Symmetry: Culture and Science

Chapters from the HISTORY OF SYMMETRY

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Interdisciplinary Study of Symmetry
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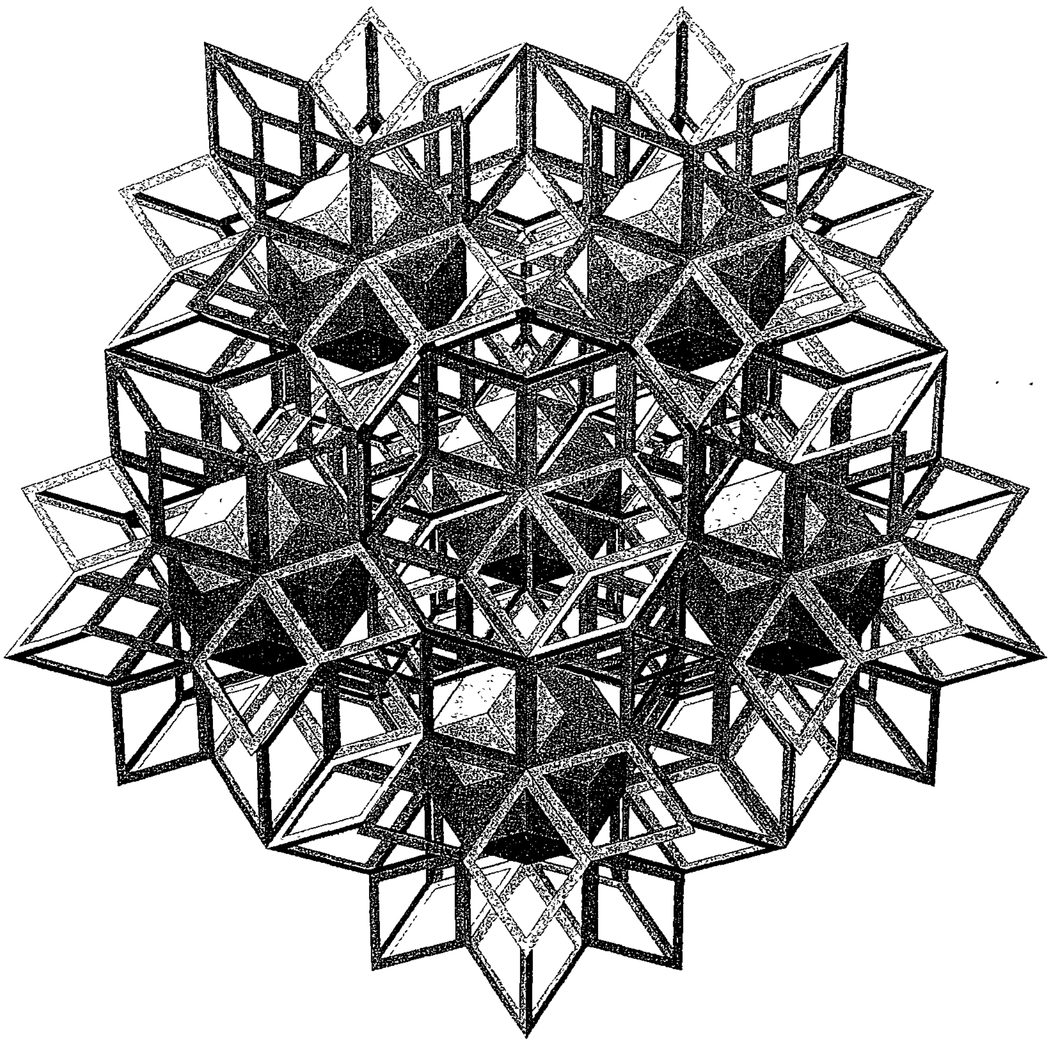

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Symmetry: Culture and Science

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CHAPTERS FROM THE HISTORY OF SYMMETRY

Edited by György Darvas

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EDITORIAL

This issue publishes a few of the papers presented in the Special Session *History of Symmetry in the Sciences, Technology, and the Arts* at the *XXth International Congress of History of Science*, held in Liège, Belgium, 22nd July 1997. The session was organised by the editor of this issue. He elaborated the concept, the structure of the session, to which he selected and invited individually the lecturers. This issue publishes nine of the papers presented at the session.

This booklet picks out a few chapters from the history of symmetry, according to the concept described by the issue editor in the *Introduction* on the next pages. Let us express our hope that further chapters will follow in later issues.

Just one year passed since the curatory of the Foundation took over the editorship of the journal. We edited two and a half volumes during these 12 months. We hope to continue the work at a similar pace to regain your faith.

February, 2002

Symmetry: Culture and Science
Vol 10, Nos 1-2, 5-6, 1999

INTRODUCTION

History of Symmetry in the Sciences and the Arts

Chapters from the history of symmetry

Traditional approach in history of science focuses either to a discipline in a period, or to the works of a certain person. We followed a different path. We aimed at the role of a phenomenon, the evolution of a concept and its applications in different disciplines both in the arts and the sciences. This concept and phenomenon was symmetry.

The historical role of symmetry can be investigated in two dimensions. One can investigate it in different historical periods, starting with the Ancient times, through the Middle ages, the Renaissance, the Baroque, and the Romantics, up to the Modern age (temporal axis). The different disciplines form an other dimension of the investigation. One can study the role of symmetry in Physics, Chemistry, Biology, Technology, and other sciences, as well as in Literature, Linguistics, Philosophy, and other humanities, and also in Fine arts, Design, Music, Dance, etc., (disciplinary axis).

Any of the two dimensions can demonstrate only a narrow slice of the subject. The concept of symmetry and its applications within a given discipline changed during the centuries. On the other side, the appearance of symmetry in a discipline in different periods has never been intact from influences of the usage of the term and applications in other disciplines.

Moreover, the essence of symmetry can be understood only in its interdisciplinary character. This interdisciplinary character means, that our interest is concentrated on how this phenomenon could mediate methods and ideas among the arts and the sciences, among disciplines, among cultures. Therefore, the authors of this collection were asked by the editor to place the subject of their papers along the diagonal between the two axes. The sequence of the papers in this issue follows more or less this line.

The first papers start from the Renaissance. *G. Hajnóczi* investigated how the ancient notion of symmetry took its roots in the Renaissance of the European culture, and appeared in the modern languages. *S. Roero* compared the treatment of the phenomenon in the Greek and in the Renaissance science and art.

A. Loeb studied the same questions through the development of the composition of music, while *I. Guletsky* compared the role of symmetry in Renaissance architecture, music and also in visual art. *D. Cohen* and *J. Cohen* dealt with the role of symmetry in music in later periods.

L. Bonpunt investigated how symmetry became the subject of the exact sciences on the example of the development of crystallography. *Y. Ne'eman* followed this line, he treated how physics took over the central role of developing symmetry studies in the exact sciences in the 20th century. This paper is unique, since one of the greatest contributors tells the story. He is really not a historian of science, rather he made the history, when he discovered (independent from another colleague) the SU(3) symmetry for the classification of certain clusters of physical particles, what influenced all further events in the latest 40 years history of the fundamental theories in physics.

F. Ilgen demonstrated how geometric composition started to prevail in 20th century art, on the example of constructivism. Finally *H. Okumura* investigated the role of geometrical symmetry in two distant cultures in nearly the same period.

György Darvas

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SYMMETRY: ART AND SCIENCE

THE CONCEPT OF SYMMETRY IN EARLY RENAISSANCE ART THEORY. LEON BATTISTA ALBERTI, *DE PICTURA*

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Although Alberti was a pupil of humanist masters of art theory (of Petrarca, for example) he created an original theory of painting, sculpture and architecture. To create a theory of painting he relied on classical rhetoric tradition (Cicero), on classical and medieval knowledge of sciences (Euclid, Vitruvius, Alhazen, Alkindi) and contemporary pictorial practice. The Greek-origin concept of symmetry was transmitted to him by Vitruvius who considered it as a harmony of members of the human body *De architectura* I. 2.) and made it the base of *compositio* (III. 1). For Alberti symmetry has two senses. In the first book of *De pictura* he uses it as a geometric rule (I. 14). In the second book symmetry is an aesthetic principle which is the reason for proportion *(compositio membrorum;* II. 36). In both cases symmetry has an anthropometric character and reveals a relation between work of art and nature. In contrast to Vitruvius, Alberti's symmetry is not a basic principle of artistic composition. The latter is a notion originating from rhetoric, and serves to organize the figures of *historia* (II. 35). For Alberti it was a great problem how to transform a humanist treatise into a practical one while editing the *volgare* version of *De pictura*. Rendering the Latin terminology in *volgare* posed a further difficulty for him.

1. A HUMANISTIC TREATISE ON A VISUAL ART

Alberti was, in all probability, the only one among humanist art theorists in the first half of 15th century who made use of the term symmetry. Humanist art theorists neglected it probably because it was missing in the terminology of rhetoric and because of its Greek origin. According to Michael Baxandall (1971), humanist writers, considering literature as a model for painting, formed the method of artistic interpretation on the basis of the terminology of rhetoric. The most important source was Quintilian's De *institutione oratoriae,* the Latin terminology of which was adapted for categories of painting. Symmetry was not a term of rhetoric. It had Greek origin and it was used in works of Plinius and Vitruvius who transformed it in a latinized orthography. Petrarch knew it from *Naturalis historia,* and in a note of a manuscript he used observed that *"simmetria latinum non est nomen."*[1]

Curiously enough Alberti uses it only in his treatise *On painting* but abandons it in his other treatise on art, *On sculpture,* and – surprisingly – also in his treatise on architecture, *De re aedificatoria.*

In this way Alberti breaks the humanist tradition of to interpreting painting with a terminology based exclusively on rhetoric. He enlarged the domain of artistic terminology and language probably because the idea he had about painting was different from that of the humanist art theorists. Although he preserves the humanistic character of his treatise (it is in Latin, based on classical rhetoric, etc.), he wants to render it suitable for the description of painting in its peculiarity. It means that painting must be considered as a visual art and proportion will be its basic aesthetic principle.

Consequently, *De pictura* has a complex character. It was written in two versions, in Latin in 1435, and in *volgare* a year later. The translation of the Latin text into *volgare* was prepared by Alberti himself and it served for artists unlearned in Latin. As Alberti says with some exaggeration in the dedicatory letter of the *volgare* version, he prepared the translation personally for Filippo Brunelleschi[2]. The work contains three books. According to Alberti's definition the first is *tutto matematico*, that is "entirely mathematical, it shows how this noble and beautiful art arises from roots within Nature herself." So the first book is dedicated to the description of perspective. The second book contains the discussion about the parts of painting. As Alberti says, this book "puts the art into the hands of the artist, distinguishes its parts and explains them all."

[1] Ms Lat. 6802 of Bibliothèque Nationale, Paris, c. 249r which refers to Plinius' *Naturalis historia,* XXIV, 65

[2] " if you should have some leisure, I shall be glad if you will look over this little work of mine on painting, which I did into Tuscan for you." (Alberti 1956)

The third book deals with the formation of the artist, offering instructions for a learned painter: "instructs the artist how he may and should attain complete mastery and understanding of the art of painting."[3] The complexity of the work then comes from both the theoretical and the practical subjects treated in it. In a very general sense we can say that theoretical problems are discussed in the first and the second book. These two books correspond respectively to the two fundamental themes of the treatise: perspective and composition.

However, even the first and the second books differ from each other. While the description of perspective is based on the Euclidean geometry the composition of the painting is defined in terms of Ciceronian rhetoric (Baxandall 1971, p. 172). In the first case Alberti codifies a method of construction of the perspective which probably comes from the experiments of Brunelleschi's circle. He unifies the proper knowledge of geometry with the empirical knowledge of these artists and for a correct discussion of the subject he relied on a Latin language which was adapted to treat the perspective by theorists like Biagio Pelacani for example (Federici Vescovini 1961, Baxandall 1971). In the second case composition is treated as one of the three components of painting, circumscription (design) and reception of light (colour) being the other two. "Composition is that procedure in painting whereby the parts are composed together in a picture. The great work of the painter is the *historia*; parts of the *historia* are the bodies, part of the body is the member, and part of the member is a surface."[4] It is then a hierarchic structure having four levels, surface-member-body and *historia*. The composition of painting, in this sense, is a category, which – as a metaphor – transfers to the picture the model of rhetoric organisation (Baxandall 1971, p. 174; Deswarte-Rosa, S. *Le De Pictura, un traité humaniste pour un art "mécanique"*, Introduction; Alberti 1993, p. 36). In this field Alberti appears as a pure humanistic theorist. For him composition is a technical concept, a method of putting together single parts in a work of art, and it has no relation at all with harmony or beauty.

In the third book, which contains instructions for learning the fundamental principles of painting, Alberti explains how the painter is to obtain the beauty, existing essentially in Nature, excellently in human body. The problem of beauty is considered as a practical rather than a theoretical question. Symmetry is also a phenomenon of Nature which is to be imitated in painting.

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[3] ibidem

[4] *"Est autem compositio ea pingendi ratio qua partes in opus picturae componuntur Amplissimum pictoris opus historia, historiae partes corpora, corporis pars membrum est, membri pars est superficies."* (Alberti 1956, p. 70.)

2. THE CATEGORY OF 'SYMMETRY' IN *DE PICTURA*

In Alberti's theory of painting symmetry is not a basic principle and it has not got the sense of *vera proportione* as it will have in Italian artistic theory some decades later.[5] For Alberti it is not a fundamental aesthetic principle rather a rule of measure and proportion existing in Nature discovered by the Greeks. He is the first to use this term in the artistic theory of early Renaissance. In the end of the fourteenth century, in his *Libro dell'Arte* Cennino Cennini, discussing the proportions of human body, uses the term *misura* and he was probably unfamiliar with the measures of Vitruvius (Cennini 1982, pp. 81-83; about Cennini and Vitruvius see Schlosser; *La Nouva Italia*, Firenze, 1979, p. 96.) Also Ghiberti, the contemporary art theorist who knew Vitruvius, uses *misura* in the sense of proportion when praises Giotto for having observed in his painting the *right proportion.*[6]

In *De pictura* symmetry is a Greek-origin category which Alberti takes from Vitruvius' *De architectura.* Similarly to humanists, like Petrarch and Landino, Alberti is also convinced that the term symmetry comes from the Greeks and he also preserves its latinized orthography. Among the instructions in the third book Alberti says to the painter: "In a standing person he will note the whole appearance and posture, and there will be no part whose function and symmetry, as the Greeks call it, he will not know."[7] This determined proportion of parts is the concept of Vitruvian symmetry which is the source for Alberti. According to this concept symmetry is the proportion, existing both in Nature and in work of art and it is the basis of artistic/architectural composition. Vitruvius says: "*Item symmetria est ex ipsius operis membris conveniens consensus ex partibusque separatis ad universae figurae speciem ratae partis responsus.*" (I. 2, 4), and "*Aedium compositio constat ex symmetria, cuius rationem diligentissime architecti tenere debent. ea autem paritur a proportione, quae graece αναλογια dicitur proportio est ratae partis membrorum in omni opere totiusque commodulatio, ex qua ratio efficitur symmetriarum.*" (III. 1, 1)[8] Although Alberti follows the Vitruvian description

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[5] Christoforo Landino in his *commento* on Dante (*Comento di Christoforo Landino Fiorentino sopra la comedia di Dante Alghieri poeta fiorentino,* Firenze, 1481) states that Cimabue rediscovered that "*vera proportione* which the Greeks call symmetry." (Panofsky 1960, 27.)

[6] "Arecò (Giotto) l'arte naturale e'lla gentilezza con essa, non uscendo delle misure." (Panofsky 1960, 27/2.)

[7] "*Notabit stantis faciem totam atque habitudinem, denique nulla erit pars cuius officium et symmetriam, ut Graeci aiunt, ignoret.*" (Alberti 1956, pp. 98-99.)

[8] "Symmetry is a proper agreement between the different parts and the whole general scheme, in accordance with a certain part selected as standard." (I 2, 4); "The design of a temple depends on symmetry, the principles of which must be most carefully observed by the architect. They are due to proportion, in Greek *αναλογια*. proportion is a correspondence among the measures of the members of an entire work, and of the whole to a certain part selected as standard. From this result the principles of symmetry " (III. 1, 1). (Vitruvii (1867), translation: Vitruvius (1960) *The Ten Books on Architecture,* translated by Morris Hicky Morgan, New York. Dover Publ., Inc., 14, 72.)

his concept of symmetry is original inasmuch it differs from its model at least in two important respects[9] First, in contrast to Vitruvius' conception Aberti's symmetry is exclusively a natural principle and which cannot be found it in the work of art. Second, in contrast to Vitruvius' opinion for him symmetry and composition are two different categories, therefore artistic composition does not include symmetry.

In the Albertian theory of painting symmetry has two aspects, geometrical and compositional. Let us see first the geometrical aspect. In the first book Alberti uses the term symmetry twice. In Chapter 14 proportions of the human body are used to explain geometrical proportionality which is the *commensuratio* (similarity) of triangles, while in Chapter 19 proportions of the human body are used to establish the measures of the painting.[10] At both places in the Latin text the word *symmetria* stands together with *membrorum* (symmetry of members), which is not a Vitruvian usage of the term, it is rather an Albertian invention. To describe geometrical elements, optical phenomena and the process of the construction of perspective he uses geometrical (Euclid), mathematical (Boethius) and optical (Euclid, Alhazen) terminology[11] in which symmetry is an alien, inappropriate term. It means real proportion existing between members of the human body which in this case is used as a metaphor explaining geometric problems.

The compositional aspect of the category of symmetry is more complicated. As we have seen, painting is divided into three parts, circumscription, composition and reception of light *(circumscriptio, compositio* and *luminum receptio).* Alberti defines the structure of the concept of the composition, as consisting of the composition of surfaces *(superficierum compositio),* the composition of members *(membrorum compositio)* and finally the composition of bodies *(corporum compositio).*[12]

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[9] He also criticizes Vitruvius in the second book where he discusses the Vitruvian canon of human proportion, and proposes the head as a human unit of measurement "I would advise one thing, however, that in assessing the proportion of a living creature we should take one member of it by which the rest are measured The architect Vitruvius reckons the height of a man in feet. I think it more suitable if the rest of the limbs are related to the size of the head. (Alberti (1956) *On Painting,* pp. 74-75; see *Introduction* by John R. Spencer, in Alberti (1956) *On Painting,* New Haven: Yale University Press, 1956, 22

[10] "Yet the proportion of the limbs of Hercules was no different from that of the body of the giant Anteus, since the symmetry from the hand to the elbow, and the elbow to the head, and all the other members, corresponded in both in similar ratio. Similarly, in triangles, there can be a certain uniformity between them, whereby the lesser agrees with the greater in all respects except in size." (I. 14) (Alberti 1956, p. 51.), "I divide the height of this man into three parts, which will be proportional to the measure commonly called a *braccio;* for, as may be seen from the relationship of his limbs, three *braccia* is just about the average height of a man's body With this measure I divide the bottom line of my rectangle into as many parts as it will hold; (...) (I 19) (Alberti 1956, 55)

[11] See *Notes* to *De pictura* by Grayson (Alberti 1956, pp 108-114)

[12] See note 8.

The difficulty arises from the diversity of these types, because it results an obscure concept of beauty. Alberti asserts that from the composition of surfaces derives harmony and grace, that is beauty *(illa elegans in corporibus concinnitas et gratia* extat, *quam pulchritudinem dicunti*; II. 35).[13] In this case the beauty consists of the surface of beautiful bodies, the model of which is to be found in Nature itself. The other way to obtain beauty *(venustas* and *pulchritudo)* is to make 'all the members accord well with one another.' This is the essence of the composition of members. The correct accord of members is the proportion, which is symmetry *(symmetria membrorum)*, which, according to Alberti, has to be studied by the painter in Nature.[14]

Summarizing what has been said above we may arrive at the following conclusions:

— Alberti wrote a humanistic treatise on painting in which he created a Latin terminology based on rhetoric. This terminology was not perfectly convenient to interpret a complex phenomenon like painting. For a correct discussion of 'proportion' he needed to integrate symmetry, an alien concept in the theory and terminology of rhetoric. By the help of the term symmetry he was able to discuss proportion both from geometrical and aesthetic point of view.

— Symmetry was a latinized Greek term which Alberti used as his own category. Although he criticized Vitruvius, his concept came from the theory of human proportions of the antique Roman architect. It was difficult to fit symmetry into the humanistic Latin categories. Alberti could not perfectly solve the problem of creating a homogeneous Latin terminology of painting.

— 'Symmetry' as a term became a category in *De pictura,* that is, in the Latin version of the treatise. Alberti could use 'symmetry' because classical authors rendered it from Greek into Latin term. Although it was not a current humanistic term, humanist readers could understand it because of their knowledge of Pliny's *Historia naturalis* and perhaps also of Vitruvius' *De architectura.* It was another problem for Alberti how to use it in the *volgare* (Italian language) version of the treatise, in *Della pittura.*

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[13] He uses the term *concinnitas* as the synonym of *pulchritudo,* which will be the fundamental category of architectural aesthetics in *De re aedificatoria,* two decades later On *concinnitas* see Michel 1930, p. 360.

[14] "As Nature clearly and openly reveals all these proportions, so the zealous painter will find great profit from investigating them in Nature for himself. Therefore, studious painters should apply themselves to this task, and understand that the more it helps them to fix in their minds the things they have learned " *"At enim cum has omnes mensuras natura ipsa explicatas in medium exhibeat, tum in eisdem ab ipsa natura proprio labore recognoscendis utilitatem non modicam inveniet studiosus pictor. Idcirco laborem hunc studiosi suscipiant, ut quantum in symmetria membrorum recognoscenda studii et operae posuerint, tantum sibi ad eas res quas didicerint memoria firmandas profuisse intelligant."* (II. 36) (Alberti 956, pp 74-75.)

The first difficulty arose from the fact that *simmetria* was not in use in contemporary artistic theory, probably because it had no equivalent in the Italian terminology of painting. The second difficulty was presented by the fact that painters, as possible readers of *Della pittura*, were not familiar with Latin terms and 'symmetry' was unknown in their artistic world. So to find a correct Italian equivalent for *symmetryia* meant not merely a linguistic but also a theoretical problem because it related to the larger problem of introducing a fundamental concept into the theory of art.

3. THE CATEGORY OF 'SYMMETRY' IN *DELLA PITTURA*

The term 'symmetry' in the text of *Della pittura* is a lacuna. While in the Latin version it is mentioned five times, in the Italian one it does not exist as a term.

Omissions in the Italian version are not surprising (Grayson 1953, pp. 54-62).

In several places Alberti abridges the original Latin text omitting sometimes relatively long passages. His intention is to simplify the theoretical discussions and to render the entire treatise more practical. So the *Della pittura* is not only a translated but also a modified version of the original Latin work (Grayson 1968, pp. 71-92; Maraschio 1972, pp. 265-273; *Id., Aspetti del bilinguismo albertiano nel "De pictura", "Rinascimento", Anno XII,* pp. 183-228.*)*

'Symmetry' presented a special problem. As it did not exist in the Italian vocabulary Alberti could not translate it but since the proportion of members was an important concept of his theory of art he had to find a solution. He had at his disposal the Italian terminology of contemporary art theory in which the term *misura* approached the sense of the term 'symmetry.'[15] Although Albertian 'symmetry' could not be identified with Ghibertian and even less with Cenninian *misura*, in the Italian text this latter term came into use instead of a missing new category.

In the Italian version Alberti demonstrates that he considers symmetry more important in aesthetic than in geometrical sense. Therefore he omits it in the geometrical discussions of the first book but interprets it into the Italian in the passages treating with the composition of members and the principles of beauty, in the second and in the third

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[15] See Note 11 For the use of *misura* in the sense of 'proportion' see Panofsky 1960, pp. 27-28

books. A comparison of the relevant passages of the two versions illustrates how Alberti transferred 'symmetry' from the Latin into the Italian text by using the word *misura*.

(...) tametsi ferunt Euphranorem Isthmium nonnihil de symmetria et coloribus scripsisse, Antigonum...	benché dicono Eufranore ıstmio scrivesse non so che delle *misure* e de' colori, e dicono che Antigono... (II. 26)
Idcirco laborem hunc studiosi suscipıant, ut quantum in *symmetria* membrorum recognoscenda studiı et operae posuerint,...	E poi che la natura ci ha porto in mezzo le *misure*, ove si truova non poca utilità a riconoscere dalla natura,... (II. 36)
denıque nulla erit pars cuius officium et *symmetriam,* ut Graeci aiunt, ignoret. (...)	né sarà ivi parte alcuna della quale non sappi suo officio e sua *misura*. 78. (Alberti 1980, pp. 46-47; 62/64-65; 96-97.)

Concluding our discussion about the symmetry ın *volgare* version we can make the following assertions.

— Although there is no correct equivalent for the term 'symmetry,' in the text of *Della pittura,* Alberti introduced this concept into his theory of art. He did not invent a new category but used the conventional term *misura* for 'symmetry,' however he filled it up with a new meaning.

— *Della pittura* represents an important stage in the evolution of the term *misura* as a category of art theory, because in this treatise *misura* became enriched by the meaning of the classical symmetry. It meant a great step toward the formation of the concept of the *vera proportione* ("just proportions") which concept became the basic aesthetic principle in the fifteenth century.

