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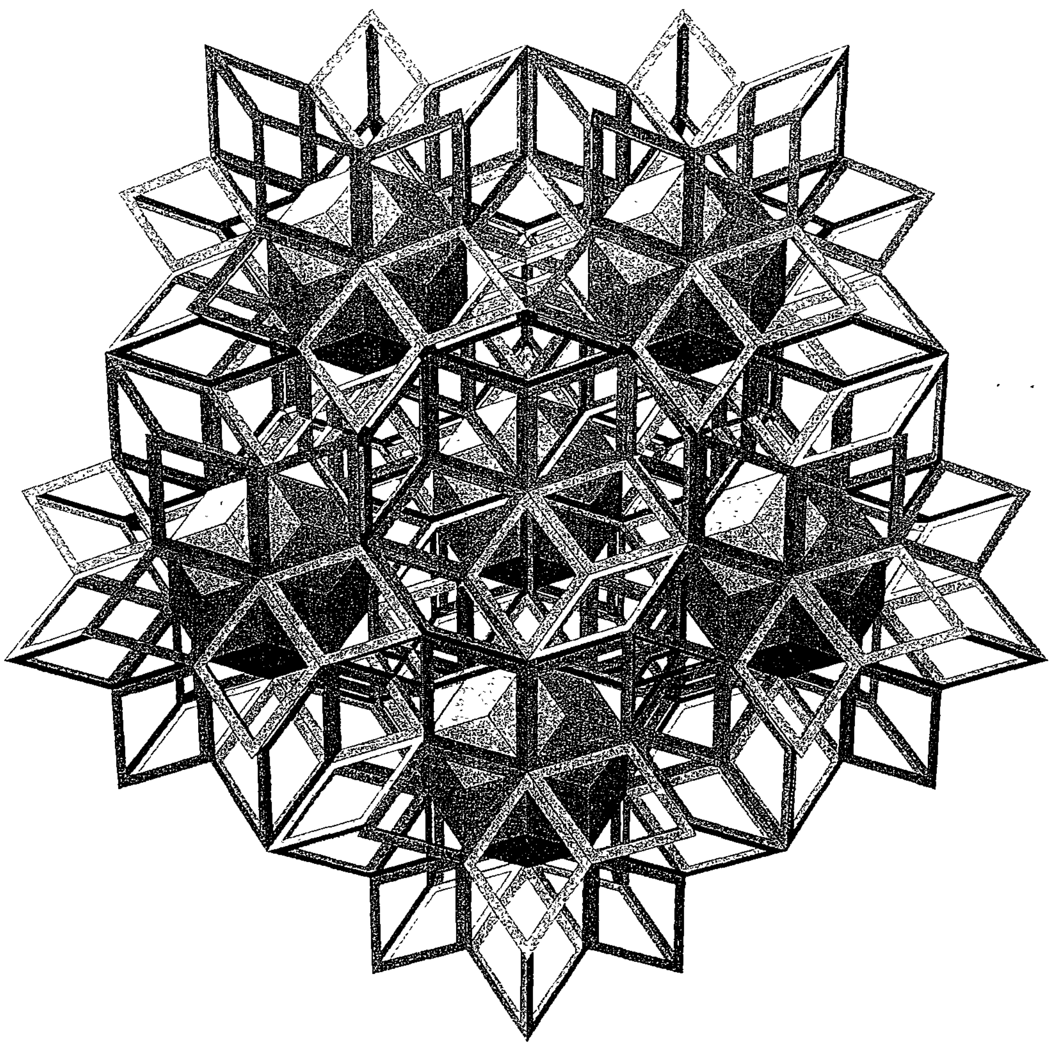

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GEOMETRIES IN THE EAST AND THE WEST IN THE 19TH CENTURY

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1 INTRODUCTION

In the 19th century several geometries were discovered, and F. Klein classified them in terms of transformation groups. Since then many publications about several such kinds of geometries have been made. But those publications only discuss geometries only in the West. Were there several kinds of geometries in the East in the same age? In this paper we will give an answer to the question by giving some examples taken from the old Japanese mathematics. Different approaches to the geometry between the East and the West and related topics will be given.

