic mean. "the most perfect proportion, that which is three-dimensional and embraces them all, and which is most useful for all progress in music and in the theory of the nature of the universe This alone would properly and truly be called harmony, rather than the others, since it is not a plane, nor bound together by only one mean term, but with two, so as thus to be extended in three dimensions, just as a while ago it was explained that the cube is harmony."

[13] Porphyry, *Commentary on Ptolemy's Harmonics*: "Archytas, in his discussion of means, writes thus: 'Now there are three means in music: first the arithmetic, secondly the geometric, and thirdly the subcontrary, the so-called harmonic. The arithmetic is that in which three terms are in proportion in virtue of some difference the first exceeds the second by the same amount as the second exceeds the third. And in this proportion it happens that the interval between the greater terms is the lesser, while that between the lesser term is the greater. The geometric mean is that in which the first term is to the second as the second is to the third. Here the greater terms make the same interval as the lesser. The subcontrary mean, which we call harmonic, is such that by whatever part of it the first term exceeds the second, the middle term exceeds the third by the same part of the third In this proportion the interval between the greater terms is the greater, that between the lesser term is the lesser.

[14] Cf. Alberti 1485 Book IX, Chap VI p. 168v-169r (1966, vol. 2, p. 833)

[15] Empedocles seems to have introduced a μεσοζ between the two extremes, Hatred and Love; Plato regarded the mean as a normative ethical principle, harmony of contrasting or differing qualities and principles, harmony of the qualities of the body (*Laws*728D-), of body and soul (*Timaeus* 87c), of joy and sorrow (*Laws*732D-), of wealth and poverty (*Republic* IV, 421 E-), of monarchy and democracy (*Laws* 756E) Aristotle too used the term μεσος in physics, collocating it at the center of the universe as that point without which circular motion would not be possible (*De cael.* II 3 286a); moreover, dealing with time in relation to present and past, he writes (Aristotle *Phys.* 251b20) '...the present is a certain 'meanity' (*μεσοτης*), and in ethics (*Et. Nic* II 5, 1106 a 29-) the term μεσος is used to describe virtue. Aristotle makes a sharp distinction between the mean in relation to things (corresponding to the arithmetical mean) and the mean in relation to ourselves (a concept which shades into relativity) (*Et. Nic.* II 6, 1106 a 26-1106 b 7)

[16] Cf. Alberti

[17] Francesco di Giorgio Martini 1482-92 *Trattato di architettura civile e militare* pp. 119-120· *"Porzione è ditta l'abitudine overo la comperazione tra due quantità. Numari proporzionali son detti quando lo primo è così al sicondo come lo terzo al quarto. Numari in continua proporzione son detti quando lo primo è così al secondo come el secondo al terzo e come el terzo al quarto E se quattro numeri seranno proporzionali, la proporzione che è dal primo al secondo si chiama prima porzione, e quella che è dal primo al terzo si chiama siconda porzione, e quella che è dal primo al quarto si chiama terza porzione. E la siconda porzione quadrato della prima, e la terza porzione sie cubo della prima "*

[18] Alberti *De re aedificatoria* Chap. II, 93v - *L'architettura*, vol. 2, 1966, pp 446-447 *("In che consistano precisamente la bellezza e l'ornamento, e in che differiscano fra loro, sarà probabilmente più agevole a comprendersi nell'animo che ad esprimersi con parole. Ad ogni modo, senza stare a dilungarci, definiremo la bellezza come l'armonia tra tutte le membra, nell'unità di cui fan parte, fondata sopra una legge precisa, per modo che non si possa aggiungere o togliere o cambiare nulla se non in peggio"*