REFERENCES

Albertı, L B. (1956) *On Painting,* New Haven. Yale University Press

Alberti, L. B (1972) *On painting* and *On Sculpture,* edited with Translations, Introduction and Notes by Cecıl Grayson, Phaıdon, London, 33

Albertı, L B. (1980 *De pıctura, a cura di Cecil Grayson*, Ed. Laterza, B.

Alberti, L. B (1993) *De la peinture,* Macula, Dédale, Paris.

Baxandall, M. (1971) *Giotto and the Orators Humanist Observers of Painting in Italy and the Discovery of Pictorial Composition 1350-1450*, Oxford University Press, 2nd chapter

Cennini, C. (1982) *Il Libro dell'Arte,* Neri Pozza Editore, Vicenza, chap. LXX Le misure che de' avere il corpo dell'uomo fatto perfettamente, 81-83.

Federici Vescovini, G. (1961) Le Questioni di "perspectiva" di Biagio Pelacani da Parma, *Rinascimento*, II, serie 1, 163-250

Grayson, C. (1953) Studi su Leon Battista Alberti, *Rinascimento, Anno IV.*

Grayson, C. (1968) The text of Alberti's De pictura, *Italian Studies.*

La Nuova Italia (1979) Firenze, 96

Maraschio, N (1972) L. B. Alberti, De pictura, bilinguismo e priorita, *Annali della Scuola Normale Superiore di Pisa.*

Michel, P -H (1930) *Un idéal humain au XVe siècle La pensée* de *L B Alberti (1404-1472),* Paris, Société d'Editions "Les Belles Lettres," p. 360.

Panofsky, E (1960) *Renaissance and Renascences in Western Art,* Stockholm. Almquist and Wiksell, p 27

Schlosser, J. (1924) *Die Kunstliteratur,* Wien

Vitruvii (1867) *De Architectura libri decem,* Ed Valentinus Rose et Herman Mueller-Struebing, Lipsiae, Teubner, 12, 65, translation Vitruvius (1960) *The Ten Books on Architecture,* translated by Morris Hicky Morgan, New York: Dover Publ., Inc., 14; 72.

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MEAN, PROPORTION AND SYMMETRY IN GREEK AND RENAISSANCE ART

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'*το γαρ ευ παρα μικρον δια πολλων αριθμων γιγνεσθαι*'
'The beautiful is achieved little by little, by means of many numbers'[1]

This is one of the very few remaining fragments of the *Canon* of Polyclitus, who was probably the first artist to codify the rules to be followed in executing a work of art. There are other classical sources which speak of Polyclitus as creating a harmonic, well-structured 'system of proportions,' which he was to express directly in his statues – especially the *Doriforus* – and in his lost treatise, the *Canon*; and these sources indicate that it was his intention to supply a model of proportions which might create a relationship among the individual parts of the human body, based on numbers.[2] Never before had anyone given artists such a corpus of numerical rules which was to be of great influence not only on Greek but also on Roman and Renaissance art. His example was followed by a host of 'theoreticians of art' who, from Vitruvius on, found in Humanism and in the Renaissance the highest peak of the fusion between the mathematical doctrines of the mean and proportion, and the doctrines of art and of music. Erwin Panofsky and Rudolph Wittkower have already shown with masterly skill that the rediscovery of the classics on the one hand, and the conception of Man as expression of the microcosm and of the macrocosm on the other, were the factors that led to an appreciation of all those classical theories and conceptions of the beautiful, perfection and harmony as the synthesis and fruit of a systematic scientific study.[3] Thus the theoreticians of art and music in the periods of Humanism and the Renaissance, if they were to arrive at the scientific knowledge which enabled them to obtain the beautiful, harmony, symmetry, perfection, had to turn for their inspiration to the Pythagorean school, the canons of Polyclitus and the philosophical teachings of Plato, handed down by Vitruvius and subsequently by L. B. Alberti. It is my intention to add to Panofsky's and Wittkower's approaches the philological and historico-mathematical analysis of the classical Greek sources in which the terms 'mean' and 'proportion' appear (§. 1) in order to assess what classical traditions or sources were most influential for the art and music theoreticians of the 15th and 16th centuries (§. 2). Furthermore I take into consideration the occurrence of the golden section in handbook for artists (§. 3) to see the role played by irrational numbers in Renaissance treatises and finally I wish to single out the presence of two factors in the writings on the portrayal of the beauty of the human body (§. 4) and to suggest a possible link between the three main means (arithmetic, geometric and harmonic) and the human figure inscribed in square and circle (Table 6). I shall point out certain passages from 15th and 16th century writings on art, where it is clear that the application of mathematics to art is mediated by what Panofsky calls 'the visual experience of the artist'. In my view, two separate scientific components can be identified in the approaches of art theoreticians on the proportions of the human body: one theoretical, rooted in the Pythagorean and Platonian teaching of means and music; the other experimental, based on the observation of reality and on the meticulous gathering of numerical data, on which the mean is calculated. Both these components are linked to the concept of the mean, which has been given an exact mathematical definition by statistics scholars[4] only in this century, but which has been in practical use from the most ancient times.

1. MEANS AND PROPORTIONS IN THE HISTORY OF MATHEMATICS

In order to arrive at the origins of the concept of the mean and the use made of it in both mathematics and practical life, we must go back to pre-Hellenic civilisations. From surviving testimony it is clear that the Babylonians used the mean, though without defining it in their calculations. An Egyptian deed from 150 B.C. was found at Edfu: this shows a number of plane geometrical figures (triangles, rectangles, trapezoids and quadrilaterals) and the rule for finding their area. This rule - which, it has been proved, the Babylonians also knew - consisted of taking the product of the arithmetic means of the opposite sides.[5] In the same way the Babylonians calculated the volume of a frustrum of a cone or pyramid by taking the arithmetical mean of the upper and lower bases and multiplying it by the height.[6] Use of the arithmetical mean can also be seen in the Babylonians' procedure for extracting the square root.[7] But it was only with the Greeks that a rigorous, systematic treatment of the means and the proportions was achieved – a treatment which subsequently appeared in the surviving texts.[8] In Table 1 you can see the names of the principal figures who dealt with this topic and the titles of their works.

PERIOD	MATHEMATICS & PHILOSOPHY	MUSIC	ART & HISTORY
6-5th B.C.	PYTHAGORAS		
5-4th B.C.	HIPPOCRATES of Chios ARCHYTAS, HIPPASUS	PHILOLAUS	POLYCLITUS *Canon* THUCYDIDES *Hist. Belli Pelop.*
4th B.C.	EUDOXUS PLATO *Tim.*, *Resp.*, *Leges* ARISTOTLE *Eth. Nic.*		PHILON *Mechan. Synt.*
3rd B.C.	EUCLID *Elements* ERATOSTHENES	MYONIDES, EUPHRANOR	HIPPODAMOS of Mileto
1st B.C.			VITRUVIUS *De architec.*
1st A.D.			PLINIUS *Nat. Hist.*
1st-2nd A.D.	NICOMACHUS *Introductio arithmeticae* THEON	NICOMACHUS *Introductio arithmeticae*	PLUTARCH *Moralia* GALEN *De temperam. Placita Hipp. et Plat.*
3rd-4th A.D.	IAMBLICHUS *Nicom. Arith. Introd.* PAPPUS *Collect. Math.*	PORPHYRY *In Ptol. Harm.*	
5th A.D.	PROCLUS *In Tim.*		
5th-6th A.D.	BOETHIUS *De Inst. Arithm.*	BOETHIUS *De Inst. Musica*	

Table 1: Ancient authors or writers on the popular theory of means and proportions

MEANS		NICOMACHUS		PAPPUS	
		Numerical examples	$a > b > c$	Numerical examples	$a > b > c$
First	*Arithmetic*	3 2 1	$\frac{a-b}{b-c}=\frac{a}{a}$	6 4 2	$\frac{a-b}{b-c}=\frac{a}{a}$
Second	*Geometric*	4 2 1	$\frac{a-b}{b-c}=\frac{b}{c}$	4 2 1	$\frac{a-b}{b-c}=\frac{a}{b}$
Third	*Harmonic*	6 4 3	$\frac{a-b}{b-c}=\frac{a}{c}$	6 3 2	$\frac{a-b}{b-c}=\frac{a}{c}$
Fourth	*Subcontrary to harmonic*	6 5 3	$\frac{b-c}{a-b}=\frac{a}{c}$	6 5 2	$\frac{a-b}{b-c}=\frac{c}{a}$
Fifth	*Subcontrary to geometric*	5 4 2	$\frac{b-c}{a-b}=\frac{b}{c}$	5 4 2	$\frac{a-b}{b-c}=\frac{c}{b}$
Sixth	*Subcontrary to geometric*	6 4 1	$\frac{b-c}{a-b}=\frac{a}{b}$	6 4 1	$\frac{a-b}{b-c}=\frac{b}{a}$
Seventh		9 8 6	$\frac{a-c}{b-c}=\frac{a}{c}$	3 2 1	$\frac{a-c}{a-b}=\frac{b}{c}$
Eighth		9 7 6	$\frac{a-c}{a-b}=\frac{a}{c}$	6 4 3	$\frac{a-c}{a-b}=\frac{a}{b}$
Ninth		7 6 4	$\frac{a-c}{b-c}=\frac{b}{c}$	4 3 2	$\frac{a-c}{a-b}=\frac{a}{c}$
Tenth		8 5 3	$\frac{a-c}{a-b}=\frac{b}{c}$	3 2 1	$\frac{a-c}{b-c}=\frac{b}{c}$

Table 2: The ten means in Nicomachus and Pappus' works

If tradition is to be trusted, Pythagoras learned in Mesopotamia of three means – the arithmetic, the geometric and the subcontrary, later called harmonic – and of the golden proportion relating two of these.[9] According to Iamblichus, the first three means were in use in the time of Pythagoras and his school, Archytas and Hippasus called the subcontrary harmonic mean, Eudoxus invented three new means: the antiharmonic or subcontrary, the fifth and the sixth and the other four were added later by Myonides and Euphranor, both Pythagoreans, in their musical studies.[10] So the means devised by Greek were ten in all (see Table 2), the number ten being "perfect" for the Pythagorean school. But it is certainly to Nicomachus and Pappus that we owe the fullest discussion on the means and on the history of the theory formulated by the Pythagoreans between the 6th and 4th centuries B.C. The importance of this theory is thus emphasised by Nicomachus:

> "After this it would be the proper time to incorporate the nature of proportions, a thing most essential for the speculation about the nature of the universe and for the propositions of music, astronomy, and geometry, and not least for the study of the works of the ancients, and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting. A proportion, then, is in the proper sense, the combination of two or more ratios [...] The first three proportions which are acknowledged by all the ancients, Pythagoras, Plato and Aristotle, are the arithmetic, geometric and harmonic; and there are three others subcontrary to them, which do not have names of their own, but are called in more general terms the fourth, fifth, and sixth forms of mean; after which the moderns discover four other as well, making up the number ten, which, according to the Pythagorean view, is the most perfect possible."[11]

He also noticed the role and importance of the means for the harmony of the cosmos:

> "Some however, agreeing with Philolaus, believe that the proportion/mean is called harmonic because it attends upon all geometric harmony, and they say that 'geometric harmony' is the cube because it is harmonised in all three dimensions, being the product of a number thrice multiplied together. For in every cube this proportion/mean is mirrored; there are in every cube 12 sides, 8 angles and 6 faces; hence 8, the mean between 6 and 12 is according the harmonic mean, for as the extremes are to each other, so is the difference between greatest and middle term to that between the middle and smallest terms, and, again, the middle term is greater than the smallest by one fraction of itself and by another is less than the greater term, but is greater and smaller by one and the same fraction of the extremes. And again, the sum of the extremes multiplied by the mean makes double the product of the extremes multiplied together. The diatessaron is found in the ratio 8:6, which is sesquitertian, the diapente in 12:8, which is sesquialter; the diapason, the combination of these two, in 12:6, the double ratio; the diapason and diapente combined, which is triple, in the ratio of the difference of the extremes to that of the smaller terms, and the diapason is the ratio of the middle term to the difference between itself and the lesser term. Most properly, then, has it been called harmonic."[12]

Porphyry in his *Commentary on Ptolemy's Harmonics* emphasised the use of the means in music by Archytas.[13] Basic musical intervals were found by the Pythagoreans studying the lengths of vibrating strings, which are expressible as ratios of integers:

1:2 for the octave, 2:3 for the fifth, 3:4 for the fourth. They would have observed that if they took three strings, of which the first gave out a note an octave below the second, while the second an octave below the third, the lengths would be proportional to 4, 2, 1, i.e., they are terms ın geometric mean. When they took the strings sounding a given note, its major fourth and its upper octave, the lengths would be proportional to 12, 8, 6, i.e., they are terms in harmonic mean and when its major fifth and its upper octave, the lengths would be proportional to 12, 9, 6, i.e., they are terms in arithmetic mean.

The last five means can be traced back to the 'moderns', hence probably to Nicomachus himself, or to Theon, or to those music theoreticians who inherited Pythagorean teaching in their own field. The sum of this tradition reached Umanism and Renaissance mainly through Boethius, who translated and spread Nichomacus'theory. A contribution to the spreading of these concepts was supplied also by Marsilio Ficino in his comments to Plato's *Timaeus* and a significant influence, particularly on atrists, was produced by L. B. Alberti, who dedicated to proportions in buildings and to means a full chapter of his *De re aedificatoria*. But out of the ten means codified by the Greeks only the first three are mentioned in the art treatises of the XV and XVI centuries, and particularly the arithmetic, the geometric and the harmonic mean. A hint for the identification of the Greek sources, which had the most significant influence during Renaissance comes from the philologic examination of the terminology used to indicate means and proportions in Greece and thereafter.

The terms used in Greek to indicate the proportion and the mean, while are distinct from each other in a strictly mathematical context, often appear as synonyms in other contexts (e.g., physics, philosophy, ethics or ordinary language). This is another factor which may serve to differentiate the traditions followed, since those closest to scientific or mathematical language are careful not to confuse the two terms. From the most philologically accurate Greek dictionaries it emerges that the term $\alpha\nu\alpha\lambda o\gamma\iota\alpha$ indicated 'proportion, i.e. the equality between two ratios', expressed in Latin by the word *proportio*. This consists of a relation among four magnitudes *a, b, c, d* having the form

$$a:b = c:d,$$

and ın this sense is used by mathematicians (Euclid, Archimedes, etc.). But philosophers, too, made use of it, though in different contexts, in connection with cosmology or physics. Plato, for example, uses it in the *Timaeus* 31c-32a in order to explain how the Demiurge began to create the universe with fire and earth: he says that these elements could not compound together without a link to unite them, and concludes by saying 'And it is proportion which naturally achieves this most beautifully.'

The term *μεσοτης* represented the 'mean', in Latin *medietas*, and sometimes central, medial or intermediate. In the mathematical context it is the abstract name given by the Pythagoreans to certain magnitudes which are intermediate between two given magnitudes, and are defined in function of these magnitudes. Prior to Euclid the term *μεσοτης* was used in Pythagorean teaching from the earliest times. The first Pythagoreans to write about means were Archytas, Hippocrates etc., to whom the historical sources attribute the ideation of the three means: the arithmetical, the geometrical and the harmonic. In the Renaissance art treatises it is often indicated with the term *mediocritas* o *mediocrità*.[14]

Since the geometrical mean is a continuous proportion, some Greek authors came to confuse the terms *analogia* and *mesotes*, treating them as synonymous – a misuse which became common in artists' writings too. Only Nicomachus of Gerasa (1st-2nd century), in his *Introductio arithmeticae*, condemned the extension of the term *analogia* to the arithmetical mean.

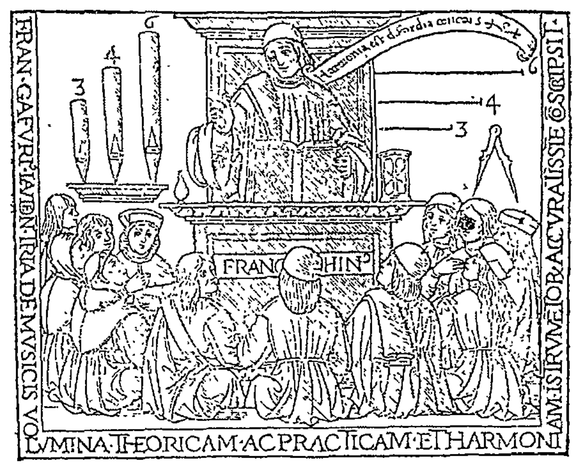
Mathematicians also made use of the terms *μεση αναλογον* for proportional mean or *ακρος και μεσος λογος* for extreme and mean ratio in the construction of that continuous proportion later named the "golden section", or "golden proportion", or "divine proportion."

In physics *mesotes* was used to reconcile opposing principles, to postulate the cosmic centre, and in ethics as the expression of moderation in two aspects: of *equilibrium* between two extremes or *moderation* exercised by the rational over the irrational.[15]

Also this aspect is transferred to art, where it is stated that beauty is balance between parts and the whole, and that for that purpose indeed the rules on proportions and means developed by mathematicians should be used.[16]

The sources of the theoreticians of art and music of the XVI century are mainly Vitruvius and Alberti. However it should be noted that there is no special care for a systematic and scientific use of the words "means" and "proportions", which are often confused. It should also be noted that, with the passing of time, the linguistic-scientific inaccuracy increases. [17] This signal indicates that, despite the fact that these theoreticians have attended abacus schools where arithmetic, geometry and the application to art of mathematical canons were taught, their interest was not in mathematical stringency, but rather in aesthetic adaptation.

1: F Gaffurio (1480) *Opera teorica della dısciplina musicale*



2: F. Gaffurio (1508) *Angelicum ac dıvinum opus musicum*

Table 3: Pythagorean means and proportions ın musıcal treatıses of XV-XVI centuries

The attitude of music theoreticians on relations between Pythagorean music and arithmetic during Humanism and Renaissance is not much different. Primary sources on the Pythagorean' musical theory have not survived, and all that remains is Boethius' account of them in his *De Inst. Musica.* The music theoretician Gafurio faithfully retells the story told by Boethius in his *Theorica musice* 1480 (published Posteriori 1492, 1508), whose frontispiece offers an effective illustration of the history of music theory in four scenes (Table 3). The characters portrayed are Tubalcain, biblical founder of music, shown in a forge where smiths are working iron; Pythagoras using a stick to touch bells of various sizes and vessels filled in varying degrees with liquid; and finally Philolaus and Pythagoras playing flutes of different lengths. In each figure appear the numbers 4, 6, 8, 9, 12 and 16, linked to the famous ratios between hammerheads (first scene), depth of liquid in vessels (second scene); weights suspended from a rope (third scene) and flutes of different lengths (fourth scene).

More precisely, in the second scene Pythagoras is shown demonstrating the octave 8:16 (i.e., 1/2) *diapason*, in the third scene Pythagoras demonstrates the fifth 8:12 (i.e., 2/3) *diapente*, and in the fourth he is holding in his hand two flutes which express the fourth 9:12 (i.e., 3/4) *diatesseron*, while Philolaus has two flutes which express the fifth 4:6. These last two figures are playing flutes one of which is twice the length of the other: Pythagoras 8, Philolaus 16, i.e. the octave 8:16 is represented once more, so that this last scene incorporates all the musical intervals. In the frontispiece as a whole, i.e. in the four scenes, two octaves thus appear, i.e., the compound chords which expressed the harmonic law of the universe, the 'Great perfect system of the Greeks': octave and fifth (1:2:3) and the two octaves (1:2:4). Raphael shows a similar scene on the blackboard in front of Pythagoras in his *School of Athens*. The fact that Plato appears in the same fresco, with the *Timaeus* in his hand, has led art historians to see in it 'Raphael's interpretation of the harmony of the universe, which Plato had described in the *Timaeus* on the basis of the Pythagorean discovery of the ratios of musical harmonies'. Indeed the inspiring sources of the artistic-musical theories were exactly Plato and Boethius.

Concerning the terminology used by art and music theoreticians to indicate fractions, it does not differ significantly from the one used by the mathematicians of the time. It is likely that in the abacus schools a similar table on proportions to the one reported by Luca Pacioli both in his *Summa* (1494) and in his *De divina proportione* (1509) (Table 4) was in use.

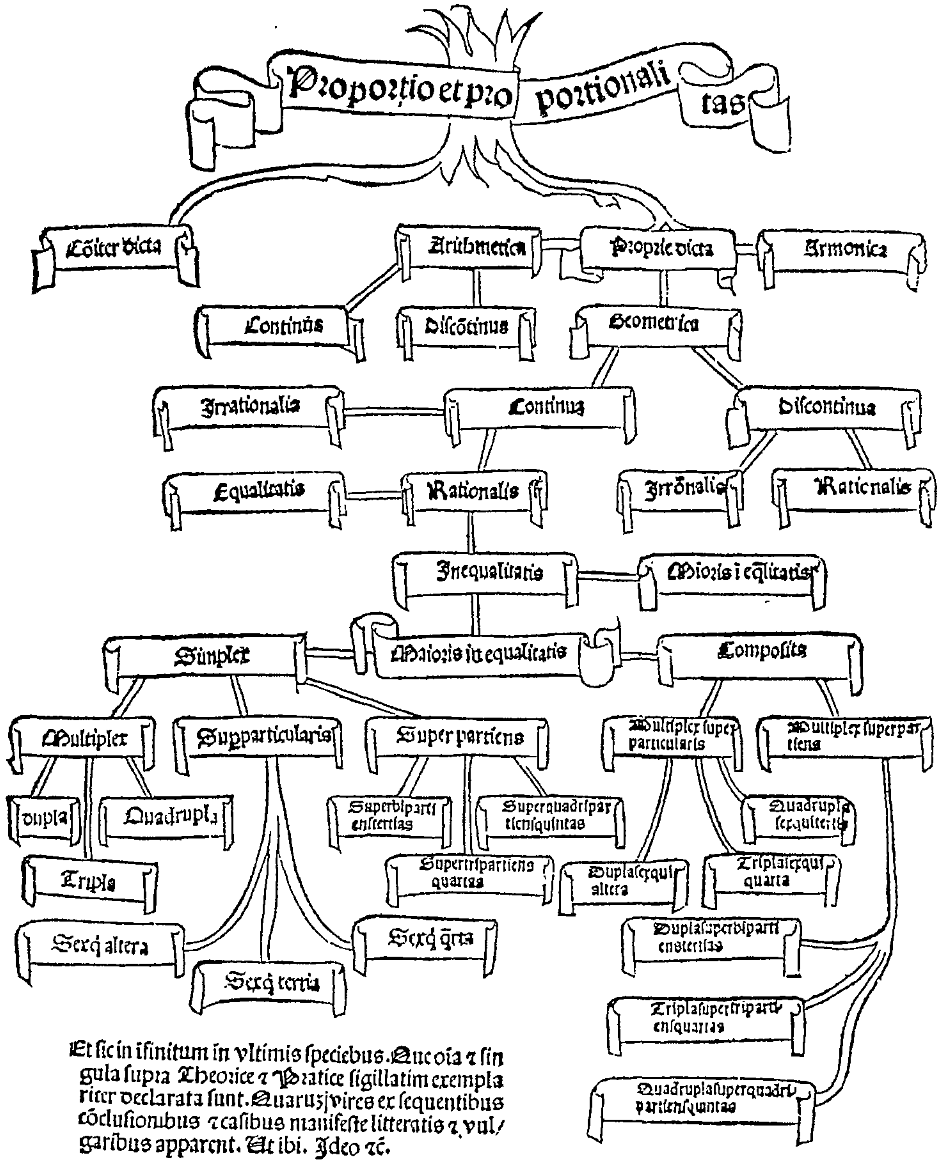


Table 4a: Diagram of proportions in L. Pacioli (1494) *Summa de arithmetica, geometria, proportioni et proportionalita*

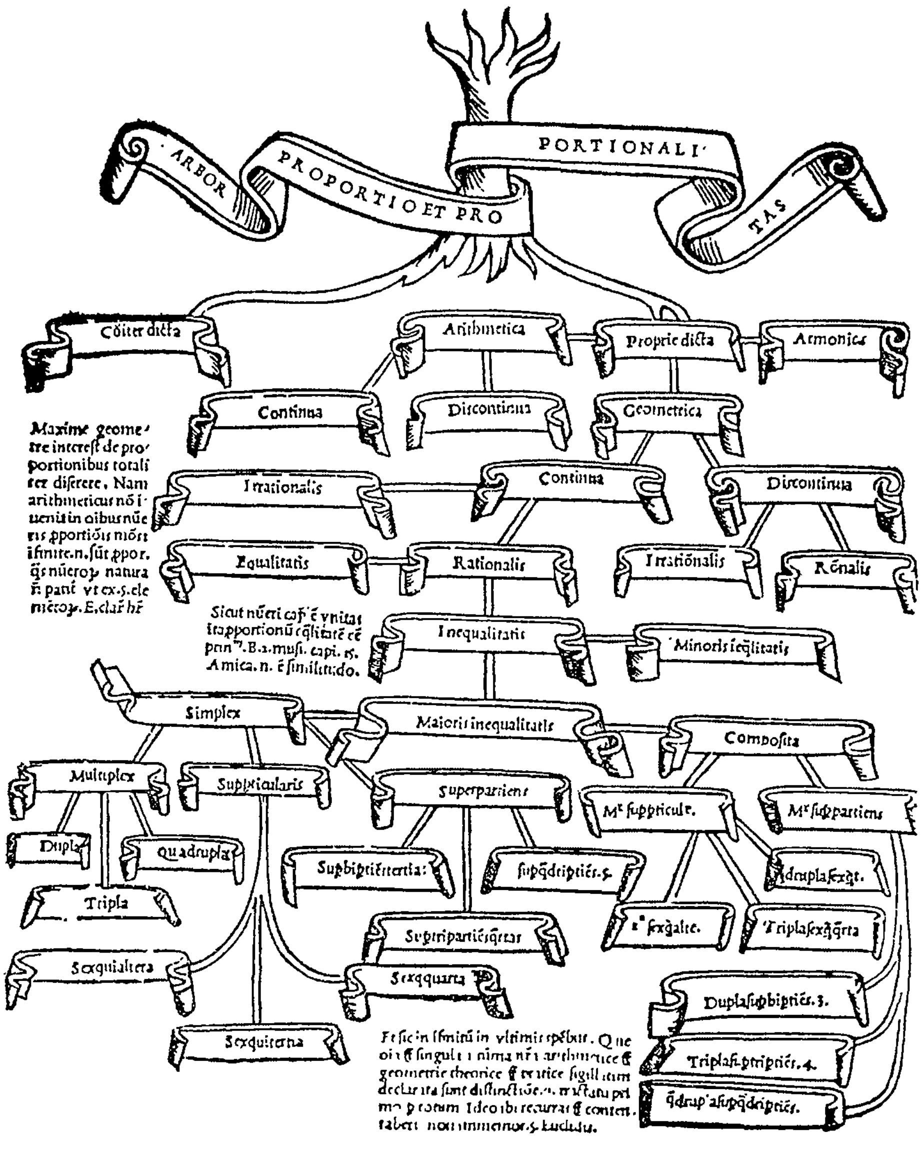


Table 4b: Diagram of proportions in L. Pacioli (1509) *De divina proportione*

2. THE LINK BETWEEN SYMMETRY, PROPORTION AND MEAN IN ART TREATISES

No doubt Nicomachus had absorbed Plato's theories (see also the initial section of the *Introductio* reproduced above) and his treatise, filtered by Boethius and flanked by Ficino and Vitruvius, marked the way which led to the birth of the eurithmy and symmetry concepts, concepts which are fundamental for research and codification of beauty in the art treatises of the XVI century. These could have been the roots of said concepts:

> Plato, *Timaeus,* "... in beginning to construct the body of the All, God was making it [...] The fairest of bonds is that which most perfectly unites into one both itself and the things which it binds together; and to effect this in the fairest manner is the natural property of *proportion.* [...] the body of the Cosmos was harmonised by *proportion* and brought into existence. [...] Whereas, because of this reasoning, He fashioned it to be One single Whole, compounded of all wholes, perfect and ageless and unailing."