The mathematics we will present here is one developed in Japan during the Edo era (1615-1868), and is called *wasan* (*wa* and *san* means Japan and mathematics, respectively). Its root is in Chinese mathematics from the late 13th century to the late 16th century which was introduced to Japan through the Korean peninsula in the late 16th century, and developed rapidly during the Edo era. At the beginning of the Meiji era (1868-1912), the new government adopted Western mathematics in the new school system, and *wasan* had to end its short life, but *wasan* tradition was maintained for a while.

They used symbolic calculation as a main tool. It was also used in their geometry. Almost all the *wasan* books have the style of problem books, and the contents need not be systematic. Hence it is rather exceptional to find a geometric *wasan* book which is written in a systematic style or on a certain geometric purpose. But in some books we can see that the problems are solved by a peculiar technique. Also there are a few which are written on a geometric purpose. We will show that we can see a certain geometric worlds in those books.

2 BŌSHA JUTSU

Bōsha jutsu is a theory of tangents of various touching circles. It was founded by Ajima [1] and developed by Umemura [16]. The basic formulas are

$$l^2 = \frac{c^2 t_{AB}^2}{(c+b)(c+a)} \quad \text{and} \quad l^2 = \frac{c^2 t_{AB}^2}{(c-b)(c-a)}$$

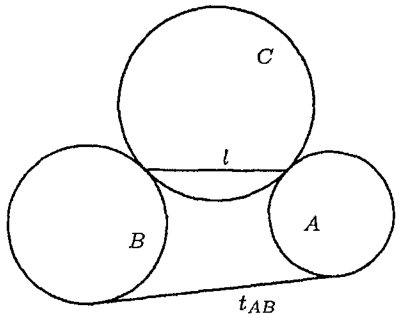

Figure 1a.

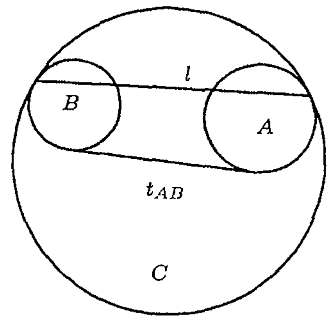

Figure 1b.

for Figures 1a and 1b, where a, b, ... are the radii of the circles A, B, ... , t_{AB} is one of the external tangents of A and B, and l is the distance between the two points of tangency of A, C and B, C. As a custom of *wasan* they used diameters of circles instead of radii in the original context to express these relationships, but we translate them using the term "radii" of circles as just shown above.

The next formulas for four tangent circles are

$$(b+d)^2 t_{CA}^4 - 8bd(b+d)(c+a)t_{CA}^2 - 16abcdt_{CA}^2 + 16b^2d^2(c-a)^2 = 0,$$

$$(b-d)^2 t_{CA}^4 + 8bd(b-d)(c+a)t_{CA}^2 + 16abcdt_{CA}^2 + 16b^2d^2(c-a)^2 = 0,$$

$$(b+d)^2 t_{CA}^4 + 8bd(b+d)(c+a)t_{CA}^2 - 16abcdt_{CA}^2 + 16b^2d^2(c-a)^2 = 0$$

for Figures 2a, 2b and 2c respectively. Umemura showed that these formulas could be applied for problems involving tangent circles by solving many problems. One of his simple results is $t^2_{BD}t^2_{CA} = 64abcd$ for Figure 3.

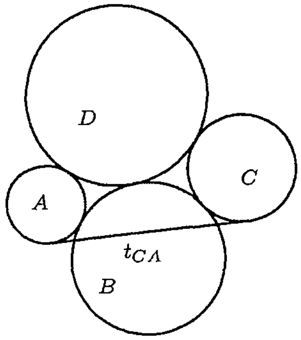

Figure 2a.

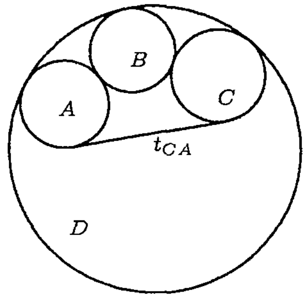

Figure 2b.