[19] Alberti *De re aedificatoria* Book VI, Chap. III, 94 - *L'architettura*, vol. 2, 1966, pp 452-453 *("In tal modo i Greci decisero che in tali imprese fosse proprio compito il tentare di superare quei popoli, non già nei doni di fortuna, che non era possibile, bensì nella potenza dell'ingegno, per quanto stava in loro. Cominciarono dunque a desumere i fondamenti dell'architettura, come di tutte le altre arti, dal seno/grembo stesso della natura, e ad esaminare, meditare, soppesare ogni elemento con la massima diligenza e oculatezza. Non trascurarono di ricercare i canoni che distinguono quali edifici siano ben eseguiti e quali sbagliati. Fecero ogni sorta di esperimenti, seguendo le orme della natura. ... non tralasciarono mai, nemmeno nelle cose più minute, di esaminare volta per volta la disposizione delle parti, di modo che quelle di destra si accordassero con quelle di sinistra, le verticali con le orizzontali, le vicine con le lontane, aggiungendo, levando, adeguando le parti più grandi alle più piccole, le simili alle dissimili, le prime alle ultime. Divenne così evidente quali criteri dovevano essere impiegati nelle costruzioni destinate a durare negli anni e in quelle realizzate soprattutto per amore della bellezza.")*

[20] Alberti *De re aedificatoria* Book VI, Chap. III, 95v - *L'architettura*, vol. 2, 1966, pp. 456-457. ("Ebbene dall'esempio degli antichi, dai consigli degli esperti e da una pratica continua, si è ricavata una esatta conoscenza dei modi in cui quelle opere meravigliose venivano condotte, e da questa conoscenza si sono dedotte delle regole importantissime Tali regole si riferiscono in parte alla bellezza e alla decorazione di ogni edificio nel suo complesso, in parte alle singole membrature di esso.")

[21] Alberti *De re aedificatoria* Book VI, Chap. IV, 95v - *L'architettura*, vol 2, 1966, pp. 458-459. ("Le caratteristiche che si apprezzano negli oggetti più belli e meglio ornati o sono frutto di ritrovati e calcoli dell'ingegno, o del lavoro dell'artefice o sono state conferite direttamente dalla natura a tali oggetti All'ingegno spetterà la scelta, la distribuzione delle parti, la disposizione e simili, col fine di dare decoro all'opera.")

[22] Alberti *De re aedificatoria* Book IX, Chap. V, 165r - *L'architettura*, vol. 2, 1966, pp. 816-817. *("Una volta acquisite queste nozioni possiamo stabilire quanto segue: la bellezza è accordo e armonia delle parti in relazione ad un tutto al quale esse sono legate secondo un determinato numero, delimitazione e collocazione, così come esige la concinnitas, cioè la legge fondamentale e primaria della natura.")*

[23] Alberti *De re aedificatoria* Book IX, Chap. V, 164r - *L'architettura*, vol 2, 1966, p. 811.

[24] Alberti *De re aedificatoria* Book IX, Chap V, 165r - *L'architettura*, vol. 2, 1966, p. 815. *("Da quanto sopra si può desumere, senza soffermarci troppo a lungo su altre considerazioni di questo tipo, che tre sono le principali leggi su cui si basa il metodo che cercavamo: numero, ciò che chiamiamo delimitazione e collocazione. Vi è inoltre una qualità risultante dall'unione e connessione di questi [tre] elementi, per cui risplende mirabilmente tutta la forma della bellezza e noi la chiamiamo concinnitas e diciamo che per suo tramite tutto è frutto di grazia e decoro. E' compito e disposizione della concinnitas l'ordinare secondo leggi precise le parti che altrimenti per loro natura sarebbero distinte tra loro, di modo che il loro aspetto presenti una mutua concordanza.").* For the finitio: *"Finitio quidem apud nos est correspondentia quaedam linearum inter se, quibus quantitates dimetiantur. Earum una est longitudinis, altera latitudinis, tertia altitudinis."* (reciprocal correspondence of lines used to measure quantities and their dimensions. One is length, the second is width and the third is hight.) This one is reguled by music. *"Hi quidem numeri, per quos fiat ut vocum illa concinnitas auribus gratissima reddatur, hidem ipsi numeri perficiunt, ut oculi animusque voluptate mirifica compleantur. Ex musicis igitur, quibus hi tales numeri exploratissimi sunt, atque ex his praeterea, quibus natura aliquid de se conspicuum dignumque praestet, tota finitionis ratio perducetur."*

[25] Alberti *De re aedificatoria* Book IX, Chap. V, 165v-166r - *L'architettura*, vol. 2, 1966, pp. 818-820.