> Vitruvius, *De Architectura, "Aedium compositio constat ex symmetria, cuius rationem diligentissimi tenere debent. Ea autem paritur a proportione, quae graece analogia dicitur. Proportio est ratae partis membrorum in omni opere totiusque commodulatio, ex qua ratio efficitur symmetriarum. Namque non potest aedis ulla sine symmetria atque proportione rationem habere compositionis, nisi uti ad hominis bene figurati membrorum habuerit exactam rationem."*

> "The planning of temples depends upon symmetry: and the method of this architect must diligently apprehend. It arises from proportion (which in Greek is called *analogia*). Proportion consists in taking a fixed module, in each case, both for the parts of a building and for the whole, by which the method of symmetry is put into practice. For without symmetry and proportion no temple can have a regular plan; that is, it must have an exact proportion worked out after the fashion of the members of a finely-shaped human body."

And here is their evolution ın Humanism and Renaissance:

> *"Sed pulchritudo atque ornamentum per se quid sit, quidve inter se differant, fortassis animo apertius intelligemus, quam verbis explicari a me possit. Nos tamen brevitatis gratia sic deffiniemus: ut sit pulchritudo quidem certa cum ratione concinnitas universarum partium in eo, cuius sint, ita ut addi aut diminui aut immutari possit nihil, quin improbabilius reddatur."*[18]

> *"Ex ea re suas esse partes instituit Grecia suscepto in opere id conari, ut quos fortunae opibus aequare non possent, hos, quoad in se esset, ingenii dotibus superaret: coepitque uti caeteras artes sic et hanc aedificatoriam ipso ex naturae gremio petere atque educere, totamque tractare, totamque pernoscere, sagaci solertia prospiciens perpendesque. Quid inter ea, quae probentur, aedificia intersit atque ea quae minus probentur, ista in disquisitione nihil praetermisit. Omnia tentavit, naturae vestigia lustrans et repetens. ...neque destitit etiam in minutissimis iterum atque iterum considerasse partes singulas, qui dextra sinistris, stantia iacentibus, proxima distantibus convenirent. Adiunxit, detraxit, aequavit maiora minoribus, similia dissimilibus, prima ultimis; quoad compertum fecit, laudari aliud in his, quae ad vetustatem perferendam quasi stataria constituerentur, aliud in his, quae nullam aeque ad rem atque ad venustatem fabricarentur."*[19]

> *"Itaque superiorum exemplis et peritorum monitis et frequenti usu, admirabilium operum efficiendorum absolutissima cognitio, ex cognitione praecepta probatissima deprompta sunt. ... Istarum rerum praecepta alia universam omnis aedificii pulchritudinem et ornamenta complectuntur, alia singulas partes membratim prosequuntur."*[20]

> *"Quae in rebus pulcherrimis et ornatissimis placeant, ea quidem aut ex ingenii commento et rationibus aut ex artificis manu deveniunt, aut a natura rebus ipsis immissa sunt. Ingenii erit electio distributio collocatio et eiusmodi, quae operi afferant dignitatem."*[21]

> *"Quae si satis constant, statuisse sic possumus: pulchritudinem esse quendam consensum et conspirationem partium in eo, cuius sunt, ad certum numerum finitionem collocationemque habitam, ita uti concinnitas, hoc est absoluta primariaque ratio naturae, postularit."*[22]

Not longer before Alberti, trying to find which is the cause which by its own nature originates beauty *("quidnam sit quod natura sui pulchritudinem efficiat")*[23] listed the criteria used by the ancients and concluded:

> *"Ex quo statuisse possumus, ne catera istiusmodi prolixius prosequar, praecipua esse tria haec, in quibus omnis, quam quaerimus, ratio consumetur: numerus, et quam nos finitionem nuncupabimus, et collocatio. Sed est amplius quippiam ex his omnibus compactis atque nexis, quo tota pulchritudinis facies mirifice collucescat: id apud nos concinnitas nuncupabitur, quam eandem profecto omnis esse gratiae atque decoris alumnam dicimus. Atque est quidem concinnitatis munus et paratio partes, quae alioquin inter se natura distinctae sunt, perfecta quadam ratione constituere, ita ut mutuo ad speciem correspondeant."*[24]

Concerning numbers,[25] after having made the distinction between even and odd and listed notions of numerology Alberti went on to define the perfect number 6=1+2+3=1x2x3. He then states that according to Aristotle the perfect one is 10 because its square 100 is the sum of the cubes of 1, 2, 3 and 4, and that in nature even numbers go up to 10 and the odd ones up to 9. Proportions appear for instance in the paragraph on the forms preferred by the ancients for the plans of rectangular buildings. Sides follow the rules of musical harmony, that is they are in 3:2 (fifth), 4:3 (fourth) , 2:1 (eight). Areas were divided in three groups: short ones, long ones and medium ones. The first ones were square (1:1) or rectangular with a length equal to $1+\frac{1}{2}=3{:}2$, or $1+\frac{1}{3}=4{:}3$. The long ones with the sides for instance in proportion 3:1 or 8:3, while the medium ones were in double 2:1 or triple 3:1 proportion.

> *"In quadrangulis ferme omnibus templis maiores observaverunt aream producere, ut esset ea quidem longior amplius ex dimidia quam lata; alii posuere, ut latitudo parte sui tertia a longitudine superaretur; alii voluere longitudinem duas capere integras latitudines."*[26]

In book IX, going back to the subject of proportions between areas[27] Alberti calls short areas those in the ratios 1:1, 2:3, 3:4, medium areas those in the ratios 1:2, 4:9, 9:16 and long areas those in the ratios 1:3, 1:4, 3:8. He then goes on to discuss buildings, that is the three dimensions, and on ratios between base areas and height he states that there also they can be in harmonic ratios, i. e. as 1:2:3:4, but we can also occur irrational numbers, for instance square roots, when the diagonal of the base quadrangle is taken as height.

"Ternatim autem universos corporis diametros, ut sic loquar, coadiugabimus numeris his, qui aut cum ipsis armoniis innati sunt aut sumpti aliunde certa et recta ratione sunt. In armoniis insunt numeri, ex quorum correspondentiis proportiones earum complentur, uti in dupla, tripla, quadrupla. [...] His numeris, quales recensuimus, utuntur architecti non confuse et promiscue, sed correspondentibus utrinque ad armoniam. [...] Diametris etiam finiundis innatae sunt quaedam correspondentiae, quae numeris necquicquam terminari possunt, sed captantur radicibus et potentiis. [...]Tales igitur, quales recensuimus, diametris finiendis et numerorum et quantitatum correspondentiae innatae sunt. Istorum omnium usus est, ut minima linea detur areae latitudini, maxima vero huic correspondens longitudini, mediae vero dentur altitudini. Sed interdum pro aedificiorum commoditate commutabuntur."[28]

Alberti goes on to discuss other devices used in art to "group dimensions in three" and introduces on this subject the means, mentioning only the first three and supplying for each numerical examples. The arithmetic mean between 4 and 8 is $6 = \frac{8+4}{2}$, the geometric mean between 4 and 9 is $6 = \sqrt{4 \cdot 90} = \sqrt{36}$, and the harmonic mean between 30 and 60 is 40. These means, Alberti states, are mainly used by architects to determine the heights of buildings:

"Quae autem diffinitionis ratio non innata armoniis et corporibus sed sumpta aliunde ad diametros ternatim iungendos subserviat, nunc dicendum est. Etenim sunt quidem trium diametrorum in opus coaptandorum annotationes quaedam valde commode ductae cum a musicis tum a geometris tum etiam ab aritmetricis, quas iuvabit recognovisse. Has philosophantes appellarunt mediocritates. Earum ratio et varia et multiplex, sed in primis apud sapientes captandarum mediocritatum modi sunt tres, quorum comparetur numerus ambobus illis positis correspondens certa cum ratione, hoc est, ut ita loquar, affinitatis quadam adiunctione.

[...] Ex tribus, quas in primis probarint philosophantes, facillima inventu mediocritas est, quam aritmetricam dicunt. Positis enim extremis numerorum terminis, hoc est hinc maximo, puta octavo, atque hinc e regione minimo, puta quattuor, hos ambos in una iungis summam: fient igitur duodecim; qua summa compositarum in duas partes divisa accipiam ex eis alteram: ea erit unitatum sex. Hunc numerum senarium istic esse aritmetrici mediocritatem statuunt, quae quidem inter illos positos extremos quattuor atque octo aequo ab utrisque distet intervallo.

Altera vero mediocritas geometrica est; ea captatur sic. Nam minimus quidem terminus, puta quattuor in maximum, puta novem, ducitur. Ex his ita multiplicatis fit summa unitatum sex et triginta; cuius summae, ut loquuntur, radix, id est lateris totiens sumptus quotiens in eo adsit unitas, ipsam compleat aream numerorum triginta sex. Erit igitur radix istaec sex: nam sexties sumpta aream dabunt triginta sex. Hanc aritmetricam [geometricam] mediocritatem perdifficile est ubivis adinvenisse numeris, sed lineis eadem bellissime explicatur; de quibus hic non est ut referam.

Tertia mediocritas, quae musica dicitur, paulo est quam aritmetrica laboriosior; numeris tamen bellissime definitur. In hac proportio quae minimi est terminorum positorum ad maximum, ista eadem proportione se habeant oportet distantiae hinc a minimo ad medium, istinc a medio ad maximum terminorum...

Huiusmodi mediocritatibus architecti et totum circa aedificium et circa partes operis perquam plurima dignissima adinvenere, quae longum esset prosequi. Atqui mediocritatibus quidem istiusmodi ad altitudinis diametrum extollendam apprime usi sunt."[29]

In the following chapter Alberti shows in which way the arithmetic mean had been used for instance to determine the dimensions of columns in the different stylesi: ionic, doric and corynthian. After having made the analogy between the human body and the column, where the base is the unit, he states that height was evaluated by the ancients sometimes 6 and some other 10, so that, in order to correct these values the arithmetic mean was used. The ionic column has a height which is the arithmetic mean between 6 and 10, that is 8. The doric one the arithmetic mean between 6 and 8, i.e. 7, and the corynthian one the arithmetic mean between 10 and 8, i.e. 9:

"Ad tales dimensiones fortassis columnas posuerunt, ut essent aliae ad basin sexcuplae, aliae vero decuplae. Sed naturae sensu animis innato, quo sentiri diximus concinnitates, tanquam istic crassitudinem et contra hic tantam gracilitatem non decere moniti, abdicarunt utranque. Denique hos inter excessus esse quod quaererent existimarunt. Ea re aritmetricos in primis imitati, ambo illa extrema in unum coegere et summam iunctorum per mediam divisere."[30]

Alberti's interest for the measurements to use in art and for mutual relations between the elements of buildings are the result of observations on works of art of antiquity as well as of the rules established by ancient artists. On this subject following statements, characterizing his concept of beauty and harmony, are particularly significant:

> *"Quare in primis observabimus ut ad libellam et lineam et numeros et formam et faciem etiam minutissima quaeque disponantur, ita ut mutuo dextera sinistris, summa infimis, proxima proximis, aequalia aequalibus aequatissime conveniant ad istius corporis ornamentum, cuius futurae partes sunt."*[31]

> *"Ergo recte asseverant, qui dicunt reperiri vitium nullum deformitatis obscenius atque detestabilius, quam ut angulos aut lineas aut superficies numero magnitudine ac situ non diligenter examinateque inter se comparatas coaequatas atque compactas intermiscere."*[32]

> *"Omnium ratio et ordo ita comparentur, ut non modo ad opus honestandum certatim conveniant, sed ne altera quidem sine alteris per se constare aut satis suam servare dignitatem posse videantur."*[33]

> *"Sic istic, quotquot ubique aderunt opinione et consensu hominum probata opera, perquam diligentissime spectabit, mandabit lineis, notabit numeris, volet apud se diducta esse modulis atque exemplaribus; cognoscet repetet ordinem locos genera numerosque rerum singularum, quibus illi quidem usi sunt praesertim, qui maxima et dignissima effecerint, quos fuisse viros egregios coniectura est, quandoquidem tantarum impensarum moderatores fuerint."*[34]

The word used by Alberti to indicate the mean is *mediocritas* and among the classical sources, where he looked for inspiration might have been Nicomachus'arithmetic. Indeed this author is mentioned as arithmetician[35] and among the other classical authors who wrote on means Plato, Aristotle, Pythagoras, Tucidides, Vitruvius, Galen and the historians Plinius, Plutarch are mentioned in Alberti's treatise. It should be noted that Alberti does not simply report the statements of the ancients, but often adds his own deductions resulting from direct observation.[36] The terminology that he uses for fractions is the one introduced by Nicomachus.

4. THE PROPORTIONS IN THE PORTRAYAL OF HUMAN BODY

Particularly concerning the rules to be applied in representing the human figure the two aspects, touched at the beginning, the theoretic and the experimental one, can be seen to surface. As regards the theoretic one the main source is Vitruvius, who uses exclusively rational ratios. The canons are modified in the course of the centuries, but the basic theory implying tht beauty is determined by the mutual relations of parts, remains.

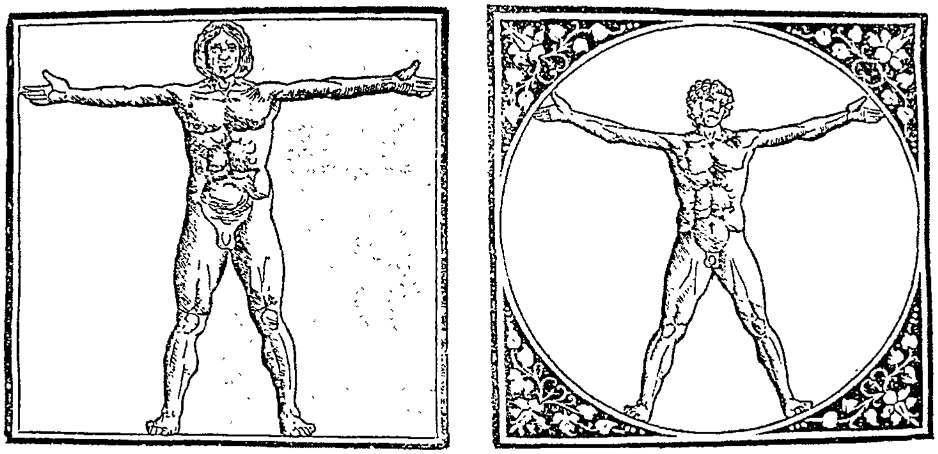
Alberti deviates somewhat from this tradition and, possibly having red Thucidides, absorbs the idea of the mean of observations, and in his *De statua* he imitates that method. Polyclitus Canon is thus recovered![37]

If at this point we go back to Polyclitus and the period in which he was writing his Canon, it is clear that he based his purpose on the philosophical, mathematical and technical knowledge of his time. His aims thus fit into Greek philosophical thinking, which aimed to find harmony and beauty of the universe; and since, according to Pythagoras, this harmony lay in musical relations, based on numbers, Polyclitus turned to these philosophico-mathematical theories in considering proportion and mean. To some extent his treatise echoed the Pythagorean theories on music, mathematics and art, as expressions of a universe ruled by number. The direct view of nature, which was certainly the inspiration for the Greek *mimesis*, i.e., the attempt to instill the naturalistic vision into art (and not the fantastic or richly imaginative vision of the Egyptians or the Babylonians or other populations bordering on the Greek world) is a further piece in this construction ruled by number and by arithmetic.

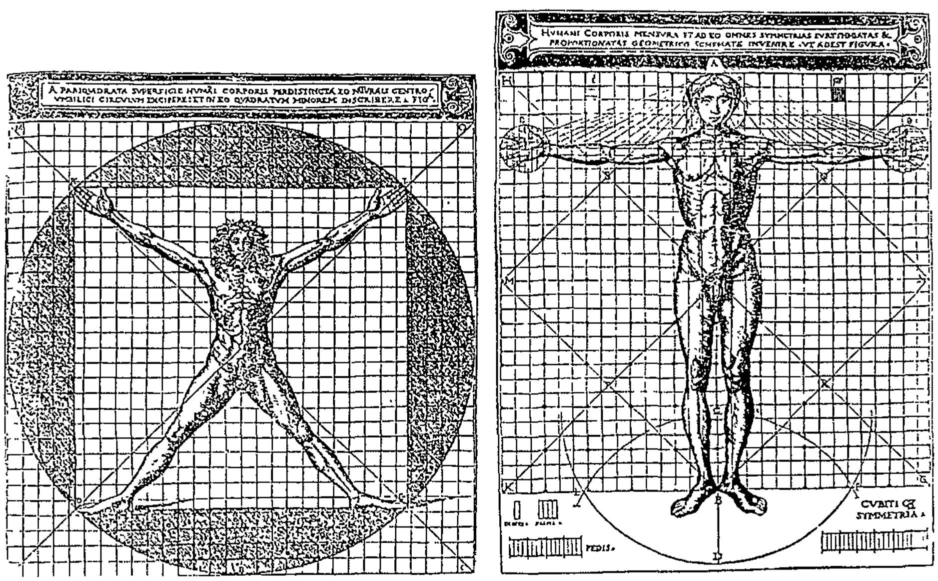
It is in this fusion of *mimesis* and mathematics/arithmetic that Polyclitus' originality lies, and this is why he had such a following not only in ancient times, but in Humanism and the Renaissance. Harmony expressed by means of numbers. Creating, on the basis of mathematical ratios or relationships, a form which, in addition to being naturalistic, might express an ideal essence. Just as music translates mathematical rules, proportions and means into perceptible form, thus becoming the expression/manifestation of theory through the senses, so the visual arts, architecture, the portrayal of the beauty of the human body require exact mathematical rules. Mathematics, the expression of exactness, of the possibility of understanding the universe regulated by number, as the Pythagoreans thought, is sublimated by Plato in the *Timaeus* and subsequently, in the Neoplatonic tradition (1st BC-5th AD)) is rediscovered and reappraised in Humanism. Plato, then, is the go-between for the transfer of the precepts of harmony and music, linked to mathematics.

Although so little survives of the 'rules' or 'canons' given by Polyclitus in his treatise, there can be no doubt that it was from him that a line of thought originated which tried to 'codify' the principles to be followed by artists, in schemes which were either mathematical (harmony linked to arithmetic and to music, following the Pythagorean tradition) or statistical (harmony which is introduced into the universe and into reality through the forms of greatest beauty is 'observed' and translated into mathematical rule with the arithmetical mean of the data observed). This is what, perhaps somewhat inappropriately, I have called experimental, linking it to some degree to the not dissimilar observation in Alberti's work *Della statua*:[38]

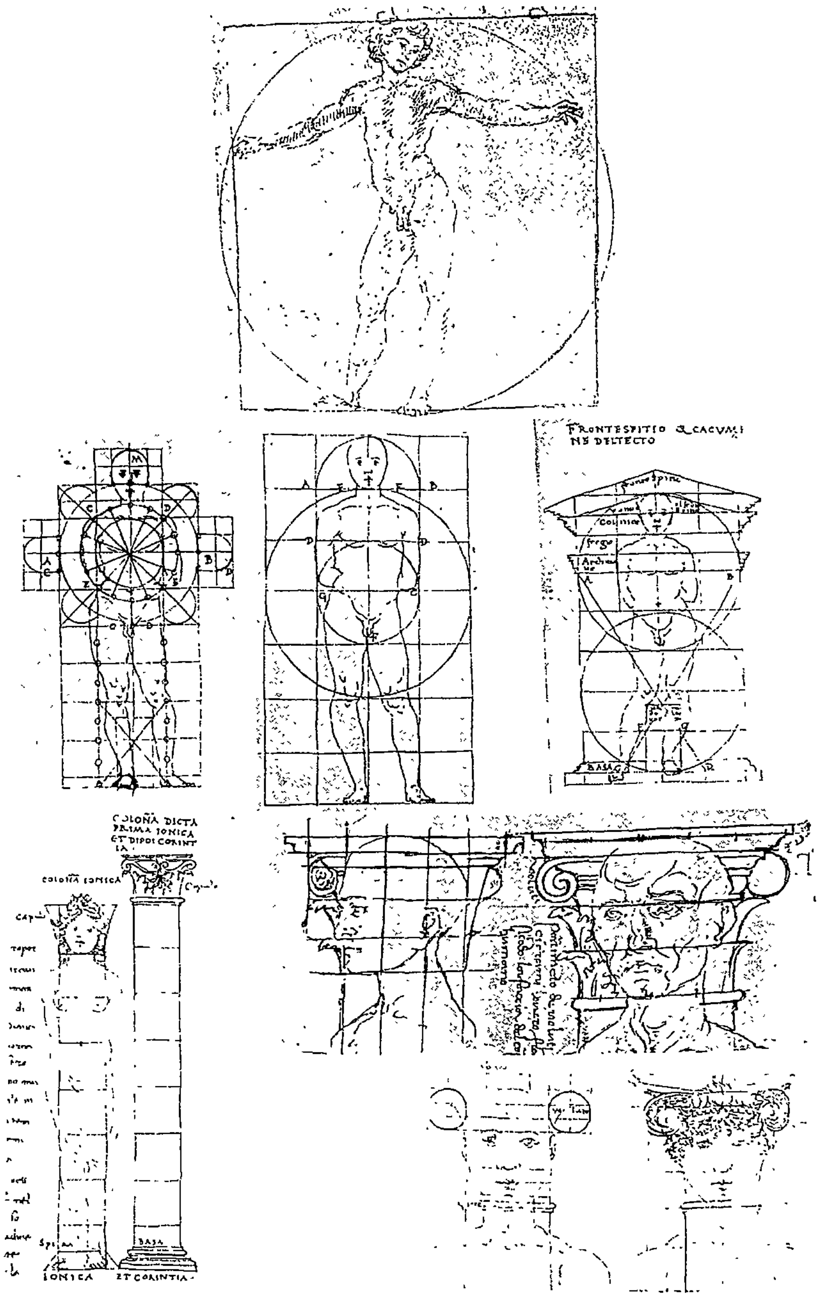
> *"Finalmente mediante tutte quelle cose che insino a qui si son dette, si vede assai manifesto, che si posson pigliare le misure, e i determinamenti da un modello, o dal vivo comodissimamente, per fare un lavoro o un'opera che sia, mediante la ragione, e l'arte perfetta. Io desidero che questo modo sia familiare a' miei Pittori e Scultori, i quali se mi crederanno se ne rallegreranno. E perché la cosa sia mediante gli esempi più manifesta ... ho presa questa fatica di descrivere cioè le misure principali che sono nell'uomo. E non le particolari solo di questo o di quell'altro uomo; ma, per quanto mi è stato possibile, voglio porre quell'esatta bellezza, concessa in dono dalla natura, e quasi con certe determinate porzioni donata a molti corpi, e voglio metterla ancora in scritto, imitando colui che avendo a fare appresso a' Crotoniati la statua della Dea, andò scegliendo da diverse Vergini, e più di tutte l'altre belle, le più eccellenti, e più rare, e più onorate parti di bellezze che egli in quelle giovani vedesse, e le messe poi nella sua statua. In questo medesimo modo ho io scelto molti corpi, tenuti da coloro che più sanno, bellissimi, e da tutti ho cavate le loro misure e proporzioni; delle quali avendo poi insieme fatto comparazione, e lasciati da parte gli eccessi degli estremi, se alcuni ve ne fossero che superassino, o fossero superati dagli altri, ho prese da diversi corpi e modelli, quelle mediocrità, che mi son parse le più lodate. Misurate adunque le lunghezze, e le larghezze, e le grossezze principali e più notabili, le ho trovate che son così fatte."*[39]



1. M. Vitruvius (1511) *per Iocundum .. cum figures et tabula ...*



2. C. Caesarirano (1521) *Di Lucio Vitruvio Pollione; De Architectura libri...*



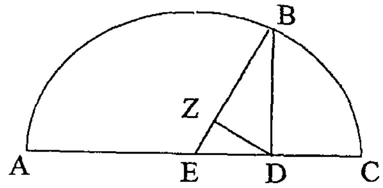
3. F. di Giorgio Martini (1487-1492) *Architettura Ingegneria, Arte militare*

Table 5: Vitruvian figures *(Homo ad circulum et quadratum)*

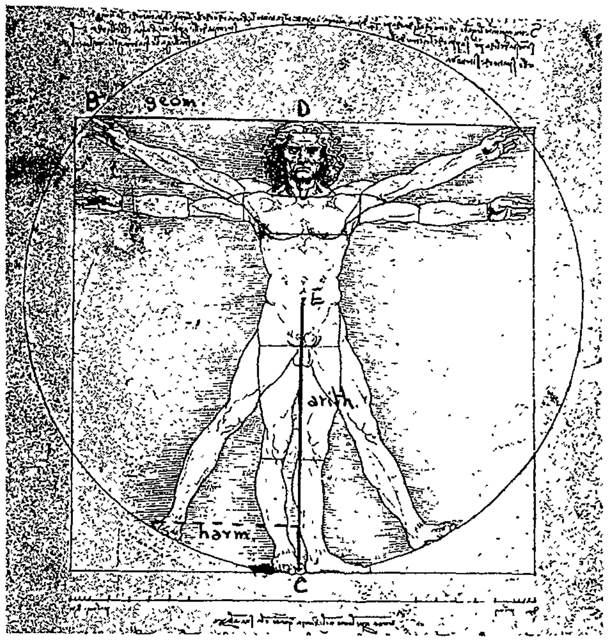
This road is followed by Leonardo[40] and finally by Dürer, with whom the most complete codification is reached. Canons are defined not only for adult men, but also for children, women and old men (Table 5). Leonardo codifies not only the proportions of the various arts of the human body, but also those resulting from the mouvement of the limbs. Beauty is enhenced through the statistical aspect. And even if, following a theorem reported by Pappus,[41] the three means can be found/seen even in the portrayal of the human body inside a circle and a square, artists do not mention it. This mathematical property was either unknown or unimportant to them! In my opinion we find here the same attitude found above on the golden mean. The rule exists and is important for the mathematician, but it is not codified by the artist and the art theoretician. It remains inside the work of art without reaching consciousness.