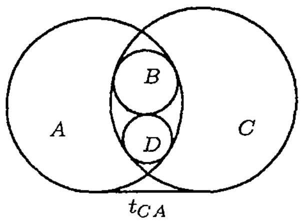

Figure 2c.

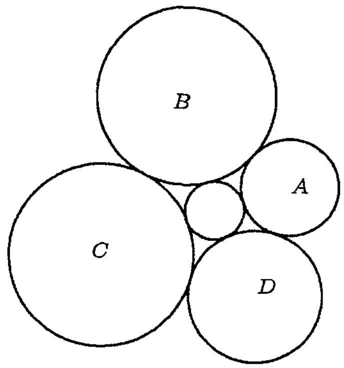

Figure 3.

As stated above, those formulas change the signs according to various tangency. It is pointed out that we must calculate to determine the signs and this is a defect of the method [7]. But those formulas can be unified if we consider oriented circles and oriented lines and if we regard tangent circles as anti touching oriented circles [13]. In this sense these results belong to the geometry of oriented circles. Since there is no idea of orientations or directions in *wasan* geometry, *bōsha jutsu* is not a counterpart of geometry of oriented circles. But it is interesting to see that such a geometry was made in Japan in the same age as Laguerre was considering oriented lines and oriented circles with his inversion.

3 *KYOKUKEI JUTSU*

Kyokukei jutsu (or *kyokugyō jutsu*) is a technique which transforms figures into symmetric ones and uses symmetric polynomials [3]. A typical problem is as follows:

Given the three exradii a, b, c of a triangle, find the inradius r (Figure 4a). The solution is consisting of several steps as follows:

1. Suppose a limiting (or an ideal) figure where the relation $a = b = c = x$ holds (in this event the triangle is equilateral) (Figure 4b).

2. Find the relation between x and r in the limiting figure (we can easily get $3r - x = 0$).

3. Regard the relation as an equation of a suitable degree ($3rx^2 - x^3 = 0$ in our case, but there in no explanation why we choose the cubic equation).

4. Substitute $(ab + bc + ca)/3$ and $abc/3$ for x^2 and x^3 respectively.

5. And we get $r = abc/(ab + bc + ca)$.

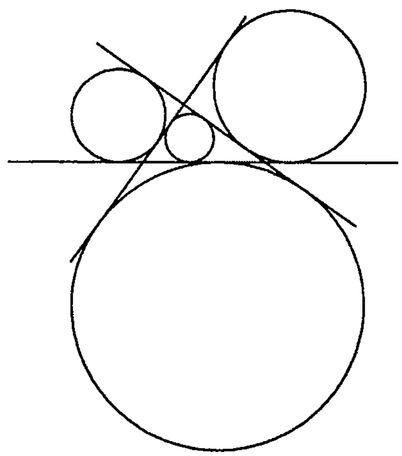

Figure 4a.

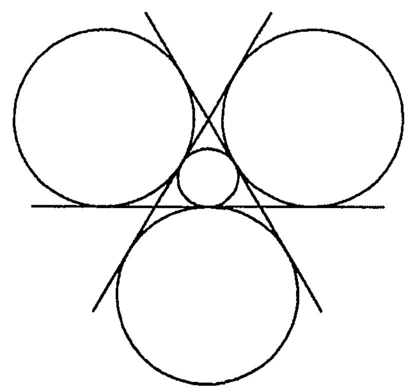

Figure 4b.