[26] Alberti *De re aedificatoria* Book VII, 114v - *L'architettura*, vol. 2, 1966, pp. 550-551 *("In quasi tutti i templi a forma quadrilatera gli antichi prediligevano la forma allungata, di modo che la lunghezza della loro area risultasse maggiore per un mezzo della larghezza; altri preferivano una forma tale che la larghezza venisse superata di un terzo dalla lunghezza; altri ancora stabilirono che questa fosse doppia di quella ")*

[27] Alberti *De re aedificatoria* Book IX, 167r-v - *L'architettura*, vol. 2, 1966, pp. 824-827.

[28] Alberti *De re aedificatoria* Book IX, 167v-168 - *L'architettura*, vol 2, 1966, pp 827-831

[29] Alberti *De re aedificatoria* Book IX, 168v-169r - *L'architettura*, vol. 2, 1966, pp 830-835.

[30] Alberti *De re aedificatoria* Book IX, 169v - *L'architettura*, vol. 2, 1966, pp 834-837

[31] Alberti *De re aedificatoria* Book IX, 170r - *L'architettura*, vol. 2, 1966, pp. 838-839

[32] Alberti *De re aedificatoria* Book IX, 171r - *L'architettura*, vol. 2, 1966, pp. 842-843.

[33] Alberti *De re aedificatoria* Book IX, 172v - *L'architettura*, vol. 2, 1966, pp. 850-851.

[34] Alberti *De re aedificatoria* Book IX, 173v - *L'architettura*, vol. 2, 1966, pp 856-857.

[35] Cf. Alberti *De re aedificatoria* Book IX, ***- *L'architettura*, vol 2, 1966, p 862.

[36] Cf. Alberti *De re aedificatoria* Book IX, ***- *L'architettura*, vol. 2, 1966, pp. 526-7, 562-3, 586-7.

[37] Plutarco, *Moralia* I, 91 45 C-D *"Perchè in ogni opera la bellezza si realizza per mezzo della simmetria ed armonia, ad esempio attraverso molti numeri che convergono nel punto giusto, mentre il brutto ha un'immediata ed improvvisa origine da un difetto casuale o da un eccesso casuale."* Galeno, (II sec. d. C) *De temperam.* I, 9, p 566 *"E viene lodata una statua, chiamata Canone di Policleto, la quale ha questo nome dal fatto di avere una perfetta simmetria di tutte le membra fra di loro "*; Galeno, *Placita Hippocratis et Platonis* V, 3 *"Crissippo invece ritiene che la bellezza non consista nei singoli elementi ma nell'armoniosa proporzione delle parti, di un dito rispetto all'altro, e di tutte insieme le dita in relazione al metacarpo ed al carpo, e di tutte queste rispetto all'avambraccio, e dell'avanbraccio rispetto al braccio, e di tutto in rispetto al tutto, secondo quanto appunto è scritto nel Canone di Policleto. Infatti egli, avendo istruito tutti noi in quello scritto sulla simmetria del corpo, rinsaldò il ragionamento con l'opera avendo creato una statua secondo i dettami del ragionamento ed avendo poi chiamata la stessa natura, come appunto lo scritto, Canone."*

[38] Probably Alberti's inspiring source was Thucidides. Indeed he mentions the Greek historian concerning the walls of the city of Platea during the siege by the Peloponnesians and consequently he must have read the famous paragraph on the mean of observations, which is commonly considered the first step in the prehistory of the law of the large numbers: "They made ladders equal in height to the enemy's wall, getting the measure by counting the layers of bricks at a point where the enemy's wall on the side facing Plataea happened not to have been plastered over. They counted the layers at the same time, and while some were sure to make mistake, the majority were likely to hit the true count, especially since they counted time and again, and, besides, were at no great distance, and the part of the wall they wished to see was easily visible. The measurement of the ladders, then, they got at in this way, reckoning the measure from the thickness of the bricks.