Things are different for the statistical mean. In this case the artist perceives something which has not yet been codified by mathematics. The law of the large numbers which surfaces in Polyclitus, in Thucidides and in Alberti will find its final formulation only in Jacob Bernoulli's *Ars Conjectandi* (1713). This passing on of 'experience' has meant that the mathematical concept of the mean, which has an important place in the calculation of probabilities and in statistics, two disciplines which date back only to the 17^{th} century, was anticipated by the artists and art theoreticians of Classical Greece (Polyclitus) and of Humanism and the Renaissance (Alberti). If we accept the anthropological point of view according to which mathematics, as an officially recognised activity, is simply the outlet, into the light of awareness, of an activity rooted in the human subconscious, or even in the unconscious, just as the tip of an iceberg is only the surface appearance of a hidden, vastly larger mass, we may see here the example of an artistic manifestation in which mathematical properties are applied even before the relevant theory has been developed. The figurative arts, in this case, may be significant for the mathematics historian in showing the sensitivity to and gift for geometry or mathematics in a particular cultural context, although they cannot in themselves provide documentation of the effective acquisition of the mathematical concept. No concept can emerge in the consciousness unless it has been made the object of research and study.

1. L. B Albertı (1435) *Della statua*

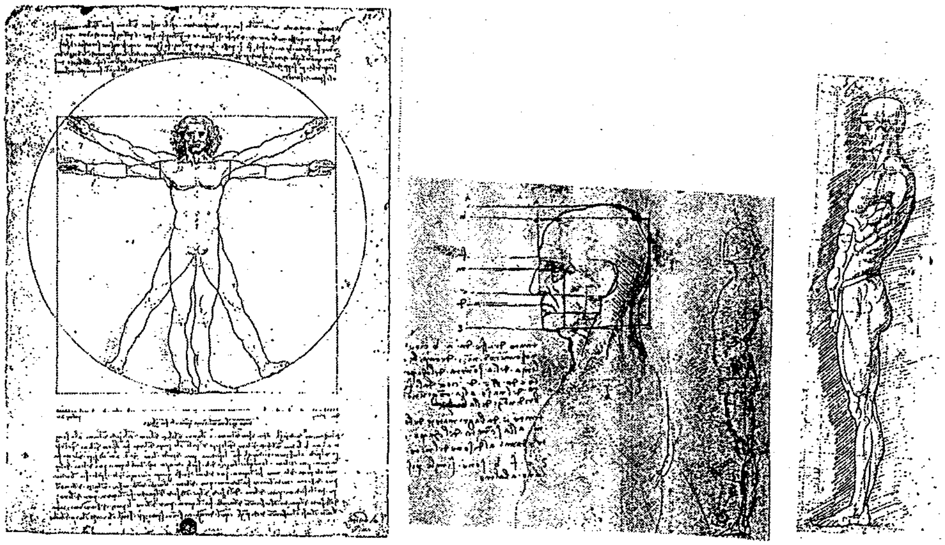


2. Pappus, *Collectio Mathematica* III.,11, 28
EC arithmetic mean
BD geometric mean
ZB harmonic mean

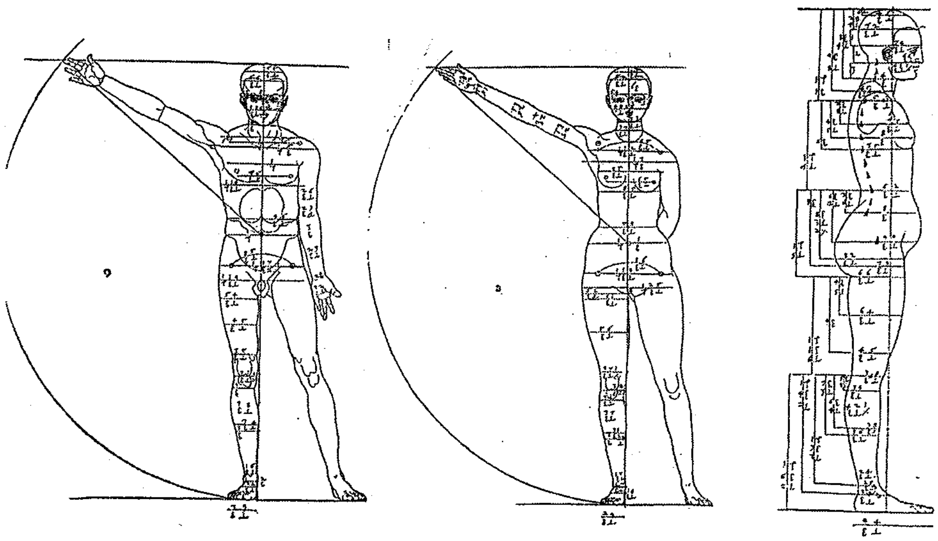


3. Leonardo (1492-1510) *Codex Urbinax Latinus 1270*

Table 6: The means and the portrayal of the beauty of the human body



1. Leonardo (1492-1510) *Codex Urbinax Latinus 1270*



2. A. Dürer (1528) *Bücher von menschlicher Proportion*

Table 7: Studies of the proportions of the human body

ENDNOTES

[1] Polyclitus' *Canon* [Philon *Mechan. synt*, IV, I, 49, 20 - Diels (1949-54, 40B2)] Cf also Diels (1889, note I, p 10); Panofski 1962, p. 73 and La Rocca (1979, pp 524-526.)

[2] Cf. Plutarch, *Moralia*, I, 91 and Galen, *Placita Hippocratis et Platonis*, V, 3

[3] Cf Panofski (1921) and Wittkower (1952, 1960), pp. 199-201 and (1962), pp. 1-29.

[4] The general definitions of the mean which are closest to the true essence of the concept are those which Oscar Chisini gave (Chisini 1929, p 106): *"Data una funzione* $y=f(x_1,x_2,\ldots,x_n)$ *di un certo numero n di variabili indipendenti* $x_1, x_2, \ldots, x_n$, *rappresentanti grandezze omogenee, dicesi media delle* $x_1, x_2, \ldots, x_n$ *rispetto alla funzione f quel numero M che sostituito alle* $x_1, x_2 \ldots x_n$ *dà il medesimo valore per la f che le* $x_1, x_2, \ldots, x_n$ *stesse, cioè quel numero M tale che* $f(M,M,\ldots,M)=f(x_1,x_2,\ldots,x_n)$. *È facile trasformare questa definizione in modo da dare l'espressione analitica esplicita della media A tale scopo si osservi che se, nella f, al posto delle* $x_1, x_2, \ldots, x_n$ *mettiamo un unico valore x, la f stessa diviene una funzione di una sola variabile; chiamiamo* f_1 *questa funzione. E' dunque* $f_1(x)=f(x,x,\ldots,x)$. *"* ("Given a function $y=f(x_1,x_2,\ldots,x_n)$ of a certain number *n* of independent variables $x_1, x_2, \ldots, x_n$, representing homogeneous magnitudes, the mean of $x_1, x_2, \ldots, x_n$, with respect to the function *f* is that number *M* which, when it replaces $x_1, x_2, \ldots, x_n$, gives the same value for *f* as those same $x_1, x_2, \ldots, x_n$, i.e that number *M* such that $f = (M, M, \ldots, M) = f(x_1, x_2, \ldots, x_n)$. This definition can easily be transformed in such a way as to give the explicit analytical expression of the mean. To this end, note that if, in *f*, in place of $x_1, x_2, \ldots, x_n$, we put a single value *x*, *f* itself then becomes a function of a single variable; we call this function f_1. Thus, $f_1(x)=f(x,x,\ldots,x)$.") and Corrado Gini's in his essay *Le medie* (1958, pp. 57-58): *"media fra più quantità è il risultato di una operazione eseguita con una data norma sopra le quantità considerate, il quale rappresenta o una delle quantità considerate che non sia superiore né inferiore a tutte le altre (media reale o effettiva) oppure una quantità nuova intermedia fra la più piccola e la più grande delle quantità considerate (media di conto) "* ("Mean among several quantities is the result of an operation carried out with a given rule on the quantities being considered, which represents either one of those quantities which should be neither greater nor smaller than all the others (real or effective mean) or a new quantity intermediate between the smallest and the greatest of the quantities considered (count mean)"). Some statistics scholars (De Finetti 1931, pp. 369-370; Jacklin 1949, pp. 3-11; Pizzetti 1950, p 428) have written that these are the first acceptable formulations to appear in the history of mathematics in that they show the intrinsic significance of the concept of the mean and the purposes it must answer They also disagree with those who regard Cauchy as the first founder of the formulation of this concept. In his *Cours d'analyse [1re partie, Analyse algebrique*, Imprimerie Royale, Paris (1821), pp. 26-29] Cauchy defined 'mean of several given quantities [as] a new quantity lying between the smallest and the greatest of those considered' Similarly vague, inexact definitions had already appeared in the history of mathematics. As Chisini himself maintains (Chisini 1929, p. 106), supported by De Finetti (1931, p. 370), Cauchy's definition 'is virtually meaningless, and defining the single kinds of mean habitually encountered (arithmetical, geometrical, harmonic, etc) is admittedly an exact operation, but merely formal and antiphilosophical, which may be used - badly - only for empirical purposes.'

[5] Boyer 1968, p. 18.

[6] Boyer 1968, p. 42.

[7] Boyer 1968, p. 31.

[8] Cf Tannery 1912, pp. 80-105, Michel 1950, pp. 365-411; Klein 1968

[9] Boyer 1968, p. 61. In modern terms if *a, b, c* are natural numbers and $a>b>c$, the arithmetical mean can be written by the proportion

$$(a-b):(b-c)=a:a=b\cdot b=c:c,$$

$$a-b=b-c \quad \text{i.e., } b=\frac{a+c}{2},\ \frac{a}{b}<\frac{b}{c}$$

the geometrical mean by

$$(a-b):(b-c)=a:b$$

$$a:b=b\cdot c \quad \text{i.e ,} \quad b=\sqrt{ac}$$

and the harmonic mean by

$$(a-b)\ (b-c)=a:c$$

$$ac-bc=ab-ac \text{ i.e , } b=\frac{2ac}{a+c},\ \frac{a}{b}>\frac{b}{c}$$

[10] Iamblichus, *In Nicom. Arith Introd.* 100, 19 ff: "In ancient days in the time of Pythagoras and the mathematicians of his school there were only three means, the arithmetic and the geometric and a third in order which was then called subcontrary, but which was renamed harmonic by the schools of Archytas and Hippasus, because it seemed to furnish harmonious and tuneful ratios. And it was formerly called subcontrary because its character was somehow subcontrary to the arithmetic. [. .] After this name had been changed, those who came later, Eudoxus and his school, invented three more means, and called the fourth properly subcontrary because its properties were subcontrary to the harmonic [..] and the other two they named simply from their order, fifth and sixth."

[11] Nichomacus Chap. XXI, p 264, Chap. XXII, pp. 266-267.

[12] Nichomacus Chap. XXVI. In chap. XXIX he again wrote on the harmonic mean. "the most perfect proportion, that which is three-dimensional and embraces them all, and which is most useful for all progress in music and in the theory of the nature of the universe This alone would properly and truly be called harmony, rather than the others, since it is not a plane, nor bound together by only one mean term, but with two, so as thus to be extended in three dimensions, just as a while ago it was explained that the cube is harmony."

[13] Porphyry, *Commentary on Ptolemy's Harmonics*: "Archytas, in his discussion of means, writes thus: 'Now there are three means in music: first the arithmetic, secondly the geometric, and thirdly the subcontrary, the so-called harmonic. The arithmetic is that in which three terms are in proportion in virtue of some difference the first exceeds the second by the same amount as the second exceeds the third. And in this proportion it happens that the interval between the greater terms is the lesser, while that between the lesser term is the greater. The geometric mean is that in which the first term is to the second as the second is to the third. Here the greater terms make the same interval as the lesser. The subcontrary mean, which we call harmonic, is such that by whatever part of it the first term exceeds the second, the middle term exceeds the third by the same part of the third In this proportion the interval between the greater terms is the greater, that between the lesser term is the lesser.

[14] Cf. Alberti 1485 Book IX, Chap VI p. 168v-169r (1966, vol. 2, p. 833)

[15] Empedocles seems to have introduced a $\mu\varepsilon\sigma o\zeta$ between the two extremes, Hatred and Love; Plato regarded the mean as a normative ethical principle, harmony of contrasting or differing qualities and principles, harmony of the qualities of the body (*Laws*728D-), of body and soul (*Timaeus* 87c), of joy and sorrow (*Laws*732D-), of wealth and poverty (*Republic* IV, 421 E-), of monarchy and democracy (*Laws* 756E) Aristotle too used the term $\mu\varepsilon\sigma o\varsigma$ in physics, collocating it at the center of the universe as that point without which circular motion would not be possible (*De cael.* II 3 286a); moreover, dealing with time in relation to present and past, he writes (Aristotle *Phys.* 251b20) '...the present is a certain 'meanity' ($\mu\varepsilon\sigma o\tau\eta\varsigma$), and in ethics (*Et. Nic* II 5, 1106 a 29-) the term $\mu\varepsilon\sigma o\varsigma$ is used to describe virtue. Aristotle makes a sharp distinction between the mean in relation to things (corresponding to the arithmetical mean) and the mean in relation to ourselves (a concept which shades into relativity) (*Et. Nic.* II 6, 1106 a 26-1106 b 7)

[16] Cf. Alberti

[17] Francesco di Giorgio Martini 1482-92 *Trattato di architettura civile e militare* pp. 119-120· "*Porzione è ditta l'abitudine overo la comperazione tra due quantità. Numari proporzionali son detti quando lo primo è così al sicondo come lo terzo al quarto. Numari in continua proporzione son detti quando lo primo è così al secondo come el secondo al terzo e come el terzo al quarto E se quattro numeri seranno proporzionali, la proporzione che è dal primo al secondo si chiama prima porzione, e quella che è dal primo al terzo si chiama siconda porzione, e quella che è dal primo al quarto si chiama terza porzione. E la siconda porzione quadrato della prima, e la terza porzione sie cubo della prima* "

[18] Alberti *De re aedificatoria* Chap. II, 93v - *L'architettura*, vol. 2, 1966, pp 446-447 *("In che consistano precisamente la bellezza e l'ornamento, e in che differiscano fra loro, sarà probabilmente più agevole a comprendersi nell'animo che ad esprimersi con parole. Ad ogni modo, senza stare a dilungarci, definiremo la bellezza come l'armonia tra tutte le membra, nell'unità di cui fan parte, fondata sopra una legge precisa, per modo che non si possa aggiungere o togliere o cambiare nulla se non in peggio"*

[19] Alberti *De re aedificatoria* Book VI, Chap. III, 94 - *L'architettura*, vol. 2, 1966, pp 452-453 *("In tal modo i Greci decisero che in tali imprese fosse proprio compito il tentare di superare quei popoli, non già nei doni di fortuna, che non era possibile, bensì nella potenza dell'ingegno, per quanto stava in loro. Cominciarono dunque a desumere i fondamenti dell'architettura, come di tutte le altre arti, dal seno/grembo stesso della natura, e ad esaminare, meditare, soppesare ogni elemento con la massima diligenza e oculatezza. Non trascurarono di ricercare i canoni che distinguono quali edifici siano ben eseguiti e quali sbagliati. Fecero ogni sorta di esperimenti, seguendo le orme della natura. ... non tralasciarono mai, nemmeno nelle cose più minute, di esaminare volta per volta la disposizione delle parti, di modo che quelle di destra si accordassero con quelle di sinistra, le verticali con le orizzontali, le vicine con le lontane, aggiungendo, levando, adeguando le parti più grandi alle più piccole, le simili alle dissimili, le prime alle ultime. Divenne così evidente quali criteri dovevano essere impiegati nelle costruzioni destinate a durare negli anni e in quelle realizzate soprattutto per amore della bellezza.")*

[20] Alberti *De re aedificatoria* Book VI, Chap. III, 95v - *L'architettura*, vol. 2, 1966, pp. 456-457. ("Ebbene dall'esempio degli antichi, dai consigli degli esperti e da una pratica continua, si è ricavata una esatta conoscenza dei modi in cui quelle opere meravigliose venivano condotte, e da questa conoscenza si sono dedotte delle regole importantissime Tali regole si riferiscono in parte alla bellezza e alla decorazione di ogni edificio nel suo complesso, in parte alle singole membrature di esso.")

[21] Alberti *De re aedificatoria* Book VI, Chap. IV, 95v - *L'architettura*, vol 2, 1966, pp. 458-459. ("Le caratteristiche che si apprezzano negli oggetti più belli e meglio ornati o sono frutto di ritrovati e calcoli dell'ingegno, o del lavoro dell'artefice o sono state conferite direttamente dalla natura a tali oggetti All'ingegno spetterà la scelta, la distribuzione delle parti, la disposizione e simili, col fine di dare decoro all'opera.")

[22] Alberti *De re aedificatoria* Book IX, Chap. V, 165r - *L'architettura*, vol. 2, 1966, pp. 816-817. *("Una volta acquisite queste nozioni possiamo stabilire quanto segue: la bellezza è accordo e armonia delle parti in relazione ad un tutto al quale esse sono legate secondo un determinato numero, delimitazione e collocazione, così come esige la concinnitas, cioè la legge fondamentale e primaria della natura.")*

[23] Alberti *De re aedificatoria* Book IX, Chap. V, 164r - *L'architettura*, vol 2, 1966, p. 811.

[24] Alberti *De re aedificatoria* Book IX, Chap V, 165r - *L'architettura*, vol. 2, 1966, p. 815. *("Da quanto sopra si può desumere, senza soffermarci troppo a lungo su altre considerazioni di questo tipo, che tre sono le principali leggi su cui si basa il metodo che cercavamo: numero, ciò che chiamiamo delimitazione e collocazione. Vi è inoltre una qualità risultante dall'unione e connessione di questi [tre] elementi, per cui risplende mirabilmente tutta la forma della bellezza e noi la chiamiamo concinnitas e diciamo che per suo tramite tutto è frutto di grazia e decoro. E' compito e disposizione della concinnitas l'ordinare secondo leggi precise le parti che altrimenti per loro natura sarebbero distinte tra loro, di modo che il loro aspetto presenti una mutua concordanza.").* For the finitio: *"Finitio quidem apud nos est correspondentia quaedam linearum inter se, quibus quantitates dimetiantur. Earum una est longitudinis, altera latitudinis, tertia altitudinis."* (reciprocal correspondence of lines used to measure quantities and their dimensions. One is length, the second is width and the third is hight.) This one is reguled by music. *"Hi quidem numeri, per quos fiat ut vocum illa concinnitas auribus gratissima reddatur, hidem ipsi numeri perficiunt, ut oculi animusque voluptate mirifica compleantur. Ex musicis igitur, quibus hi tales numeri exploratissimi sunt, atque ex his praeterea, quibus natura aliquid de se conspicuum dignumque praestet, tota finitionis ratio perducetur."*

[25] Alberti *De re aedificatoria* Book IX, Chap. V, 165v-166r - *L'architettura*, vol. 2, 1966, pp. 818-820.

[26] Alberti *De re aedificatoria* Book VII, 114v - *L'architettura*, vol. 2, 1966, pp. 550-551 *("In quasi tutti i templi a forma quadrilatera gli antichi prediligevano la forma allungata, di modo che la lunghezza della loro area risultasse maggiore per un mezzo della larghezza; altri preferivano una forma tale che la larghezza venisse superata di un terzo dalla lunghezza; altri ancora stabilirono che questa fosse doppia di quella ")*

[27] Alberti *De re aedificatoria* Book IX, 167r-v - *L'architettura*, vol. 2, 1966, pp. 824-827.

[28] Alberti *De re aedificatoria* Book IX, 167v-168 - *L'architettura*, vol 2, 1966, pp 827-831

[29] Alberti *De re aedificatoria* Book IX, 168v-169r - *L'architettura*, vol. 2, 1966, pp 830-835.

[30] Alberti *De re aedificatoria* Book IX, 169v - *L'architettura*, vol. 2, 1966, pp 834-837

[31] Alberti *De re aedificatoria* Book IX, 170r - *L'architettura*, vol. 2, 1966, pp. 838-839

[32] Alberti *De re aedificatoria* Book IX, 171r - *L'architettura*, vol. 2, 1966, pp. 842-843.

[33] Alberti *De re aedificatoria* Book IX, 172v - *L'architettura*, vol. 2, 1966, pp. 850-851.

[34] Alberti *De re aedificatoria* Book IX, 173v - *L'architettura*, vol. 2, 1966, pp 856-857.

[35] Cf. Alberti *De re aedificatoria* Book IX, ***- *L'architettura*, vol 2, 1966, p 862.

[36] Cf. Alberti *De re aedificatoria* Book IX, ***- *L'architettura*, vol. 2, 1966, pp. 526-7, 562-3, 586-7.

[37] Plutarco, *Moralia* I, 91 45 C-D *"Perchè in ogni opera la bellezza si realizza per mezzo della simmetria ed armonia, ad esempio attraverso molti numeri che convergono nel punto giusto, mentre il brutto ha un'immediata ed improvvisa origine da un difetto casuale o da un eccesso casuale."* Galeno, (II sec. d. C) *De temperam.* I, 9, p 566 *"E viene lodata una statua, chiamata Canone di Policleto, la quale ha questo nome dal fatto di avere una perfetta simmetria di tutte le membra fra di loro "*; Galeno, *Placita Hippocratis et Platonis* V, 3 *"Crissippo invece ritiene che la bellezza non consista nei singoli elementi ma nell'armoniosa proporzione delle parti, di un dito rispetto all'altro, e di tutte insieme le dita in relazione al metacarpo ed al carpo, e di tutte queste rispetto all'avambraccio, e dell'avanbraccio rispetto al braccio, e di tutto in rispetto al tutto, secondo quanto appunto è scritto nel Canone di Policleto. Infatti egli, avendo istruito tutti noi in quello scritto sulla simmetria del corpo, rinsaldò il ragionamento con l'opera avendo creato una statua secondo i dettami del ragionamento ed avendo poi chiamata la stessa natura, come appunto lo scritto, Canone."*

[38] Probably Alberti's inspiring source was Thucidides. Indeed he mentions the Greek historian concerning the walls of the city of Platea during the siege by the Peloponnesians and consequently he must have read the famous paragraph on the mean of observations, which is commonly considered the first step in the prehistory of the law of the large numbers: "They made ladders equal in height to the enemy's wall, getting the measure by counting the layers of bricks at a point where the enemy's wall on the side facing Plataea happened not to have been plastered over. They counted the layers at the same time, and while some were sure to make mistake, the majority were likely to hit the true count, especially since they counted time and again, and, besides, were at no great distance, and the part of the wall they wished to see was easily visible. The measurement of the ladders, then, they got at in this way, reckoning the measure from the thickness of the bricks.