Another example is as follows: Given three inradii of the four incircles of the four inscribed triangles as in Figure 5a, find the remaining inradius. Let the radii of the circles A, B, C, D be a, b, c, d. The solution is:

1. Suppose a limiting figure where the relations $a = c = x$, $b = d = y$ hold (see Figure 5b).

2. Then we have $x = y$.

3. Replacing x and y by $(a + c)/2$ and $(b + d)/2$, we have $(a + c)/2 = (b + d)/2$.

4. Hence $a = b + d - c$.

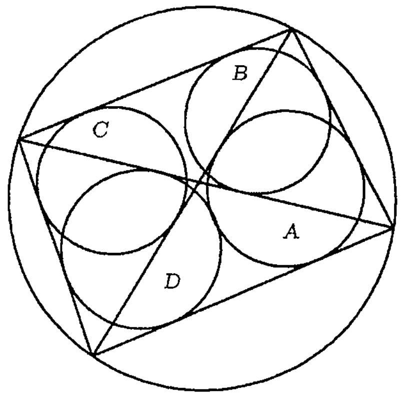

Figure 5a.

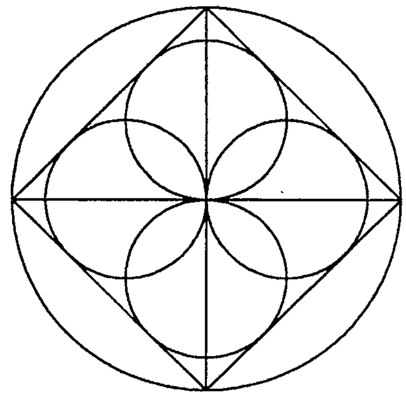

Figure 5b.

It is easy to point out several defects of this method: The metric relations in the limiting figures are not preserved in the original figures in general. There are many ways to replace variable x and y in the examples by symmetric polynomials. Indeed the method had drawn serious criticism from other *wasan* mathematicians in the same age. We can even find an incorrect answer to a problem in the book. However there are serious defects in this method, the method requires certain symmetry to the figures. Therefore the book is a collection of certain symmetric figures consequentially. It seems that the author of the book had already known the answers to most of the problems, or had got answers by another way, then he adopted these method to obtain the answer. It is very interesting that the author of the book considered symmetry of the figures connecting with such symmetric polynomials. We cannot find Western counterpart of this technique.

4 *SAMPENHŌ* AND OTHER SIMILAR TECHNIQUES

Sampenhō (or *Sanhenhō*) is a technique due to Hōdoji, which transforms tangent circles into parallel lines to obtain a certain metric relationship of the figures [8]. Similar techniques were also used by a couple of other *wasan* mathematicians in the same age. Let us see an easy example: Given two internally tangent circles, let us inscribe successively touching n small circles as in Figure 6a, where the figure is symmetric in the vertical line through the two centers of the given circles. Find the radii of the smallest circles A and B at the both ends. To obtain the common external tangent t_{AB} of A and B of radii r, Hōdoji considers a figure such that the radii of the two large circles become larger and larger and gets Figure 6b. Since the tangent in the transformed figure is $2(n - 1)r$, Hōdoji concludes that $t_{AB} = 2(n - 1)r$ also holds in the original figure (Figure 6a) with no explanation. Then by the formula of *bōsha jutsu* for Figure 2b, he obtains the relationship between the radii of the two given radii, t_{AB} and r. Eliminating t_{AB} by replacing it by $2(n - 1)r$, he gets the final answer.

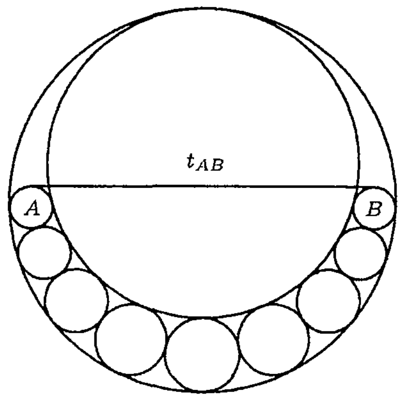

Figure 6a.

Figure 6b.

As in the example, the feature of the method is the way to get tangents of two circles. He gets common tangents of several kinds of tangent circles in the transformed figures, then he uses them as the relation in the original figures with no explanation. Those transformed figures exactly coincide with the ones obtained by suitable inversions. Let us see other examples. In the following four examples, we will cite only the transformed figures, and state what relations are derived from them. Those relations are also used as ones in the original figures.