[39] Leon Battista Alberti (1435) *Della statua.*

[40] Leonardo 1492-1510 *Codex Urbinax Latinus 1270*, Chap. X-XI - *Scritti d'arte del Cinquecento*, vol. II ed. P. Barocchi 1973, pp. 1720-1731: X. *Se l'omo di 2 braccia è picolo, quello di quattro è troppo grande, essendo la via di mezzo laldabile; il mezo in fra 2 e 4 è 3· adunque piglia un omo di 3 braccia e c'quello misura colla regola ch'io ti darò. Se tu mi dicessi, io mi potrei ingannare, giudicando uno bene proporzionato che sarebbe il contrario, a cquesta parte i' rispondo che tu debbi vedere molti omini di 3 braccia e c'quella maggiore quantità che sono conformi di membri: sopra uno di quelli di migliore grazia piglia tue misure; la lungheza della mano è 1/3 di braccio e entra 9 volte nell'omo, e così la testa è da la fontanella della gola a la spalla e da la spalla a la tetta e da l'una all'altra tetta e da ciascuna tetta alla fontanella. XI. Vetruvio arhitecto mecte nella sua opera d'architectura, chelle misure dell'omo sono dalla natura distribuite in questo modo, cioè che 4 diti fan uno palmo, e 4 palmi fan uno piè, 6 palmi fan un cubito, 4 cubiti fan uno uomo e 4 cubiti fan uno passo, e 24 palmi fan uno uomo, e queste misure son ne' sua edifizi. Se tu apri tanto le gambe che tu cali da capo 1/14 di tua altezza e apri e alza tanto le braccia che colle lunghe dita tu tochi la linia della sommità del capo, sappi che 'l cientro delle stremità delle aperte membra fia il bellico, e lo spazio che si trova infra le gambe fia triangolo equilatero. Tanto apre l'omo nelle braccia quanto è la sua altezza."*

Leonardo treats the following subjects· "I Figura e sua divisione, II Proporzione di membra, VIII Dell'attitudine e movimenti e loro membra, IX Dello imparare li movimenti dell'ommo, XII Delle mutazioni delle misure dell'uomo pel movimento delle membra a diversi aspetti, XIII Delle mutazioni delle misure dell'uomo dal nascimento al suo ultimo accrescimento, XIV Delle prime quatro parti che si richiedono alla figura, XV De la convenzione delle membra, XVI Della grazia delle membra, XVII De la comodità delle membra." Mario Equicola s l (1526) *Libro di natura d'amore,* 73v-79v [*Scritti d'arte del Cinquecento,* vol II ed. Barocchi, P. (1973), pp. 1615-1621] *"Per la eccellenzia ai Crotoniati piacque che [Zeusi] pingesse loro alcuna cosa, e la immagine di Elena. Disseli che volea vedere alcune lor virginelle; Crotoniati, per conoscerlo eccelente pittore di donne, volentieri li consentirno, e monstrateli (che così consultaro), le più belle scelse; per dimostrar la singular grazia in una non ritrovarse, tolse da ciascuna la più egregia parte, che beltà compitamente non se vede in una sola. Così finì la sua leggiadra opera e tante bellezze vive in una figura accolse. [. .] La bellezza del corpo ricerca che le membra siano ben collocate con debiti intervalli e spazii, ciascuna parte sia con sue tempre, commensa proporzione e conveniente qualità. .Plinio, Varrone e Gellio scriveno il corpo umano non posser crescere sopra 7 piedi in longhezza .. Mettendo l'uomo con le braccia estese, tirando dall'umbilico [...] linee all'estremità di piedi et de dita della mano, trovaremo fanno un circulo perfetto. Vitruvio il corpo dell'uomo dice esser stato da natura così composto, che la faccia tutta, cioè la punta del mento sin dove finiscono li capelli nella fronte, è la decima parte del corpo, dal commo petto, cioè dove finisce il collo sino alla sommità del capo parti quattro: se il corpo è ben quadrato e robusto di 7 teste il trovarai· se è dilicato di 8 e 9, le donne di 7 il più delle volte.. .";* Nifo, Agostino or Augustinus niphus (1549) *De Pulcro primus, De Amore secundus - Scritti d'arte del Cinquecento,* vol. II ed. Barocchi, P. (1973), pp. 1647-1652: *"Quod simpliciter pulchrum sit in rerum natura, ex illustrissimae Ioannae pulchritudine hic probatur".*

[41] Pappus *Coll.* III (ed Hultsch 68, 17-70. 8) "A certain other [geometer] set the problem of exhibiting the three means in a semicircle. Describing a semicircle ABC, with center E and taking any point D on AC, and from it drawing DB perpendicular to EC, and joining EB, and from D drawing DZ perperpendicular to it, he claimed simply that the three means had been set out in the semicircle, EC being the arithmetic mean, DB the geometric mean and BZ the harmonic mean."

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