[39] Leon Battista Alberti (1435) *Della statua.*

[40] Leonardo 1492-1510 *Codex Urbinax Latinus 1270*, Chap. X-XI - *Scritti d'arte del Cinquecento*, vol. II ed. P. Barocchi 1973, pp. 1720-1731: X. *Se l'omo di 2 braccia è picolo, quello di quattro è troppo grande, essendo la via di mezzo laldabile; il mezo in fra 2 e 4 è 3· adunque piglia un omo di 3 braccia e c'quello misura colla regola ch'io ti darò. Se tu mi dicessi, io mi potrei ingannare, giudicando uno bene proporzionato che sarebbe il contrario, a cquesta parte i' rispondo che tu debbi vedere molti omini di 3 braccia e c'quella maggiore quantità che sono conformi di membri: sopra uno di quelli di migliore grazia piglia tue misure; la lungheza della mano è 1/3 di braccio e entra 9 volte nell'omo, e così la testa è da la fontanella della gola a la spalla e da la spalla a la tetta e da l'una all'altra tetta e da ciascuna tetta alla fontanella. XI. Vetruvio arhitecto mecte nella sua opera d'architectura, chelle misure dell'omo sono dalla natura distribuite in questo modo, cioè che 4 diti fan uno palmo, e 4 palmi fan uno piè, 6 palmi fan un cubito, 4 cubiti fan uno uomo e 4 cubiti fan uno passo, e 24 palmi fan uno uomo, e queste misure son ne' sua edifizi. Se tu apri tanto le gambe che tu cali da capo 1/14 di tua altezza e apri e alza tanto le braccia che colle lunghe dita tu tochi la linia della sommità del capo, sappi che 'l cientro delle stremità delle aperte membra fia il bellico, e lo spazio che si trova infra le gambe fia triangolo equilatero. Tanto apre l'omo nelle braccia quanto è la sua altezza."*

Leonardo treats the following subjects· "I Figura e sua divisione, II Proporzione di membra, VIII Dell'attitudine e movimenti e loro membra, IX Dello imparare li movimenti dell'ommo, XII Delle mutazioni delle misure dell'uomo pel movimento delle membra a diversi aspetti, XIII Delle mutazioni delle misure dell'uomo dal nascimento al suo ultimo accrescimento, XIV Delle prime quatro parti che si richiedono alla figura, XV De la convenzione delle membra, XVI Della grazia delle membra, XVII De la comodità delle membra." Mario Equicola s l (1526) *Libro di natura d'amore,* 73v-79v [*Scritti d'arte del Cinquecento,* vol II ed. Barocchi, P. (1973), pp. 1615-1621] *"Per la eccellenzia ai Crotoniati piacque che [Zeusi] pingesse loro alcuna cosa, e la immagine di Elena. Disseli che volea vedere alcune lor virginelle; Crotoniati, per conoscerlo eccelente pittore di donne, volentieri li consentirno, e monstrateli (che così consultaro), le più belle scelse; per dimostrar la singular grazia in una non ritrovarse, tolse da ciascuna la più egregia parte, che beltà compitamente non se vede in una sola. Così finì la sua leggiadra opera e tante bellezze vive in una figura accolse. [. .] La bellezza del corpo ricerca che le membra siano ben collocate con debiti intervalli e spazii, ciascuna parte sia con sue tempre, commensa proporzione e conveniente qualità. .Plinio, Varrone e Gellio scriveno il corpo umano non posser crescere sopra 7 piedi in longhezza .. Mettendo l'uomo con le braccia estese, tirando dall'umbilico [...] linee all'estremità di piedi et de dita della mano, trovaremo fanno un circulo perfetto. Vitruvio il corpo dell'uomo dice esser stato da natura così composto, che la faccia tutta, cioè la punta del mento sin dove finiscono li capelli nella fronte, è la decima parte del corpo, dal commo petto, cioè dove finisce il collo sino alla sommità del capo parti quattro: se il corpo è ben quadrato e robusto di 7 teste il trovarai· se è dilicato di 8 e 9, le donne di 7 il più delle volte.. .";* Nifo, Agostino or Augustinus niphus (1549) *De Pulcro primus, De Amore secundus - Scritti d'arte del Cinquecento,* vol. II ed. Barocchi, P. (1973), pp. 1647-1652: *"Quod simpliciter pulchrum sit in rerum natura, ex illustrissimae Ioannae pulchritudine hic probatur".*

[41] Pappus *Coll.* III (ed Hultsch 68, 17-70. 8) "A certain other [geometer] set the problem of exhibiting the three means in a semicircle. Describing a semicircle ABC, with center E and taking any point D on AC, and from it drawing DB perpendicular to EC, and joining EB, and from D drawing DZ perperpendicular to it, he claimed simply that the three means had been set out in the semicircle, EC being the arithmetic mean, DB the geometric mean and BZ the harmonic mean."

SOURCES

Agrippa, C (1533) *De Occulta Philosophia*

Alberti, L. B (1435) *Della statua.*

Alberti, L. B. (1485) *De re aedificatoria* Editio princeps Florence.

Alberti, L B (1966) *L'architettura* (ed. Orlandi, G), 2 vols , Il Polifilo, Milano

Averlino, A (Filarete) (1457-1467) *Trattato di architettura.*

Barbaro, D. (1556)

Cesariano, C (1521) *M. Vitruvio Pollione De Architectura* Como.

Danti, Vincenzio (1567) *Trattato delle perfette proporzioni.*

Dürer, A (1528) *Vier Bücher von menschlicher Proportion*, reprint A. Uhl, Nördlingen

Equicola, M (1526) *Libro di natura d'amore.*

Ficino, M., (1484, 1525) *Banquet*

Ficino, M. (1576) *Opera*, vol. 2.

Ficino, M., *Commentary on Plato's* Timaeus.

Gafurio, F. (1480, 1492, 1508) *Theorica musicae.*

Galen, *De temperam.*

Galen, *Placita Hippocratis et Platonis*

Gaurico, P. (1504) *De Sculptura* Florence, Cap II *De simmetria*

Giorgi, F., *De Harmonia mundi totius,* Venice (1525), Paris (1545), French transl., Paris (1579).

Giorgi, F (1535) *Memorandum.*

Giorgio Martini, F. di (1482-92) *Trattato di architettura civile e militare.*

Iabmlichus, *In Nicomachi Arithmetica Introductio*

Leonardo (1492-1510) Codex Urbinax Latinus 1270 [*Scritti d'arte del Cinquecento*, vol. II., ed. Barocchi, P (1973, pp. 1720-1731)]

Lomazzo (1591) *Idea del tempio della pittura.*

Nicomachus of Gerasa (1926) *Introduction to arithmetic*, Engl. Transl. M. L. D'Ooge Robbins and Karpinski, New York: The Macmillan Company.

Nifo, A. (1549) *De Pulcro primus, De Amore secundus.*

Pacioli, L. (1494) *Summa de arithmetica, geometria, proportioni et proportionalita*, Venetiis

Pacioli, L. (1509) *De Divina Proportione.*

Palladio, A. (1570) *Quattro libri dell'architettura.*

Pappus, *Collectio Mathematica.*

Philon, *Mechan. synt.*

Pino, P. (1548) *Dialogo di pittura*

Plato (1961) *Timaeus* (ed. and transl. Bury, R. G.), London: Heinemann.

Plinius, *Naturalis, Historia*

Plutarch, *Moralia*

Porphyry, *Commentary on Ptolemy's Harmonics*

Serlio, S. (1537) *Sull'architettura*

Theon of Smyrna (1892) *Exposition des connaissances mathématiques utiles pour la lecture de Platon*, ed. Dupuis, J., Paris: Hachette.

Thucidydes (1958) *Historia Belli Peloponesiaci History of the Peloponnesian War* Books III and IV, ed and transl. Forster, C., London: Heinemann.

Vitruvius (1486) *De Architectura,* Editio princeps by Sulpicio Verulano, Roma; (1511, 1513) ed. Giocondo, F.; (1521) ed. Cesariano, C.; (1556) ed. Barbaro, D

Vitruvius (1945, 1962) *On Architecture*, ed and transl. Granger, F., 2 vols., London: W. Heinemann.

REFERENCES

Bailly, A., *Dizionario greco francese*

Boyer, C. (1968) *A History of Mathematics*, New York.

Chisini, O. (1929) Sul concetto di media, *Periodico di Matematiche* 4, IX, pp. 106-116

De Finetti, B. (1931) Sul concetto di media, *Giornale dell'Istituto Italiano degli Attuari*, II, pp 369-396.

Diels, E (1889) ?, *Archäologischer Anzeiger.*

Diels, H., Hermann, G. and Krantz, W. (1949-54) *Die Fragmente der Vorsokratiker*, 3 vols, Berlin: Weidmann.

Garuti, G. (1957) *L'uso e il significato di μεσος (μεσοτης) e μετρον (μετριος) nella filosofia greca fino ad Aristotele*, Atti e mem, Accademia Scienze Lettere e Arti di Modena, 1-*.

Gini, C. (1958) *Le medie*, Torino. Utet.

Giusti, E. (1993) *Euclides Reformatus La teoria delle proporzioni nella scuola galileiana*, Torino: Bollati Boringhieri.

Heath, T (1921) *A History of Greek Mathematics*, 2 vols., Oxford: Clarendon Press

Heath, T (1956) *Euclid's Elements*, 3 vols., 2nd ed , Dover, New York.

Invrea, R. (1936) La legge dei grandi numeri era nota a Tucidide?, *Giornale dell'Istituto Italiano degli Attuari*, 14, pp. 229-230

Jacklin, H. (1949) La notion de moyenne, *Metron* 15, pp 3-11.

Klein, J. (1968) *Greek Mathematical Thought and the Origin of Algebra*, Cambridge, Massachussetts: MIT

La Rocca, E. (1979) Policleto e la sua scuola, *La Grecia nell'età di Pericle. Le arti figurative* (ed. Bianchi Bandinelli, R.), Milano: Bompiani, pp. 516-556.

Laserre, F. (1964) *The Birth of Mathematics in the Age of Plato*, Hutchinson of London.

Liddell, H G. and Scott, R. (1948) *A Greek English Lexicon*, Oxford.

Michel, P H. (1950) *De Pythagore à Euclide*, Paris.

Mugler, C. (1958) *Dictionnaire historique de la terminologie geometrique des Grecs*, Paris.

Panofski, E. (1921) Die Entwicklung der Proportionslehre als Abbild der Stilentwicklung, *Monatshefte für Kunstwissenschaft*, 14, pp. 188-219, reprinted in *Meaning in the visual arts*, (1955) pp 76-91., New York (Italian translation *La storia della teoria delle proporzioni del corpo umano come riflesso della storia degli stili*, Panofski, *Il significato nelle arti visive*, Einaudi, Torino (1962), pp. 59-106).

Pizzetti, E (1950) *Considerazioni in margine al concetto di media*, Statistica, X, pp. 428-437.

Szabó, A. (1964) Die frühgriechische Proportionenlehre im Spiegel ihrer Terminologie, *Arch Hist Ex Sci.* 2, pp. 197-270.

Tannery, P (1880) L'arithmétique des Grecs dans Pappus, *Extrait des Mémoires de la Societé des sciences physiques et naturelles de Bordeaux* 3, pp. 351-371

Tannery, P. (1912) *Sciences exactes dans l'antiquité*, vol 1, Paris.

Thomas, I. (1957) *Selections illustrating the history of Greek Mathematics with an English Translation*, vol, 1, *From Thales to Euclid*, London: Heinemann.

Vacca, G. (1936) Sul concetto di probabilità presso i Greci, *Giornale dell'Istituto Italiano degli Attuari*, XIV, pp 231-234

Wittkower, R. (1952) Su alcuni aspetti della Proporzione nel Medioevo e nel Rinascimento, Summary of the lecture held at the 'Primo Convegno sulle Proporzioni nelle Arti, Milano 27-29 9 1951', *Atti e Rassegna Tecnica della Società degli Ingegneri e degli Architetti in Torino*, 6, p 121.

Wittkower, R. (1960) *The Changing Concept of Proportion*, Daedalus, 199-215

Wittkower, R. (1973) *Architectural Principles in the Age of Humanism* 4th ed., London, [Italian transl. *Principi architettonici nell'età dell'Umanesimo*, Einaudi, Torino (1964)].

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SYMMETRY IN MUSIC AS A STYLISTIC INDEX FOR THE TRANSITION FROM THE MIDDLE AGES TO THE RENAISSANCE

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Music exists in an anisotropic three-dimensional space whose independent variables are *time, pitch* and *polyphony.* Figure 1 shows part of a score of a chanson by the 15^{th}-century Netherlandic composer Jacob Obrecht, which is a projection of this three-dimensional space onto two dimensions; time runs from left to right, but pitch and polyphony are both projected vertically. Polyphony, the coexistence of several *parts* or *voices,* is indicated on the various staffs, the horizontal barlines in the score. Actually, the different staffs should be projected into a third dimension, perpendicular to the sheet of music, more or less as shown in Figure 2.

Figure 3 shows the three-dimensional music space: the different parts, here called *soprano, alto, tenor* and *bass,* are stacked perpendicular to the polyphony axis. Generally *symmetry* is the imitation of a motiv, theme or module in a pattern. In the three-dimensional music space symmetry may occur as follows:

Any one voice may imitate a given pattern, either at the same pitch, or at higher or lower pitch. In the first case we have translational symmetry along the time axis, in the second and third cases translational symmetry in the time-pitch plane.

Any one voice or part may imitate a theme with all intervals reflected in the time axis, in other words *upside down.* The symmetry in such a case would be *glide-reflection symmetry.* The motiv or theme might be imitated backwards in time; in this case the symmetry of the musical pattern would be that of *reflection in the pitch axis.* Finally, the theme may be imitated backwards as well as upside down, in which case the symmetry is two-fold rotational: the pattern would be invariant to a 180° rotation. These symmetries are shown schematically in Figure 4, with the notes designated by white rectangles.

J'ay pris amours

Jacob Obrecht

Figure 1

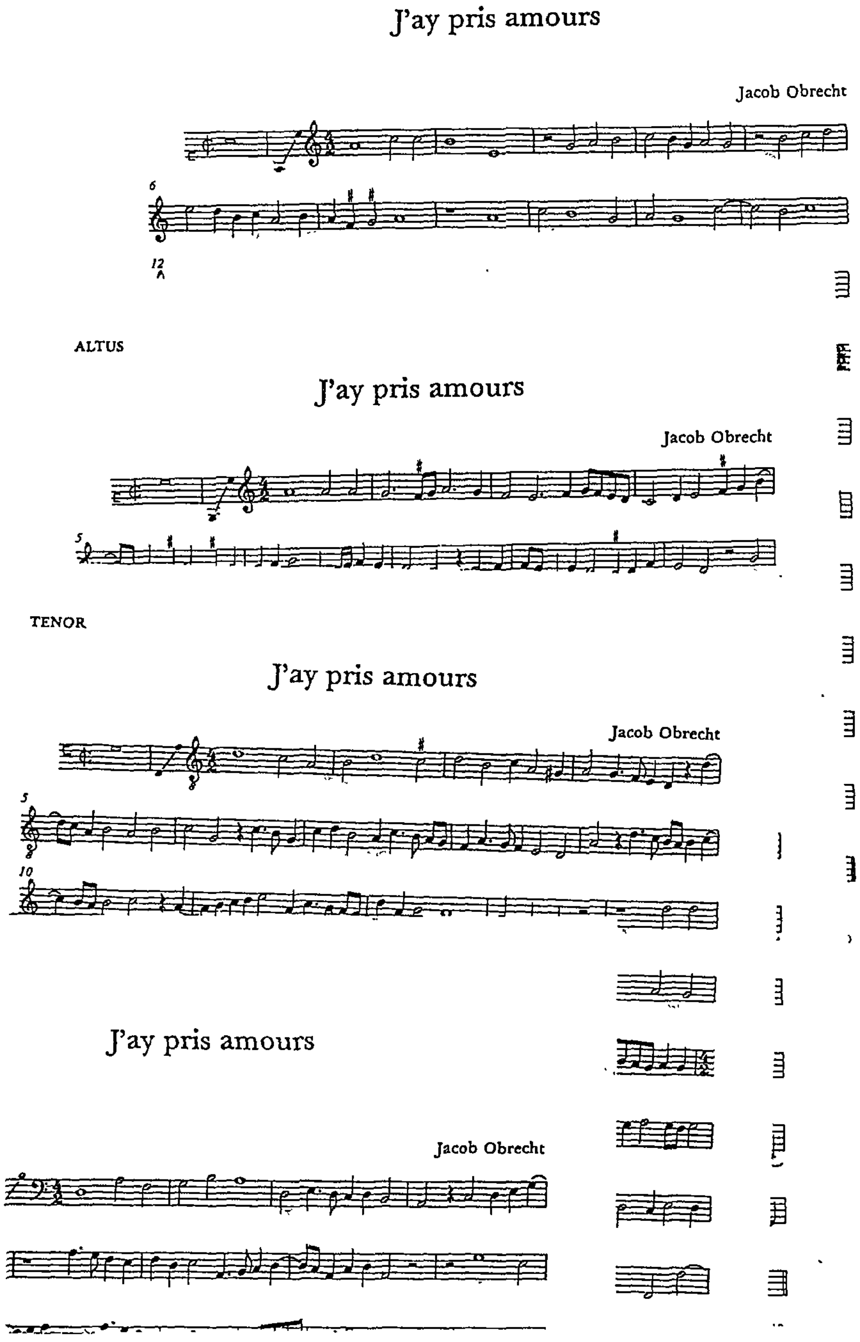

J'ay pris amours
Jacob Obrecht
ALTUS
J'ay pris amours
Jacob Obrecht
TENOR
J'ay pris amours
Jacob Obrecht
J'ay pris amours
Jacob Obrecht

Figure 2

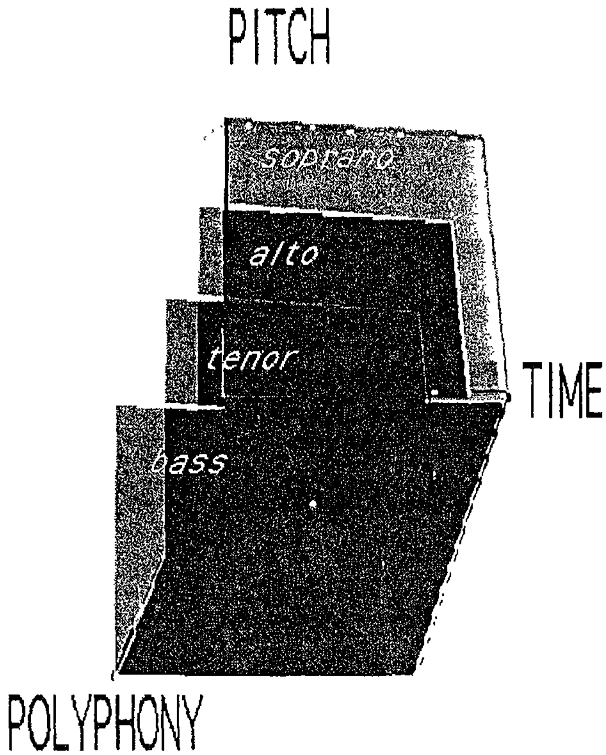


Figure 3

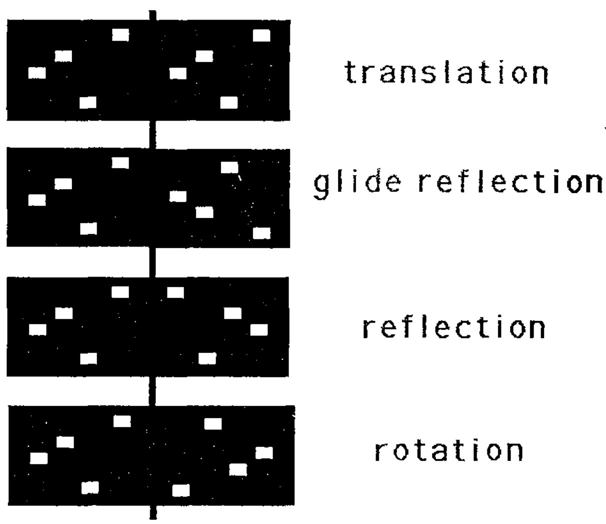


Figure 4

The theme may also be imitated among different parts or voices: the symmetry then has a component along the *polyphony*-axis. Figure 5 shows the Polyphony-Time plane. *Counterpoint* describes the sequence in which the different parts imitate the theme(s); the relation between pitches in different parts at any given time is called *harmony.* The use of symmetric imitation in different parts or voices gives rise to musical forms such as the *canon, ricercar* and *fugue.* Although Johann Sebastian Bach's *Kunst der Fuge* and the canons in his *Musikalisches Opfer* represent a culmination of the use of the symmetries illustrated in Figure 4, the historical origin of the use of these symmetries lies in the early renaissance in the northern part of the medieval Burgundian realm. This is not the place for an exhaustive enumeration of all possible symmetry patterns in the three-dimensional music space; rather we shall focus on the historical development of polyphonic music, and the stylistic parameters which characterize the Renaissance.

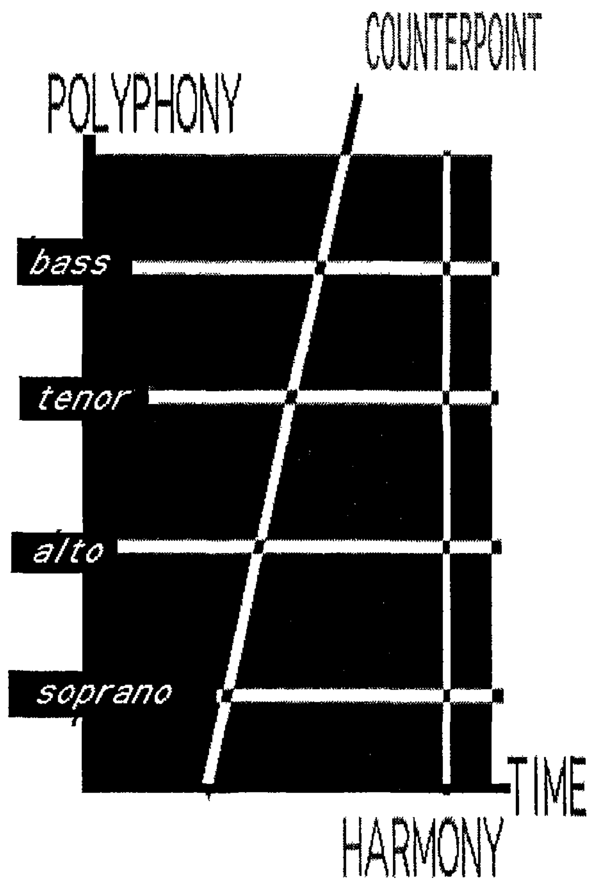


Figure 5

Polyphony, the simultaneous sounding of different voices, developed only gradually. Monophonic chant existed in the synagogue before Christ, and developed in the Catholic liturgy into Ambrosian and Gregorian Chant. To the *Cantus* was added a *Discantus*; the two parts could be very similar (Figure 6) but in most medieval music a traditional melody was sustained in long notes, with two or three freely ornamented parts added (Figure 7). The sustained part, called the *tenor*, was rarely the highest in pitch: there were usually one or two higher parts, called either *discant* or *duplum* and *triplum,* or a lower one, the *bassus*.

The *Caccia* in 14th-century Italy was highly symmetrical: two upper parts moved in strict canon over a sustained line. By the second half of the 15th century, the *Caccia* had become obsolete, however, and is therefore no longer relevant to the transition from Middle Ages to Renaissance.

The famous chanson *De Plus en Plus* by Gilles Binchois, born in Mons in 1400, illustrated in Figure 8, demonstrates the typical unsymmetrical medieval structure, in which each part reveals its own individual history, and the parts do not share thematic material.

Along the time axis, this chanson is highly structured, being a *rondeau*, the most complex of the medieval *formes fixes* or poetic forms. The *music* is in two sections, which we shall denote by lower case *a* and *b*. The *text* uses four different modules, denoted by upper case *A, B, C* and *D*; the repeat pattern of each is shown in Figure 9. Both text and music sequence are highly unsymmetrical. It would go too far here to go beyond a mere hint that medieval architecture, too favored unsymmetrical forms. By contrast, Figure 10 shows a chanson, *Files à marier,* unusual for Binchois. It is not in any of the *formes fixes*, and it is in *four* parts. The tenor is a popular song, *Se tu t'en marias,* but when the composer adds a fourth part, instead of using new thematic material, he writes a *canon*, somewhat reminiscent of the ancient *Caccia*: the two top parts are related by translational symmetry!