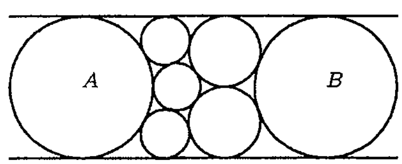

Figure 7: $t^2_{AB} = 2(\sqrt{2} + 1)^2ab$

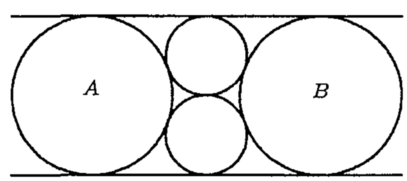

Figure 8: $t^2_{AB} = 8ab$

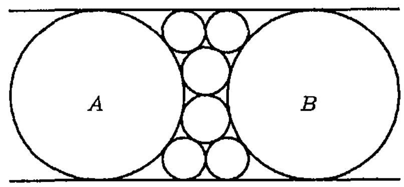

Figure 9: $t^2_{AB} = (25/4)\cdot ab$

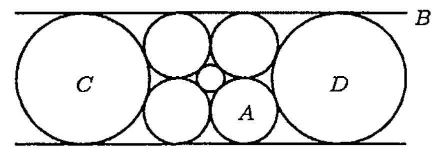

Figure 10: $t^2_{AB} = 8ab$, $t^2_{CD} = (4\cdot\sqrt{2} + 9)cd$

However there is also a gap or a jump of logic, it seems that the *wasan* mathematicians did not criticize the method. What is the Western counterpart of this? It is very similar to inversive geometric technique to consider parallel lines instead of tangent circles. Is Hodoji's method inversion? Does he use inversion? This point was controversial between Hayashi [4, 5, 6] and Mikami [11, 12]. Hayashi states that the method was correspond to inversion [4] at first. Countering this, Mikami mentions that there is no reason that the method is inversion [11], and he concludes that it is not inversion [12]. Hayashi also writes that the method is using not transformation as inversion, but deformation [5]. Also he says that the method is not inversion but it corresponds to inversion [6]. However there were several arguments between them, it seems that both grant that the method is not inversion, as a result.

In [10, p. 77] it is said that there is no evidence that the method is using inversion, and it is conjectured that they had already known the relationships between the tangents and the radii in the original figures and Hodoji tried to derive the relationships from the transformed figures. But in [7, p. 135], it is said that the method corresponds to inversion and since it is discussing metric relationships, there are a lot of difficulties, therefore some unknown parts are still left even by today's mathematics.

It seems that he points out the difficulty of how to prove results after the gaps of the method. However proofs that the results after the gaps being true can be found in [12], [10] (some can also be found in [4]). In [14], it is stated that Hodoji's acumen founded inversive method between circles and lines. Quoting the above difficulty in [7], it is shown that Hodoji's method can be explained by suitable inversions [15], and the authors conjecture that the method may be done by such inversions.

As mentioned above, recent publications following to [7] are affirmative on the question while earlier ones are not. After seeing a simple history of the issues on the method, let us give our evaluation. We agree with [10]. In the terms of inversive geometry, the ratio of the square of the tangent of two circles and the product of their radii are preserved by inversion [9, p. 122], and we can show that the relations obtained by transformed figures are also true in the original figures in fact. But we can find no reason that the method is inversion. In the preface of the manuscript, Hodoji says that his method is useless for beginners, so it is not allowed to tell the method to such persons. There are several versions of the manuscript, and among them we can find a sentence in which he says that if one had not solved a thousand problems, one could not be allowed to see the manuscript. It seems that he thought that after enough experiments, the method can be applied correctly. In other words one must know many relationships between the radii and tangents of circles to use the method. If some rule, how to apply his technique, was given in his manuscript, one could use it even without enough experiments. But we cannot find such rules, and can guess that he could not establish such rules. It seems that the sentences in the preface suggest this.