Antoine Busnois, born a generation later than Binchois in Busne, French Flanders in 1430, wrote two versions of the chanson *Ha que ville et abominable*, whose title is a pun on the name of its dedicatee, Jacqueline d'Hacqueville. Whereas the first version is a straightforward *Rondeau*, the second, shown in Figure 11, attempts to be a *Rondeau* as well as a Canon. However, in the *b* music the top part goes its own way, and if the chanson is to use the *Rondeau* sequence, the canon is effectually ruined. Frequently, music was to be sung *or* played, and the canonic effect may have been intended for an instrumental version played straight through. This example illustrates the difficulty in combining the highly unsymmetrical *formes fixes* with the symmetrical canon form.

A solis ortus cardine (three-voice conductus cum cauda)

Florence, Bibl. Laur., Pluteus 29.1 (F), fols. 242'–43'.

Figure 6

Perotin, *Alleluia: Posui adiutorium*

Figure 7

De plus en plus


[DISCANTUS]
TENOR
CONTRATENOR
A
1 4.7 De plus en plus se re - nou - vel - le, Ma doul - ce da - me gen - te et bel - le, Ma vo - lon - té de vous ve - ir.
3. nw cui - diés vous que re - cel - le, Comme a tous jours vous es - tes cel - le Que je vueil de tout o - bé - ir.
5. Hé - las, se vous m'es - tés cruc - el - le, J'au - roie au cueur an - gois - se tel - le, Que je vou - droie bien mo - rir,
B
2 8. Ce me fait le très grant de - sir Que j'ai de vous ou - ir nou - vel - le.
6. Mais ce se - roit sans des - ser - vir, En sous - te - nant vo - stre que - rel - le.

Figure 8

TEXT : A B C A D B A B
MUSIC: *a b a a a b a b*

FORME FIXE: RONDEAU

Figure 9

Gilles Binchois, 8 chansons

Filles a marier

Figure 10

Antoine Busnois, 5 Songs a 3

2(b). Ha que ville et abhominable

Figure 11

Although music in many more than three voices was not uncommon, the texture of a tenor on a well-known theme with two more florid, thematically independent, parts does characterize the High Middle Ages. Binchois's *Filles à Marier* points the way to a four-part texture in which some or all voices are thematically related. As the number of parts increases, keeping the voices thematically distinct becomes more problematical. The transition from Middle Ages to Renaissance is in any case aesthetically characterized by a change from a heterogeneous to a more homogeneous, harmonious texture.

Margaret of Austria's favorite composer, Pierre de la Rue, born only thirty years after Busnois, wrote in a typically early renaissance style (Figure 12). The chanson *Mijn hert altijd heeft verlangen* starts canonically with the *altus* and *bassus*. This beginning is deceptive, however, for when the *tenor* enters, it alone continues the original chanson in the medieval tradition, while the other three parts, once the tenor has taken hold of the original melody, perform elaborations and variations based on the traditional melody. These *tenor songs* with their deceptive canonic pre-imitation and retention of the cantus, became a very common form characteristic of the early renaissance.

The great Josquin des Prés, who was born in the middle of the 15th century and died in 1521, went further, as seen in his chanson *Baisés moy* , a double canon. (Figure 13) Here both the tenor and the bass carry the melody in strict canon, whereas the top two parts, also canonically, ornament the melody. Around measure 10 it appears for a moment that all four parts will enter in canon, but the top entrances are again deceptive, for soon they are off on their own. Since there are two separate but simultaneous canons, this example is called a *double canon.*

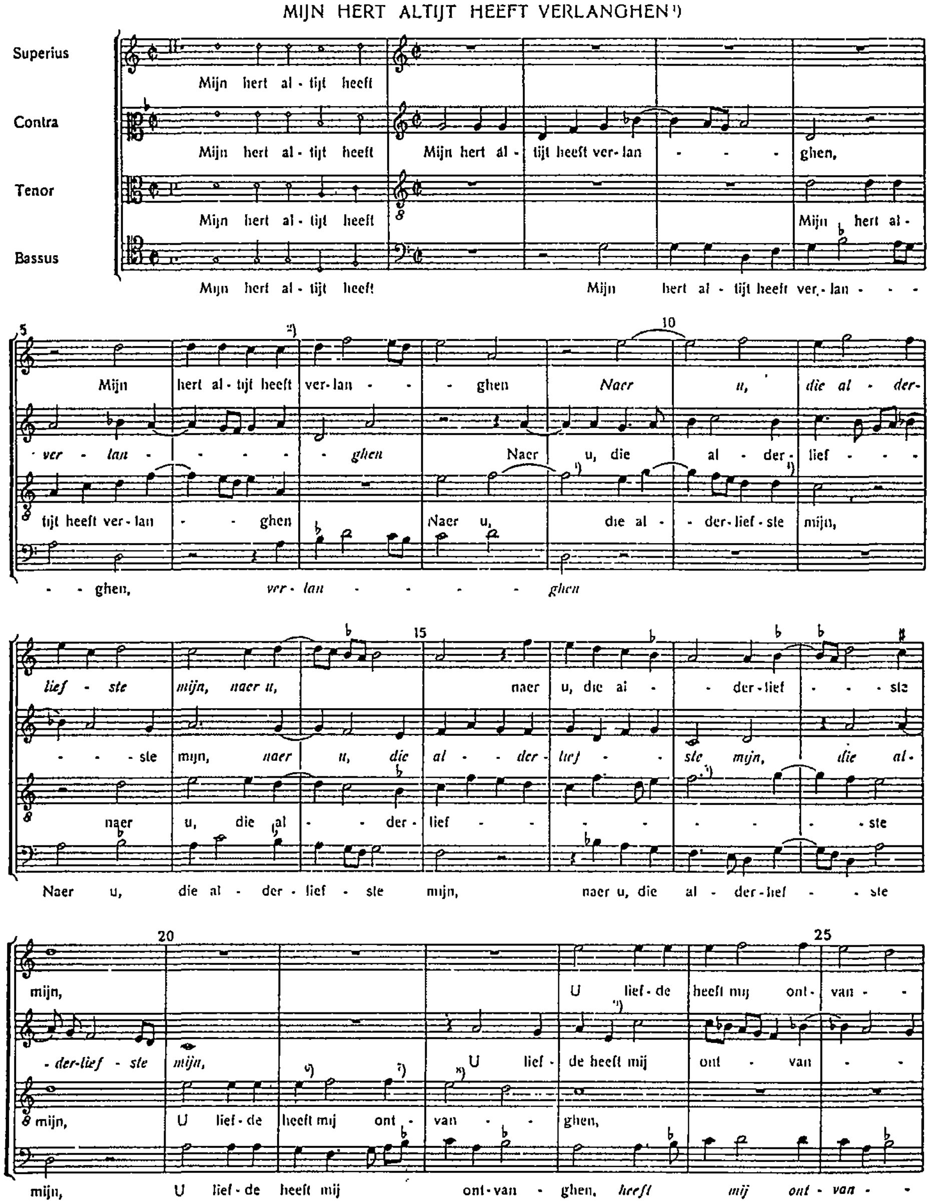
Note that in each of the two canons the two partners enter and remain a fourth apart. *Alto*, or more properly counter tenor, and bass voices are an octave apart, as are *soprano* and *tenor*. As the thematic material in the different parts became less diverse, the desire was to have similar instrumentation or voices in each of the parts. Families of wind and string instruments developed, each family having members of three or four different sizes, hence ranges. Sopranos and altos, and equally tenors and basses tend to be about a fourth or a fifth apart, and therefore will pair off in canons at the fourth and fifth rather than the unison or octave. Note that in *Baisés moy* the soprano and tenor have a flat in their key signature, the alto and bass do not.

Adriaan Willaert lived from 1485 to 1562, and studied with Josquin's student Jean Mouton. In his *Inter natos* (Figure 14) also a double canon, the two inner parts combine into one canon, the two outer voices into a second one. Whereas the two outer voices are an octave apart, Willaert attempts a canon for the two inner parts a minor third apart.

38 Pierre de la Rue

(± 1460-1518)

MIJN HERT ALTIJT HEEFT VERLANGHEN [1]



[1])In partituur gebracht volgens Brussel, Koninklijke Bibliotheek, Ms. 228, fol. 16v-17r Anoniem.
[2])In plaats van semibrevis c. [3])Niet gebonden [4])In plaats van semibrevis d [5]) Semibrevis f, gevolgd door minima e.
[6])20_4-21_1: gebonden. [7]) Minima f, gevolgd door semibrevis f. [8]) In plaats van 2 minimae e. [9]) In plaats van minima f.

Figure 12

2. BAISES MOY

Figure 13

1. Inter natos

Figure 14

However, he runs into a problem here, which we can understand by considering Figure 15, *the circle of fifths* or *quintencirkel,* which shows the key relationships between the different scales. The scale of what we presently call *C major* has no sharps or flats; neither has the scale of *a minor*. Note that *a* is ninety degrees to the right of *C*. Every time we move one place (30°) clockwise, we add a sharp or remove a flat from the key signature. Analogously, a counterclockwise move removes a sharp or adds a flat to the key signature.

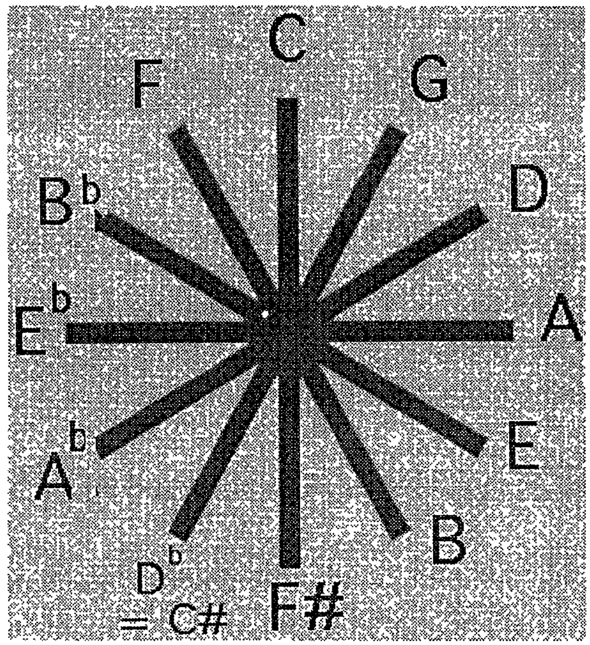


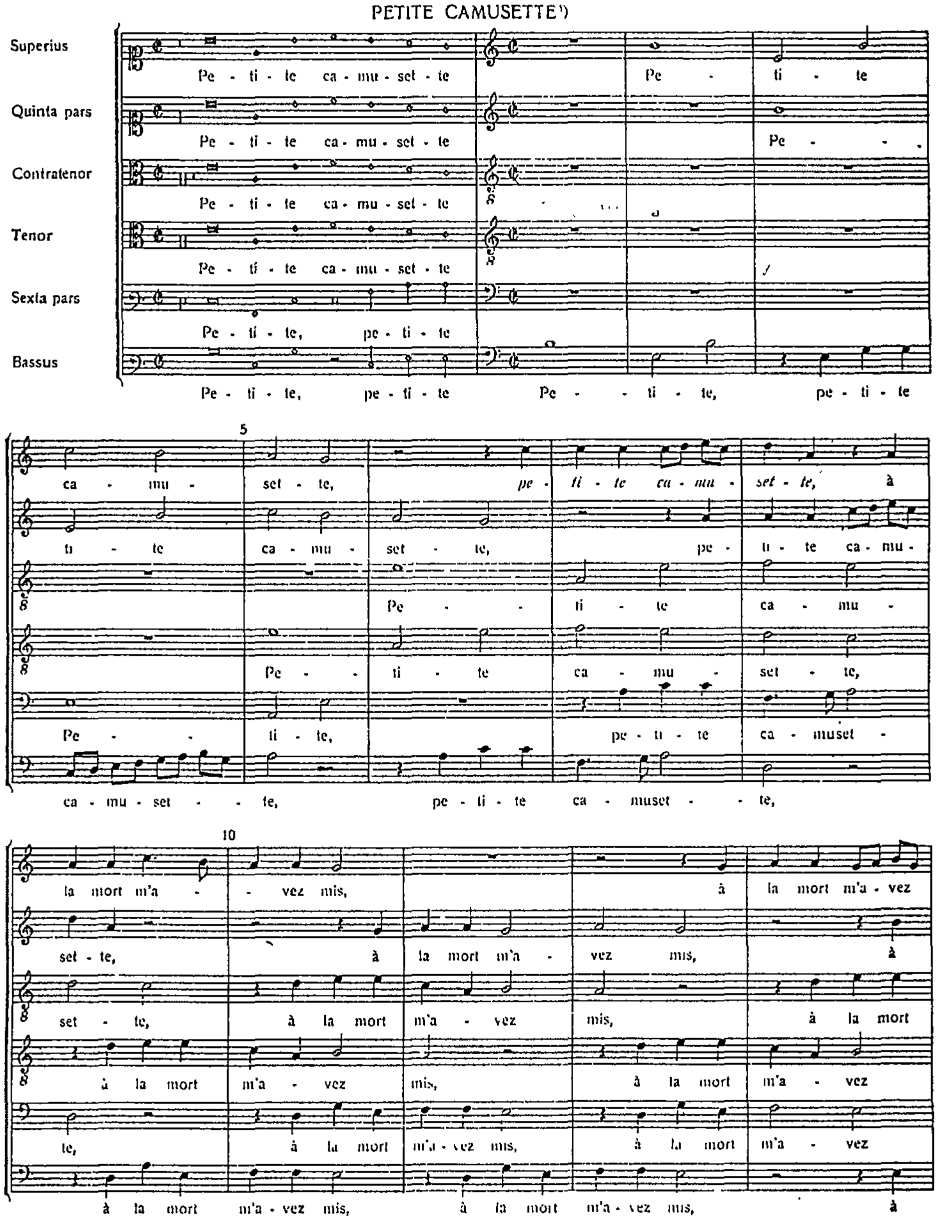
Figure 15: Quint Circle (Circle of Fifths)

Note that adjacent positions in the circle of fifths are a fourth or a fifth apart. In the Middle Ages the intervals of a fourth and fifth as well as the octave were considered consonant, but the third was still a dissonant. In a canon at the fourth or fifth the key signatures of the parts differ by only one sharp or flat. Returning to Willaert's *Inter Natos*, (Figure 14) we note that the middle two parts start respectively on *g* and *b flat*, a minor third apart. Note, however, that the tenor starts with a minor third upward (*g* to *b flat*), whereas the alto starts with a major third (*b flat* to *d*). In point of fact, the inner parts are actually related by something akin to color translation symmetry: the translation transposes the parts from what is presently called the minor mode to the major one. In the circle of fifths, *g minor* and *B flat major* are just 90° apart, and have the same key signature!

47 Josquin des Prez

(± 1450-1521)

PETITE CAMUSETTE[1]



[1]) Overgenomen uit de Werken van Josquin des Prés, vijfde aflevering, blz. 43 - 44.

Figure 16

2. Mir ist ein rot Goldfingerlein

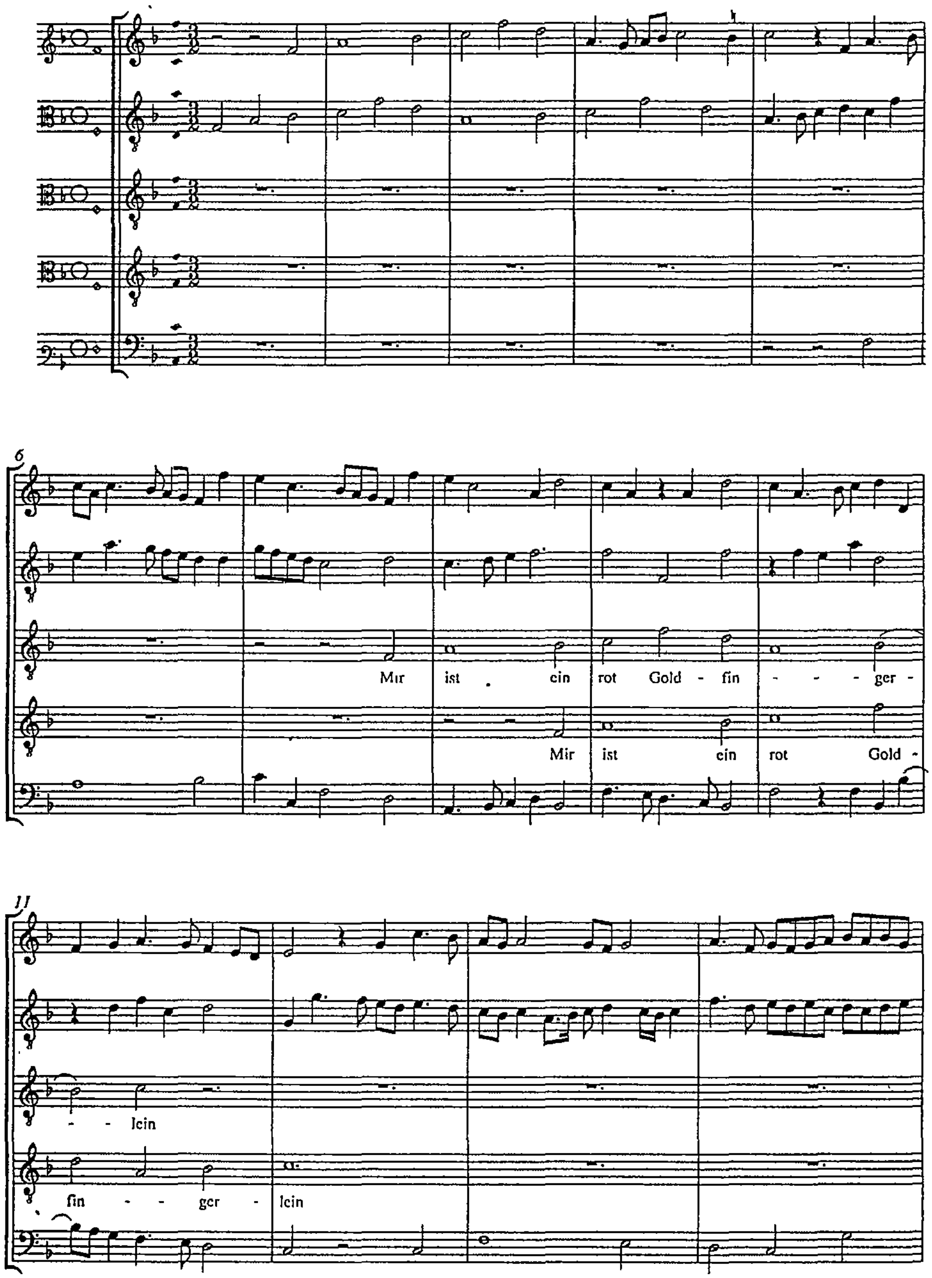


Figure 17

1. SCARAMELLA

Figure 18

Adriaen Willaert experimented with double canons, which became very fashionable in the early 16th century. With the introduction of the interval of a third and the major-minor relation in his canons, he was far removed from the medieval texture of distinct parts and voices. In this he was not alone, but it does appear that the Burgundian Low

Countries were the birthplace of this new style of composition. A particularly notable example is the beautiful and ingenious chanson *Petite Camusette,* by Josquin (Figure 16), in which the two middle parts are strictly canonic, the top and bottom nearly, but not completely so, but elaborate of the original melody in the middle parts. Note that, as in the Middle Ages, the tenors still carry the unadorned traditional tune.

Most of the composers from the Burgundian *Pays LàBas* traveled widely, exporting their canonic counterpoint. Henric Isaac followed Marguerite's father, the emperor Maximilian, to Salzburg, where the Swiss composer Ludwig Senfl took up his counterpoint (Figure 17), and Adriaen Willaert in Venice became the spiritual ancestor of Andrea and Giovanni Gabrieli, composers of multichoral music for San Marco. Conversely, these Netherlandic composers returned home bringing a yet more symmetrical form of composition, the Italian Renaissance madrigal. Figure 18 is an example of such an Italianate composition, Josquin's *Scaramella,* in which three of the four parts start and finish their phrases simultaneously rather than entering in counterpoint, only the tenor maintaining its traditional autonomy for a bit. The parts are linked harmonically, that is to say that at any given time the four parts sound the notes of a major or minor triad, by then the most harmonious consonance.

In too few minutes we have traversed a period of radical change in musical texture, characterized by an increase in symmetry. Although the Burgundian counterpoint became superseded by Italian homophony, polyphonic counterpoint survived in the baroque, culminating in the works of Johann Sebastian Bach, surfacing again in the organ music of César Franck, and experiencing a renaissance in the 20th century.

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THE CONCEPT OF THE MASS FORM WITHIN THE CONTEXT OF VISUAL ART AND ARCHITECTURE OF THE ITALIAN RENAISSANCE

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In a letter to Baldassare Castiglione, Raphael once wrote: "In order to paint a beauty, I have to see many beauties, on the condition that Your Worship is beside me to help me make the best choice. However, due to the shortage in both good judges and beautiful women, I am applying a certain idea that has crossed my mind. Whether it contains within itself any artistic perfection, I do not know, but I am striving very hard to achieve it." (Golzio 1936, p. 31)

This certain idea conceived by Raphael is a type of ideal beauty that is the cherished goal of any artist, and that he attempts to realise above all on the level of composition.

Unity of artistic, aesthetic, philosophical and religious ideals of the times determines the corresponding homogeneity and similarity of structural planes and geometric patterns of different works of art, among various artists, and even in various art forms.

In this context, the era of the Italian Renaissance provides ample material for comparative analysis and search for various types of interconnections, material that has been widely consulted by numerous art experts. It would appear that in this area no paths have been left untrodden. That is definitely true in respect to visual arts - painting and architecture - whose compositional planes are easily accessible to the eye, and consequently to understanding and comprehension. These art forms are initially connected through geometry. As far as music is concerned, however, we will always sense a certain metaphorical, subjective element in making such comparisons, until the musical form as a process has been translated into a geometrical plane, into a crystal-like form.

Thus the large-scale musical form of the Italian Renaissance – its emergence, its distinctive features, its relation to the ideals of the time, finally its similarity (or dissimilarity) to the contemporary forms of art and architecture - these are the main issues I have raised in this study.

At the outset, it is important to note that during the time of Leonardo da Vinci, a scientific subject like the theory of musical composition or of musical form was non-existent. Whereas architects and artists had the benefit of dozens of voluminous treatises as well as simple manuals devoted to the issues of perspective, art composition, building design, nothing of the kind was produced in the field of music theory. In this respect, music found itself in the position of a "poor relative" as it were. At the same time, as paradoxical as it may sound, music was the "ruling mistress", setting the laws, occupying a privileged position; in other words, it was one of the "liberal arts", while painting and architecture were considered professions. In the minds of the Renaissance man, music was identified with mathematics and imbued with a divine spirit. After all, according to the Pythagorean doctrine and Plato's *Timaeus*, the proportions of musical consonances are the cornerstones of the universe. The theory of musical proportion was so popular among artists and art experts that it largely set the tone for the main developments in these fields (Wittkower 1949).

At the same time, the theory of musical proportions primarily referred to intervals and rhythm. The musical form, the laws of inner structure, compositional technique – these "holy of holies" of musical art were hidden from the uninitiated, setting music apart and veiling it in a cloak of mystery.

The first attempts to shed light on the problem of the form-making principles of Renaissance cyclical works only took place towards the middle of the present century. These works are few in number. Among the most important, I will cite the study by Marcus van Crevel of the secret structure of two *Marienmasses* by J. Obrecht, as well as Marianna Henze's book on masses by J. Ockeghem (van Crevel 1959, 1964; Henze 1968).

The secret structure of masses by 15th century composers was analysed by these scholars only from the point of view of proportions applied to the duration of *cantus firmus* – the main voice of the choral score.

However, the study of architectonics is essentially the study of all, or at least of the key composition structures, followed by combination and comparison of the yielded results - blueprints for a sort of recreation of the overall design. Only in this case do we obtain a complete picture of the musical form and an opportunity to conduct an objective comparison between music and visual arts.

I developed and applied this method in the analysis of masses by G. P. da Palestrina, a prominent 16^{th}-century composer and leader of the Roman school.

Palestrina's 104 masses are but the quantitative contribution made by the composer to the development of the genre. 91 of them, as well as numerous works by the master's contemporaries and predecessors, make up the material on the basis of which I build the overall concept of the form of 16^{th}-century mass in the period of its "classical" maturity.

The music form is realised in nine structures, each of them having a semantic aspect of its own. These are:

1. The cycle's overall structure (related to the text)
2. Mathematical structure - proportions
3. Ensemble-choral structure
4. Polyphonic[1]
5. Textural
6. Mode-cadence
7. Motif-thematic
8. Functional
9. Form as symbol – as a result of the manifested idea.

In addition, the cycle contains:

1. The composition's main level (two versions)
2. The macro-level (two versions)
3. The sub-level (three versions)

---

[1] Polyphony, mode, texture and thematism represent primarily elements of the musical language, and may be studied as separate disciplines and key aspects of the composer's style At the same time, in a musical composition they often acquire formative constructive functions as well. Their role within the composition may be likened to the vital systems of a living organism in which, while being autonomous, they act only in conjunction with each other.

Let us examine some of the structures.

The form of the mass (not the entire liturgy but only its permanent music part – the so-called *Ordinarium missae* represented a composition made up of five parts: *Kyrie, Gloria, Credo, Sanctus,* and *Agnus Dei* (Diagram 1). The second, six-part version was formed when the *Benedictus* section of *Sanctus* was singled out and treated as an independent part. The macro-level is formed by the fusing of extreme paired parts into macro-parts, which is due to the arrangement of the *Ordinarium* parts in the liturgy.

Correspondingly, *Kyrie* and *Gloria* on one side, *Sanctus* and *Agnus Dei* on the other form macro-sections. The result is a three-part (instead of five-part) structure.

Also common since the 15th century was the two-part macro-structure, formed as a result of the cycle being treated as a six-part structure. In this case the cycle was divided into two macro-parts: the *Gebetsmesse*, made up of *Kyrie, Gloria* and *Credo*, and the *Opfermesse*, consisting of *Sanctus*, *Benedictus*, and *Agnus Dei*[2]. Thus the two macro-structures as it were merged together.

The semantics of the overall structure is as follows:

Parts 1 and 5 - *Kyrie* - *Agnus Dei* - prayer

Parts 2 and 4 - *Gloria* - *Sanctus* - glorification

Part 3 - *Credo* - benediction, symbol of faith.

Graphically, what distinguished the semantics of the overall structure is that the cycle is formed hierarchically, with each part placed according to the level of its importance; thus the most important is the *Credo* - the central part, followed by the glorification sections, and finally the prayer sections. The central part of the *Credo* consists of three sub-sections - *Patrem, Crucifixus* and *Et in Spiritum* - symbolising the Holy Trinity.

The second structure is mathematical, related to proportions, to scale correlations, i.e. to the duration of parts in relation to each other and the entire structure. This is the first architectonic plane of the form.

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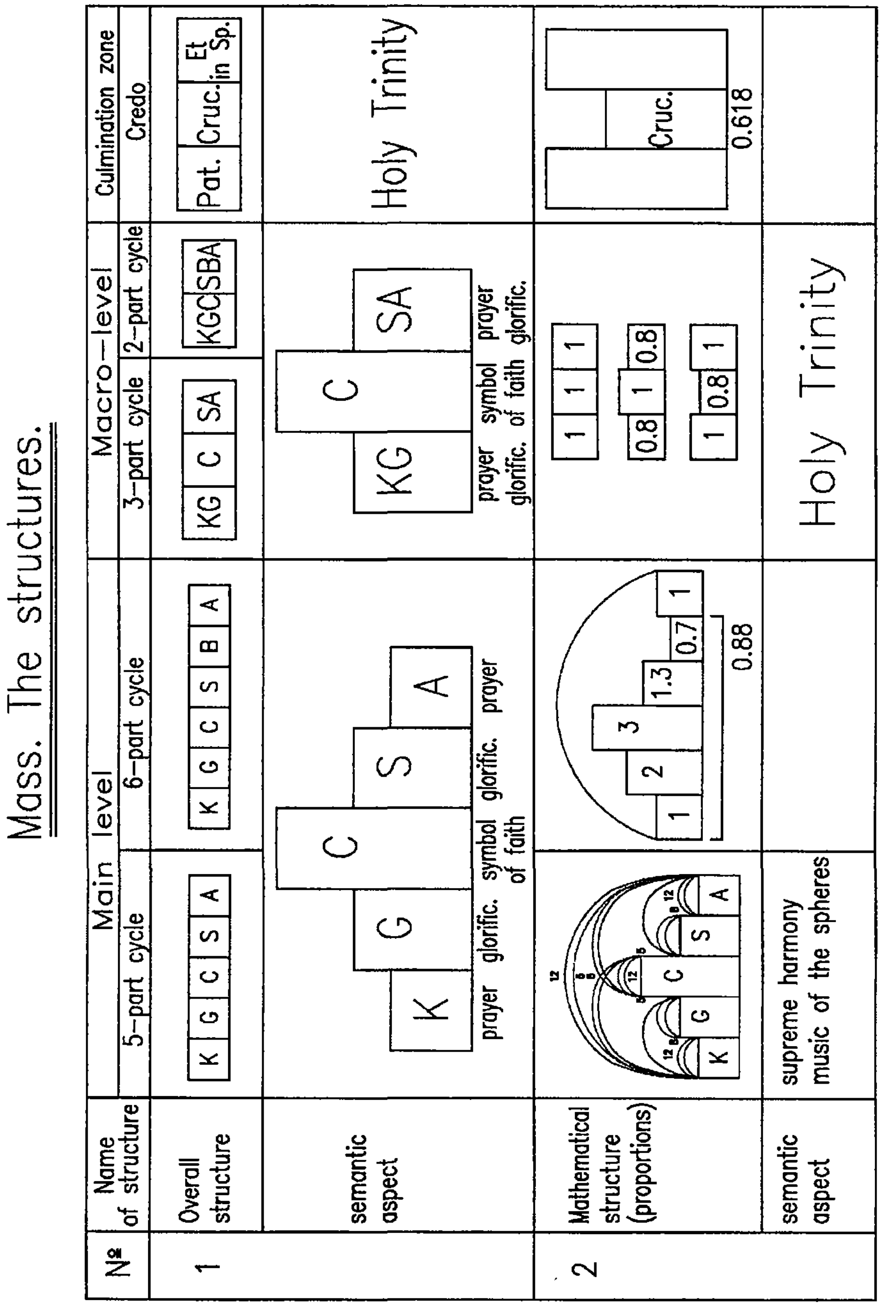
[2] *Gebetsmesse* - the prayer part of the mass; *Opfermesse* - the sacrifice.



Mass. The structures.
№
Name of structure
Main level
5–part cycle
6–part cycle
Macro–level
3–part cycle
2–part cycle
Culmination zone
Credo
1
Overall structure
K G C S A
K G C S B A
KG C SA
KGCSBA
Pat. Cruc. Et in Sp.
semantic aspect
K G C S A
prayer glorific. symbol of faith glorific. prayer
KG C SA
prayer glorific. symbol of faith prayer glorific.
Holy Trinity
2
Mathematical structure (proportions)
K G C S A
1 2 3 1.3 0.7 1
0.88
1 1 1
0.8 1 0.8
1 0.8 1
Cruc.
0.618
semantic aspect
supreme harmony music of the spheres
Holy Trinity

Diagram 1

Thus what we have before us is an absolutely symmetrical composition constructed in accordance with the supreme laws of musical consonances – octaves (the ratio of 1:2) and fifths (the ratio of 2:3), combined with a sequence of golden sections (Diagram 2).

The semantics of such a structure is supreme harmony, music of the spheres. Furthermore, the number 5 (the total number of parts) is open to a variety of Christian and neo-Platonic interpretation, symbolising both Christ and the Virgin, as well as being the perfect Pythagorean number (Bossuyt 1994; Elders 1967, 1969; Timmers 1974).

In the six-part cycle the *Sanctus*, breaking up into two parts no longer corresponds to *Gloria*; consequently the structure loses its perfect symmetry, retaining only a partial one in the *Kyrie* - *Agnus* correlation (Diagram 1).

The (three-part) macro-level yields three schematic versions. In the first case, we can see that all the macro-parts are equal, i.e., the ration is 1:1:1. This equally balanced interval – the most perfect of consonances – symbolises the Trinity in the given context. The remaining two schemes indicate that the extreme parts are larger or smaller than the middle part by the ratio of 0.88 to 1. The number 888 is known to be the numerological equivalent of the name of Christ.

The *Credo* structure demonstrates a mirror-symmetry and a golden section correlation between the *Crucifixus* and the extreme sections.

The mass sub-level is related to the separation of mass parts into sub-sections, whose total number is usually 13-15. However, regardless of how many sections the cycle contains, it exhibits fascinating regularities. In the mathematical sense, practically all three versions represent the Fibonacci series (Diagram 3)[3]. Thus 13 sections yield the main sequence: 1, 1, 2, 3, 5, 8, 13.

14 sections are the sequence of the Evangelist - 7, 7, 14.

15 sections go as far as to represent two sequences moving towards each other. From left to right is the main Fibonacci series (the first 7 numbers), while from right to the left is the Evangelist series - 2, 5, 7, 12 (Math. 14:15, M. 6:38; L. 9:12; I. 6).

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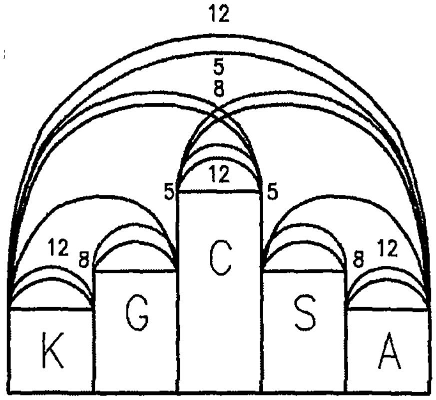
[3] About Fibonacci series and their manifestations in Renaissance music, see N. Powell's article "Fibonacci and Golden Mean...", 1979.

Palestrina. Mass "Sacerdotes Domini"

(perfect consonances)



$$\underbrace{\frac{K}{G}=\frac{A}{S}=\frac{1}{2}}_{8}\;;\quad \underbrace{\frac{G}{C}=\frac{S}{C}=\frac{2}{3}}_{5}\;;$$

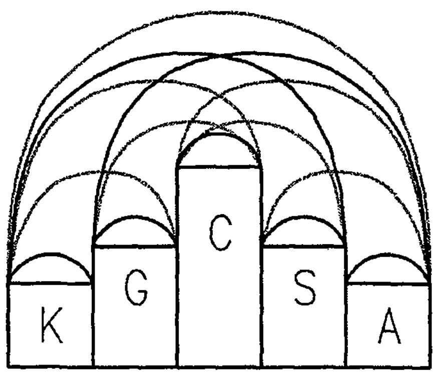
$$\underbrace{\frac{K+G}{C+S+A}=\frac{S+A}{K+G+C}=\frac{1}{2}}_{8}\;;$$

$$\underbrace{\frac{K+G+C}{M}=\frac{C+S+A}{M}=\frac{2}{3}}_{5}\;;$$

$$\underbrace{\frac{K}{C}=\frac{A}{C}=\frac{C}{M}=\frac{1}{3}}_{12}$$

Mass "Pater noster"

(golden section)



$$\frac{K}{G}=\frac{G}{C}=\frac{S}{C}=\frac{A}{S}$$

$$\frac{G}{K+G}=\frac{C}{G+C}=\frac{C}{C+S}=\frac{S}{S+A}$$

$$\frac{K+G}{G+C}=\frac{K+G}{C+S}=\frac{S+A}{G+C}=\frac{S+A}{C+S}$$

$$\frac{G+C}{K+G+C+S}=\frac{G+C}{G+C+S+A}=\frac{C+S}{K+G+C+S}=\frac{C+S}{G+C+S+A}$$

$$\frac{K+G+C}{M}=\frac{C+S+A}{M}\qquad \boxed{=0.618}$$

Diagram 2

Mass. The structures.

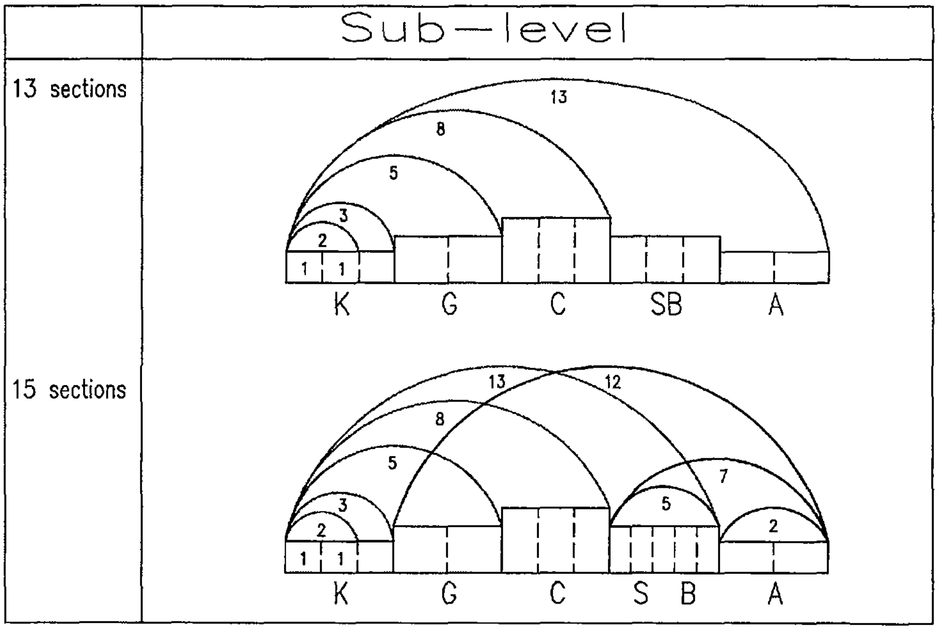


Diagram 3

The semantics of these added series and numbers is obviously an illustration of the Biblical episode of Jesus feeding the 5,000 and 4,000 - the miracle of bread and fishes containing profound esoteric meaning. It is also a symbol of the Creator and the Creation.

The *Kabbala* interprets the number 13 as the One, as well as Love - the creative energy that sustains the universe. The Christian reading is that of Jesus and the 12 disciples (the Last Supper composition).

The number 14, made up of 7 + 7, symbolises mourning, agony – the seven words spoken by Jesus on the cross, the 14 stations of the Via Dolorosa (Bossuyt, Elders).

Finally 15 sections, broken up into 7 + 8, are, according to Plato's *Timaeus*, the 1st and 2nd stages of the emanation of the universal soul, as well as Death and Resurrection. As we can see, the sub-level's mathematical structure provides a rather fruitful ground for interpretations.

The ensemble-choral structure shows a reduction in the number of choral voices in *Crucifixus* and *Benedictus*, and a possible increase in the *Agnes Dei* (Diagram 4). This scheme is asymmetrical, acting as it were for the benefit of the asymmetrical six-part structure. On the whole, the reduction in voices in *Crucifixus* and *Benedictus* creates a lightening effect in these sections, something that is related to a certain theological dogma, *Deus est lumen purus*[4] and the light doctrine of a leading neo-Platonist and Christian mystic Pseudo-Dionissius.

The scheme of the textural structure enables us to trace the most general plane of the sequence of imitative-polyphonic and chordal-harmonic sections, and to observe the increasing importance of the "vertical" towards the *Credo*, as well as the reversion to the linear at the end of the cycle. The linear and the vertical have a rather concrete meaning, symbolising the heavenly (–) and the militant earthly (|) church. Combined, they form a cross.

Thus the textural plane forms 5 signs – 3 crosses and two H's (*Crucifixus* and *Benedictus*). This structure has a dual nature, combining both symmetry and asymmetry.

The symmetrically enclosed cadence-harmonic structure (Diagram 5) demonstrates intensifying cadence-harmonic development towards the center and the appearance of a new cadence in the *Crucifixus*, which emphasises this section by means of a new harmonic illumination. Thus the *Credo*, possessing the most vivid contrasts of texture, harmony, choral ensemble, and, as we will see later, theme, constitutes the emotional-semantic and structural axis of the composition.

The motif-thematic structure of the mass (Diagram 5) is the most "material" structure.

The foundation of a thematic structure is the thematic complex (TC) consisting of 4 to 5 motifs taken from the original source. The entire or almost the entire TC is presented in the *Kyrie*, and from there distributed throughout the remaining parts, transforming and changing the order of sequence. Those parts of the form that contain themes represent a sort of tectonic milestones of the composition, forming melodic arches linking the sections.

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[4] "God is light, and in Him is no darkness at all." (John, 1:5)

Mass. The structures.

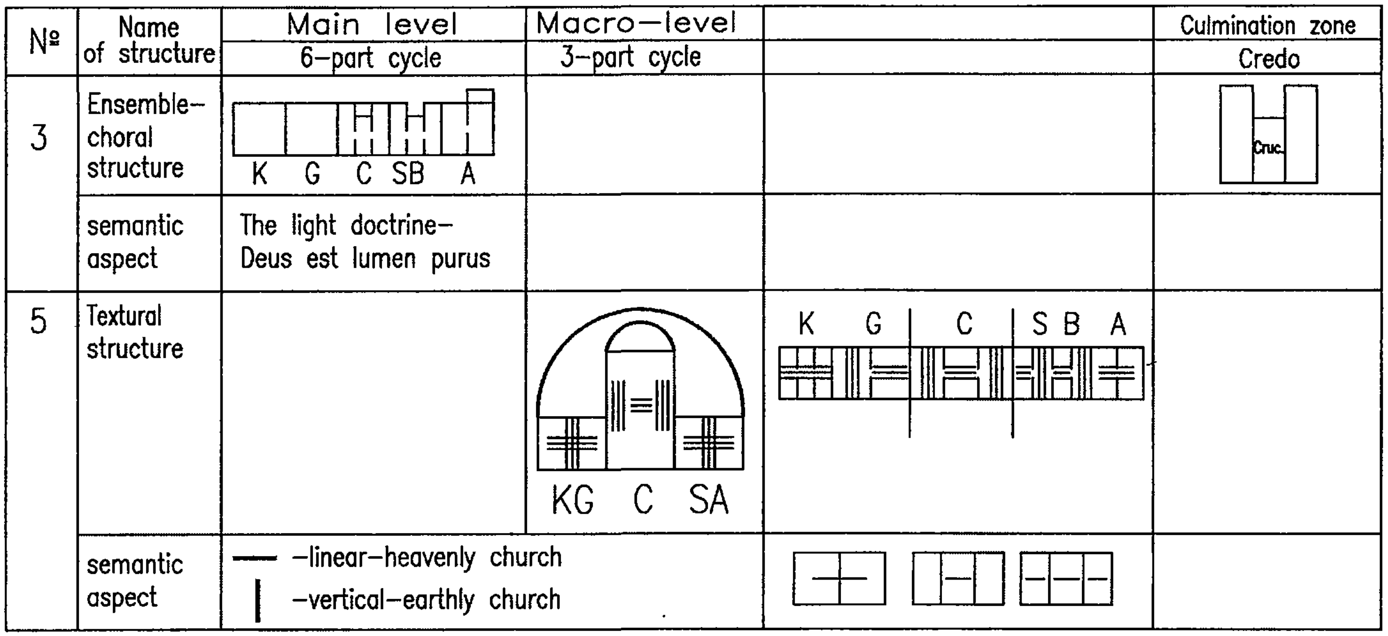


Diagram 4

Mass. The structures.

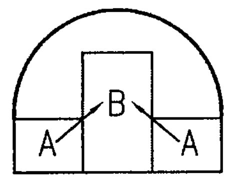
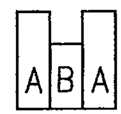
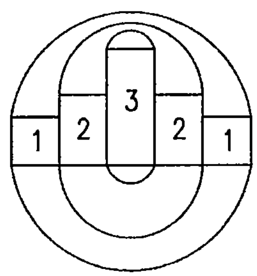
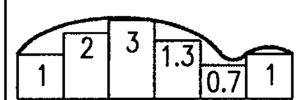
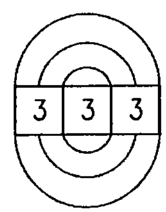
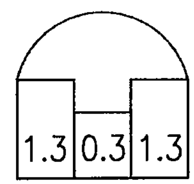
№	Name of structure	Main level: 5–part cycle	Main level: 6–part cycle	Macro–level: 3–part cycle	Culmination zone: Credo
6	Mode–cadence structure				 Cruc.
	semantic aspect			a new harmonic illumination in the Credo (Crucifixus)	
7	Motif–thematic structure				 Pat. Cruc. Et in Sp.
	semantic aspect	Virgin Mary – the Queen of Heaven Heavenly Kingdom without end		The nine angelic hierarchies	

Diagram 5

The numbers signify the number of times the TC is presented in each part; i.e., in *Kyrie* and *Agnus Dei* it appears once, twice in *Gloria* and *Sanctus*, and three times in *Credo*. The result is a perfectly symmetrical form that fuses not only parts that correspond in relation to the center, but all of the parts as a whole. In its schematic expression, such a form represents a concentric *mandala* made up of 9 circles, and at the same time a sphere (any two points can be joined). The semantics of this structure: the nine heavenly circles (nine angelic hierarchies), Virgin Mary the Queen of Heaven, and the Heavenly Kingdom without end (Timmers).

The thematic structure of a six-part cycle forms a separate pattern which corresponds to its asymmetrical proportional structure.

Before embarking on the final, summing-up stage of the form-symbol, we must compare and contrast all the resulting structures, to – as it were – juxtapose them against each other.

Having compared thus all the structures of the 3–5-part composition, we can see that all of them, with the exception of the ensemble structure, represent a single, strictly symmetrical form that can be expressed by the following schematic symbols (Diagram 6):

1. a 3–5-nave cathedral

2. a dome

3. a concentric *mandala* made up of 3 or 9 circles

4. a sphere

5. the Latin cross (*crux immissa*)

6. and the cross of St. Anthony (*crux comissa*)

7. the Greek cross inserted in three concentric circles

8. the letter H

9. a cup

Mass. Symbols of form.

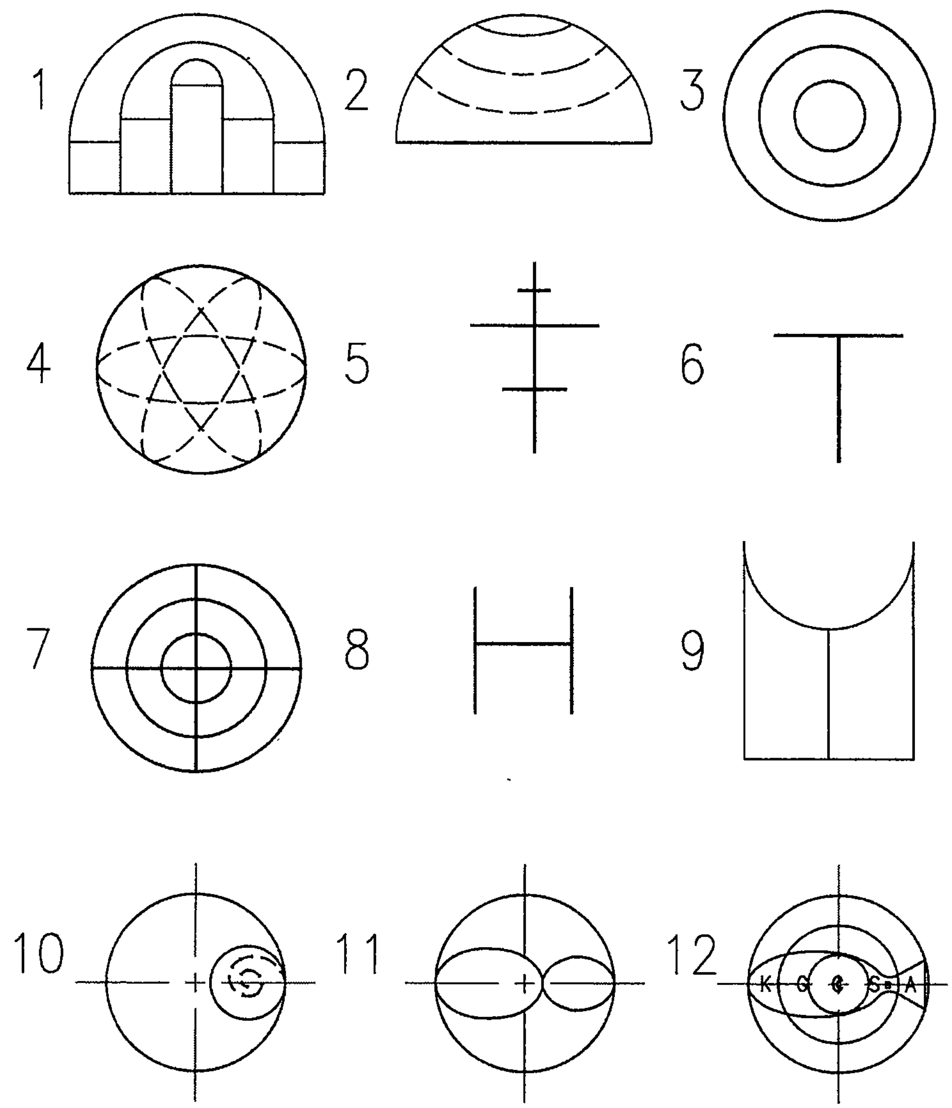


Diagram 6

The first four symbols were vividly demonstrated in the preceding diagrams. As for the others, let us examine the way they are produced.

The Latin cross is produced if we imagine the *Credo* as the vertical axis of the composition, which conforms with its overall structural position and semantic function as the center of the cycle and of the entire liturgy. This is tangibly confirmed by quantitative and qualitative characteristics of the *Credo* in all the structures. The remaining parts represent a horizontal temporal axis of the composition, reflecting its evolution. Moreover, if we mark along the vertical *Credo* axis the proportional, textural and thematic levels of the development of each of the corresponding pairs:

Kyrie - Agnus, Gloria - Sanctus

we will receive two transverse crossbars. The third one is formed by the symmetrical structure of the *Credo* itself, with the *Crucifixus* (the crucifixion) at its center. A similar process leads to the formation of the cross of St. Anthony in the macro-structure, in the case where all the three parts are equal. The depiction of a cross within a circle is quite justified as well, since a combination of these two symbols provides a yet fuller reflection of the overall construction.

The letter H is among the most significant Christian symbols, especially during the Middle Ages. This letter stands in the center of Christ's initials, JHS - *Jesus Hominem Salvator*, translated as Jesus the Savior of Mankind. It also stands for Helios - the Sun - the abode of the star spirit and the astrological sign of the Pisces. Fish is a known symbol for purity and depth of spirit, and was widely used by the early Christians. The Greek word for fish - *ichtis* - is an anagram deciphered as "Jesus Christ the Son of God the Savior" (Timmers, p. 53).