5 CONCLUSION

Since the three techniques are collecting problems which are able to be solved by each of the methods, they are resulting to correct figures which are suitable for the techniques. Therefore each of [1], [3], [8], [16] consists of figures with certain common properties respectively. Hence they have their own geometric worlds in this sense. But there was no notion of transformation. As mentioned in § 4, Hayashi says that sampenho is using not transformation as inversion, but deformation. It seems that he notes that there is no notion of mapping figures in the method. Indeed as we have shown, those techniques to deform figures only just occurred at the close of *wasan* age.

Since *wasan* geometry was not influenced by Euclid, axiomatic treatment was not made. Results were always expressed as problems and answers, which was far from theorems and proofs. Also *wasan* geometry was concerning about metric relationships, they did not care for projective properties of figures.

Rhombuses frequently appear in *wasan* problems while parallelograms do not. In most cases rhombuses are drawn so that one of the diagonals is horizontal. Ellipses can also be seen though neither parabolas nor hyperbola. It has been said that one of the reason is that they thought that ellipses were sections of cylinders. In addition to this, it seems that the Japanese prefer to consider an inner world of a closed figure. Indeed most *wasan* figures are of such kind.

Wasan left many results and some of them were made earlier than those of Western's. For example, a solution to Malfatti's problem; to draw three circles in a given triangle each of which is tangent to the other two and to two sides of the triangle, can be found in [2]. Malfatti solved this in 1803. But *wasan* geometries were very primitive if we regard them as systematic ones. It might be natural if we take its too short history, two centuries, and a dearth of Euclid into consideration. There were neither mathematics journals nor universities in those times. Since the Japanese attached little importance to mathematics, one could not live by mathematics. The situation forms a striking contrast to the fact that a European mathematician could devote himself to study mathematics as a professor at a university or at an academy.

REFERENCES

[1] and [2] Ajima, N. (1739-1798) Shi-enbosha no Kai (undated) in Naonobu Ajima's complete works (1966), Fuji Junior College, Tokyo. Fuji Junior College (compiled by Hirayama, A and Matsuoka, M.)

[3] Akita (1835) *Sampō Kyokukei Shinan*

[4] Hayashi, T (1937) On Hodoji's sampenhō I, *Wasan kenkyū shūroku (Collected papers of wasan) Volume 1*, Tokyo Tokyo Kaiseikan, 878-889.

[5] Hayashi, T. (1937) On Hodoji's sampenhō II, *Wasan kenkyū shūroku (Collected papers of wasan) Volume 1*, Tokyo: Tokyo Kaiseikan, 890-906.

[6] Hayashi, T. (1937) Problems in *wasan* taste I, *Wasan kenkyū shūroku (Collected papers of wasan) Volume* 2, Tokyo: Tokyo Kaiseikan, 825-832

[7] Hirayama, A. (1965) *Peoples in wasan history*, Tokyo: Fuji Junior College

[8] Hodoji (1860) *Kanshinko Sampen* (manuscript).

[9] Johnson, R. A. (1960) *Advanced Euclidean Geometry*, Dover, New York.

[10] Katoh, H. (1956) *A study of wasan miscellanea III*, Tokyo. Maruzen.

[11] Mikami, Y (1932) On deformation methods by Ushijima, *Hōdōji and Tsuchiya*, Tokyo Butsuri Gakkō Zasshi, Oct , 419-428; Dec., 534-540.

[12] Mikami, Y (1933) On deformation methods by Ushijima, *Hōdōji and Tsuchiya, part 2*, Tokyo Butsuri Gakkō Zasshi, Jan , 13-19, Feb , 51-60, Mar , 106-110, Apr., 146-154; May, 188-190

[13] Okumura, H. (1990) Generalization of the results in the old Japanese incircle geometry by the terms of cycles and rays, *Sūgakushi Kenkyū*, No. 124, 24-53 (with English summary).

[14] Shimodaira, K. (1970) *A history of wasan, volume* 2, Tokyo. Fuji Junior College

[15] Suzuki, F and Michiwaki, Y. (1988) On Hōdoji's "Sanhenhō" and Casey's inversion theorem, *Sūgakushi Kenkyū*, No. 116, 62-77 (with English summary).

[16] Umemura (1804-1884) Bōsha Shōkai (undated manuscript).

All the references except [9] are in Japanese.