The H-shape is assumed in almost all structures by the *Credo*, and in several cases by the cycle's three-part macro-structure.

The second interpretation of the form of the *Credo* is a "cup" – the cup of sorrow referring to Christ's agony and prayer in the New Testament.

Having once given my thought to the perfection and obvious wholeness of the five-part composition, I asked myself the following natural questions:

1. What accounts for the additional treatment of the cycle as six-part?

2. Why was the emphasis placed on the Benedictus as opposed to some other section?

Only after comparing all the resulting structures did I realise that the strange shape that had emerged is some of them was the shape of a fish, that same fish whose image recurs in all canonical texts and repeatedly appears on all the levels in the mass. What brought this concept about was the almost total identity between the second half of the word *Benedictus* – *ictus* – and the Greek *ichtis* mentioned above. After that everything fell into place, with all the elements balanced in an amazing harmony: the contents of this section, where a joyous chorus of disciples greets Jesus as he rides into Jerusalem (*Benedictus qui venit in nomine Domine, Osanna*); the reduction in the number of voices in the choral ensemble - an illumination (the same as in the *Crucifixus*); the second H appearing in the textural scheme (I assume that the first H appeared in the *Crucifixus*); the golden section proportions; the number 0.88 that emerges here; even the linearity of imitative polyphony common for this section (once again, similar to the *Crucifixus*), creating the feeling of fluidity, is associated with the image of water.

The two sections form two semantic centers - the first for the entire cycle, the second for the small cycle of *Opfermesse.*

Thus we obtain another two symbols belonging to the macro-level of the form (Diagram 6):

10. an egg - a spiral (meaning a small cycle within a large one)

11. an inverted figure eight within a circle (i.e., *Gebetsmesse* and *Opfermesse*).

The former is known to be the most effective structure for storing information; the latter is the symbol of em - eternity.

Thus the composition of the mass is a combination - conjunction of two totally different, even opposing concepts: a perfect circle (or concentric circles) and a fish. Of these two, the former is explicit, i.e., *fabula, involucra,* while the latter is concealed. Together they are the "spiritual bread", which is what the mass means for every Christian.

The 12^{th} key symbol of the mass is a fish within a circle. The mysterious duality and uniqueness of this symbol are amazing. On the one hand, its sights are strongly set on the future, since it contains the embryonic elements of all the forms to come: the fugue, the variations, the sonata-symphonic cycle, etc. (after all, it is no accident that its 10^{th} sign is the embryonic egg!). On the other hand, as symbolised by the fish swimming against the current, it is firmly rooted in the past traditions, as if struggling against the inexorable time to save the indestructible spiritual values for the future generations. Externally, this form represents the perfect symmetry of the circle symbolising tranquillity; while internally it contains the dynamic energy of great tension and force, since the fish and the water element which it inhabits strive to overcome and destroy structure as such.

The 12 symbols of the mass that have been discovered are the quintessence of the experience of comprehending the world and G-d, of esoteric knowledge originating in the ancient past. Their deciphering provides the key to understanding form and its semantic meanings in all their shades: both as musical form proper and form as a genre, in perspective and retrospective, and in various contexts: religious, philosophical, ethical, aesthetic, artistic, and so on.

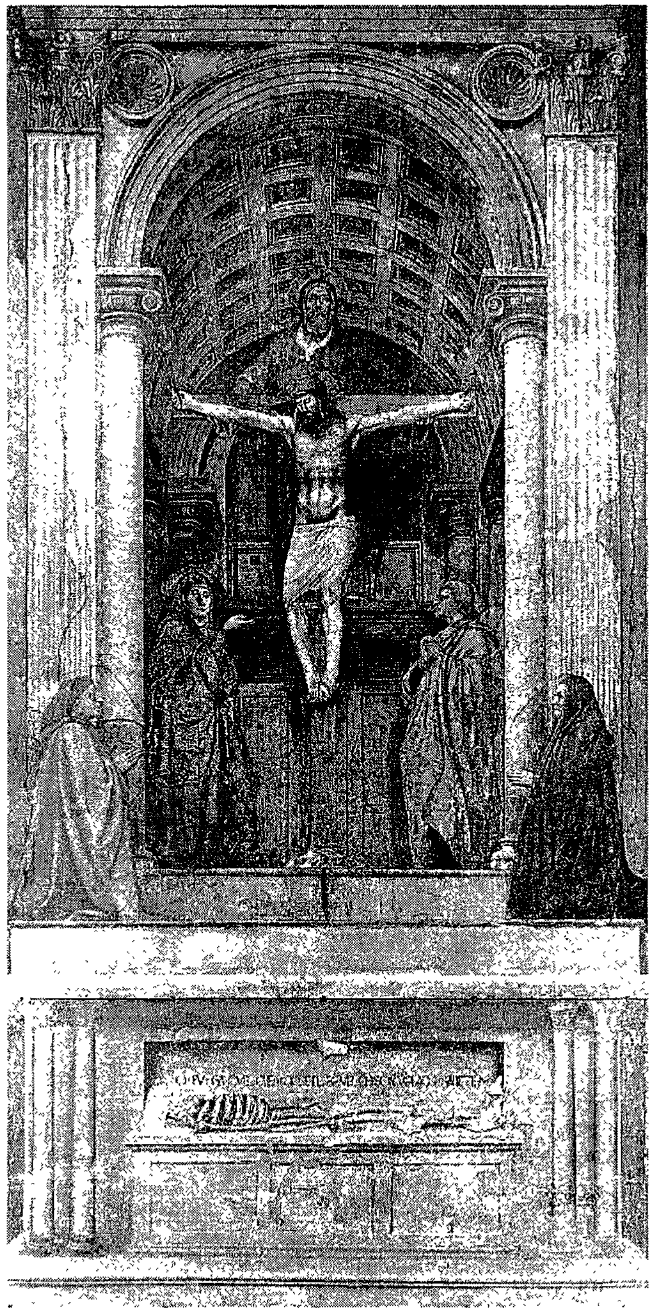
We have now reached the stage where it would be appropriate to draw certain analogies and comparisons between mass forms, visual art and architecture, i.e., to tackle the very task we had posed at the start of this paper.

The compositional commonality between various art forms that emerged during the Renaissance was much more pronounced that in any other periods. Thus, architecture and painting borrowed from music the proportions of perfect consonances (Alberti 1985; Wittkower), which accounted for the unique lucidity and pure harmony of Renaissance forms. In its turn music, obviously under the influence of visual art and especially architecture, evolved the laws of compositional symmetry and strict tectonics of the form.

The history of music knows no other form that would embrace to such a glorious extent the principles of other art forms.

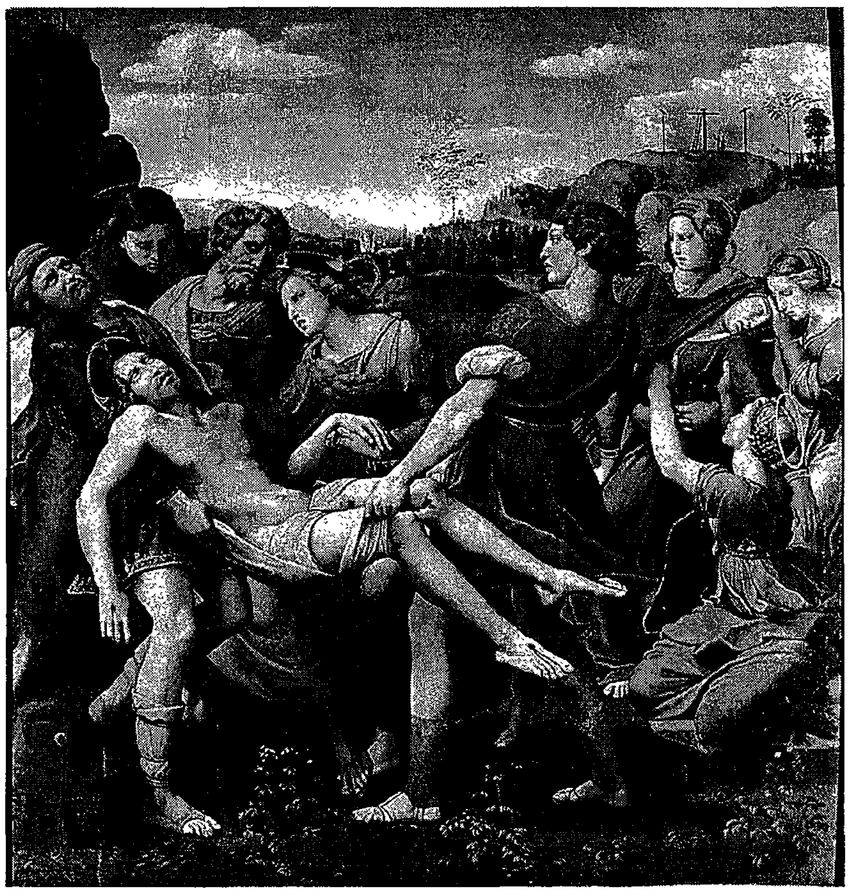
I will permit myself to present several examples.

Thus, the 3–5-part mass form corresponds to the design of a 3–5-nave cathedral, as to the art composition of the "revelation" inherited by the Renaissance from the medieval altar triptych. The strict symmetry and semantic hierarchy of mass parts are analogous to the hierarchical arrangement of figures in a painting in relation to the central figure of Madonna or Christ. For example, in Masaccio's Trinity the crucifix corresponds to the location of the *Credo*, the flanking saints to the *Gloria* and *Sanctus*, and the praying good givers to the prayer sections of *Kyrie* and *Agnus Dei.*



Masaccio, *The Trinity*, c. 1425-28, Fresco, 667-317 cm, Florence, Santa Maria Novella
[from Toman, R., ed. (1995) The Art of the Italian Renaissance, Konemann.]

The cross and the circle are probably the most frequently recurring shapes used by artists and architects of the Renaissance. If the shape of the cross appearing in 15^{th}–16^{th}-century cathedrals had its origins in the Middle Ages, the circular forms of the churches enthusiastically acclaimed and springing up in great numbers during the late 15^{th} – early 16^{th} centuries, the magnificent domes erected over Roman basilicas and medieval cathedrals (recall the famous Brunelleschi dome over the Santa Maria del Fiore) are undoubtedly owed to the Renaissance. Speaking of circular compositions, we cannot avoid mentioning the numerous *tondi* that emerged during that period – the Madonnas by Perugino, Raphael, Michelangelo and others. For a Renaissance artist, the idea of the circle spanned a range of associations – neo-Platonic, Christian, aesthetic – that was probably unequalled in its popularity.



Raphael, Deposition [from B. Santi Raffaello (1991) Raffaello, Scala, Firenze.]

The symbol of the cup was reflected in compositional ideas of Lamentation, Deposition, and Crucifixion. In Raphael's Deposition, the shape of the cup formed by the group of people surrounding Christ's body, is highlighted by the landscape relief enhancing the emotional charge and the mourning atmosphere.

The letter H is encountered quite frequently in Renaissance Annunciation compositions, but in others as well. We may see it appear in most facades of Gothic cathedrals. The depiction of fishes is featured occasionally in the interiors of medieval churches.

In conclusion, summing up all of the above, I would like to stress once again that the link between various art forms is made primarily through compositional unity of forms which, though present in every period, achieved its highest force and conviction during the Renaissance. The unity of forms is the unity of ideas, the powerful field of human consciousness that foments creativity. The world of the Renaissance, resounding through all the subsequent periods all the way to this day, is a world of harmony and creativity, of divine artistic endeavour, a world of unity, of complete fusion between the universe and a flower.

However, a discussion of the link between various arts of the period – which is the subject matter of so many studies – cannot be complete without an understanding of the role of music among these art forms, of the common features it shares with them, while maintaining its unique and separate nature. The duality of composition we have discovered to exist in the mass, the two primal forces inherent in it – the symmetrical, the rational, the explicit – and the asymmetrical, the irrational, the transcendental - these forces are the clear indicators of the planes containing the links in question, and of the borderline which defines the music and sets it infinitely apart from other art forms, as an art that had reached the highest summits of the spirit.

REFERENCES

Aaron, P. (1969) *Thoscanello de la musica*, New York, (printed by facs. of the 1523 Venice edition).

Alberti, L. B. (1985) *L'architettura*, Bologna, (orig. *De re aedificatoria* promulgated in 1452, published in 1485, Florence).

Busse Berger, A. M. (1990) Musical Proportions and Arithmetic in the Late Middle Ages and Renaissance, *MD*, 44, 89-118.

Bossuyt, J. (1994) *De Vlaamse Polyfonie*, Leuven, pp. 41-44, 63-67.

Crevel, M. van (1959, 1964) *Jacobus Obrecht: Opera Omnia, Missae*, vi and vii, ed. Crevel, M. van, Vereniging voor Nederlandse Musiekgeschiedenis, Amsterdam.

Evdokimova, Y. (1989) *Istoriya poliphonii*, vol 2, Moscow.

Elders, W. (1969) Das Symbol in der Musik von Josquin des Prez, *AM*, vol. 41, 164-185.

Elders, W. (1967) *Studien zur Symbolik in der Musik der alten Niederlander*. Bilthoven

Fellerer, K G. (1982) *Palestrina-Studien*, Baden-Baden.

Golzio, V (1936) *Rafaello nei documenti, nelle testimonianze dei contemporanei e nella letteratura del suo secolo*, Citta del Vaticano.

Henze, M. (1968) *Studien zu den Messenkompositionen J. Ockeghems*. Berlin

Pacioli, L (1974) "*Divina Proportione*" *in Qullenschriften fur Kunstgeschichte und Kunsttechnik des Mittelalters und der Neuzeit*, vol. 2, transl and ed. by Winterberg, C , New York, (orig. Leonardo Lauterano, Venice, 1509)

Plato (1994) *Collected Works*, vol. 3, Moscow, pp. 437-438, 440-442.

Powell, N. W. (1979) Fibonacci and the Golden Mean: Rabbits, Rumbas and Rondcoux, *JMT*, 23(2), 227-273

Timmers, J. J. M. (1974) *Christelijke symboliek en iconografie*, Bussum, pp. 217-221.

Tinctoris, J. (1979) *Proportions in Music*, New York, (Proportionate Musices, ca. 1473-74)

Wittkower, R. (1949) *Architectural Principles in the Age of Humanism*, New York.

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SYMMETRY IN MUSIC: A HISTORICAL PERSPECTIVE

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INTRODUCTION

Compared to other arts, symmetry phenomena in music are not immediately perceptible. They are mainly human-made, and reflect the artistic ideal in a more hidden and complex way than those in other arts. Whereas in the other arts symmetry may be part of the ready-made raw material detected in nature (e.g., the human body, colors and shapes of many objects, etc.), musical raw materials (e.g., scales and rhythmic patterns) are created by human beings. They therefore require learning – also in respect to symmetry – and differ from culture to culture.

We believe that the very existence of symmetry or asymmetry, whether conscious or unconscious, and the specific ways of their realization, together form one of the most important characteristics of style in its various manifestations – of a culture, a period, a composer or an individual composition.

Verbal references to symmetry in music appeared relatively late. Admittedly, already in classical Greece the general concept of symmetry, as embodied in the dualism of Apollo versus Dionysus, was also reflected in the classification of musical instruments and scales; and, likewise, in ancient China discussions of music merged with the broad dualism of *yin* and *yang* (Danielou 1943). But these are examples of very broad concepts and philosophies.

While in the realm of visual arts symmetry figured as a prominent aesthetic ideal of works of art in the ancient world (Weyl 1952), and was clearly expressed in theoretical writings and discussions of various periods, we do not know of any parallel written reference to symmetry in music. This is especially intriguing in the Renaissance, when despite music being undoubtedly guided by the notion of overall symmetry (explicit or implicit), this principle was not mentioned at all in theoretical musical writings.[1] Interestingly, in the Baroque era, too, where many manifestations of musical symmetry (according to our definition of the concept) may be detected, they were not accompanied by theoretical statements.[2]

In music, symmetry may be manifested not only in the manner in which the various parameters are distributed on the time axis, but also in their distribution on the pitch axis as well as in groups of elements serving as raw material. However, until recently symmetry in music was discussed mainly in regard to the time axis. Curt Sachs, for instance, while drawing broad comparisons between symmetrical phenomena of various arts and of different periods, related only to this tacit definition of symmetry (Sachs 1946).[3]

Symmetry on the time axis may be subdivided into three kinds of manifestations:

(1) "Mirror" on various levels of the time axis (e.g., retrograde motion, also termed "cancrizans" or "crab"; "chiasmus", etc.). This kind of "mirror" may be realized by pitches, tonal centers, formal structures, textures, etc. On a more abstract level, the structure *a-b-c-...-b-a* is also a mirror on the time axis, although not on the immediate level, since the repetitions of *a* and *b* maintain the same order of events within units.

(2) One repetition (*a-a*) or division into two similar parts ("period", and in an extreme case a continuous division into two, according to the formula 2^n; see Figure 1).

(3) Various kinds of division of the whole, made in reference to a central point on the time axis (e.g., the "golden section", or other divisions in accordance with various mathematical formulae).

To these symmetries a fourth, accepted one should be added:

(4) "Mirror" with regard to the pitch axis ("Mirror", "Inversion").

One of the detailed theoretical discussions that entails the concept of symmetry on the pitch axis (i.e., the vertical axis) is that of J. P. Rameau (1722), who presumed the existence of undertones as a mirror of overtones in order to explain the minor-major dualism.[4] In addition, he defined the symmetry existing between the dominant and the subdominant harmonic degrees, that are placed a fifth above and below the tonic accordingly.

As far as we know, music theorists explicitly referred to musical symmetry as late as in the eighteenth century (e.g., Mattheson 1739, Riepel 1752, Chastellux 1765, Daube 1773, La Cépéde 1785, Koch 1787, and others; also summarized in Ratner 1980). However, they considered solely symmetry that results from a twofold division of phrases (i.e., No. 2 in the above list of symmetry manifestations).

In the last decades, the concept of musical symmetry has been expanded, under the influence of three main factors:

(1) The perception of symmetry as a comprehensive phenomenon, manifested both in nature and in human activities;[5]

(2) The new research directions, that regard musical activity as cognitive activity;

(3) The increasing awareness of twentieth-century music (which rejected the learned schemata on which tonal music was based) of symmetrical operations functioning as basic procedures in the formation of a musical work (e.g., Oppenheim 1989).

In our research we refer to the relatively new definition of symmetry, which is common to various domains of human culture and of natural phenomena: "Symmetry [is] no more than statements as to the operations that have no effect upon the systems that we consider" (Wilkinson 1989). The adoption of this definition to the realm of music may, of course, be made only on the basis of a primary discussion of the significance of the various musical systems. In this way we hope to shed additional light on phenomena that have already been accepted as symmetrical, and also to expose latent symmetries and other similar phenomena. With the help of the new definition we shall try to examine symmetry in various styles as reflecting the change in the aesthetic ideal of different historical periods.

I BACKGROUND CONSIDERATIONS

A. Symmetry Phenomena in Music

The main concepts by which we describe symmetry in music – schema (learned and natural), transformation, and categories of operations – are frequently used in the analysis of specific musical compositions and in general analytical researches pertaining to musical theoretical principles. As yet, however, there is still no consensus as to their application, and they may often be encountered in a different terminology. In the following, we shall summarize their characteristics where relevant to our discussion (for more details see Cohen—Dubnov 1997, and Cohen 1996).

A1. Schemata

The concept of schema represents an organizational principle that may be realized in many ways, and may be seen as derived from the operation of "fusion" (see below). The schemata create links between the events and may appear on various hierarchic levels. They contribute to the creation of a system of expectations and to intelligibility, and they make complex organization possible. They originate consciously or unconsciously in our mind, and all our aural impressions are perceived through comparisons with them. Their special importance in music stems from the fact that in music all meanings are based on organizational rules rather than on semantics.

We may distinguish between learned and natural schemata. The learned schemata (intervals, scales, chords, meters, etc.), which are expressed mainly in theory, are expressed in exact magnitudes of specific parameters, and differ from culture to culture.[6] In contrast, what we call "natural schemata" are not culture-dependent, and are not characterized by precise quantification. They are known to us from outside music, too, and they appear on various levels of abstraction (e.g., ranges of occurrences in various parameters, with reference to the normative range; curves of changes in time; types of operations; degree of definability; etc.). They contribute to basic and natural sensations of excitement.[7]

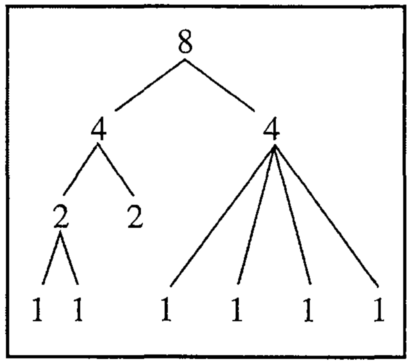


Figure 1: The Symmetrical 2^n Schema One Possibility

Let us specify two natural schemata which enable us to predict the continuation of the musical course, and whose presence or absence in different periods and cultures is very significant: (1) The 2^n formula, i.e., a continuous duple division of musical phrases, or a continuous multiplication (1 + 1 + 2 + 4 + 8 + ...).[8] As we shall see later, this schema may be seen as one manifestation of the operation of segregation/fusion. (2) A convex curve (in all parameters: low-high-low; slow-fast-slow; soft-loud-soft, etc.). For example, most popular tunes of the world that do not aim to arouse excitement adhere to the convex curve (Nettl 1964; Huron 1996).

These two schemata function as important characteristics of style. Another natural schema which contributes to stylistic characterization is the principle of concurrence and non-concurrence between events of different parameters[9]: a full concurrence, for instance, would mean a convex curve in all the parameters simultaneously, while non-concurrence means a convex curve in one parameter and a concave curve in the other. The principle of concurrence/ non-concurrence may also apply to the relation between various natural schemata, between various learned schemata, and between both learned and natural ones. Non-concurrence may contribute to complexity and to uncertainty as to the limits of the various units and their characterization, and therefore of a sense of tension, too. This schema can be seen as one manifestation of the degree of definability.

A2. Symmetry as a Natural Schema

In this study we discuss symmetry (or a-symmetry) phenomena in their broad meaning, as manifested in three interrelated domains:

(1) Learned schemata of human-made raw materials, in the cognitive stage, as opposed to the raw materials in the physical-acoustic stage;

(2) Forms that may be regarded as modes of organization of musical units in terms of similarity and difference, such as *a-a* or *a-b-a*, the units being defined by learned and/or natural schemata;

(3) Compositional rules (explicit or implicit, conscious or unconscious ones) as manifested in categories of operations, that are applied to learned or natural schemata or to various events on the immediate level.

From our point of view, the principles of organizing the variables on the basis of similarity and difference (Tversky 1977) may be considered natural schemata of high abstraction, as are the categories of operations, since they represent natural procedures of our cognitive activity. As we shall see later, formal organization is one of the manifestations of one of the operations (segregation and fusion).

Operations are, of course, the main characteristic of transformation.

A3. Transformation in Music

Transformations preserve the two essential conditions of any kind of organization: similarity (= repetition) and difference (resulting from a well-defined operation). They have frequently been discussed in musical literature.[10] In order to determine their musical significance, we have examined their manifestations, while taking into account their variables. These are:

(1) The kind of operation in which symmetry is manifested;

(2) The parameters and the components derived from them, that are subjected to operations (e.g., interval, scale, and chord, derived from the parameter of pitch; rhythm, meter, and tempo, derived from the parameter of duration; and loudness and timbre);

(3) The level of musical organization (a transformation of an immediate event or of a schema on a deeper level, such as a chord, a scale, or even a harmonic phrase);

(4) The degree of change (large or small);

(5) The contribution to clear/unclear directionality or to the sense of relaxation/tension.

A4. Categories of Operations

The operations may be grouped into five categories, that may appear in various units and dimensions, and whose principles are those of cognitive operations (Appel 1992). They may therefore be considered natural schemata that make complex and intelligible organization possible. The categories may appear in various units and dimensions.

These are the categories of all the operations: (1) contrast; (2) shift; (3) augmentation and diminution; (4) fusion and segregation; (5) equivalence. In part, these operations are analogous to mathematical ones (Weyl 1952; Leyton 1992), and may be applied to other subjects, including arts in general and especially visual arts (Avital 1996, 1998).

(1) Contrast may be manifested in many parameters and in three main kinds of operation:

a. A simple contrast with regard to two related points on one scale with different parameters (e.g., high/low; loud/soft, fast/slow); or with regard to two directions on one scale (up/down); or in regard to the location on the time axis: *a-b-c-d-c-b-a* (for details see the discussion of form). This contrast is analogous with the mathematical operation called "reflection."

b. A compound complementary contrast that agrees with a few conditions not specified here (e.g., harmony/melody; a chord/a melodic second; harmony/polyphony);

c. A binary contrast between two options only (major/minor scale; stressed/unstressed beat, minor/major second; double/triple meter).

(2) A shift within a cyclic system, such as the three inversions of the triad chords; modal scales (in the diatonic system); melodic sequence (in the diatonic system); transposition (in the chromatic system); rhythms pertaining to the metric cycle (such as the various poetic meters). This operation is analogous with "translation" or "reflection".

(3) Augmentation and diminution, such as durational multiplication/division; addition/subtraction (including condensation/reduction of the melodic line); or expansion/contraction of a musical period in Classical style).

(4) Fusion and segregation (or grouping/splitting, which is a basic operation for any kind of organization, e.g., grouping into schemata; grouping into "variations" – as opposed to "family resemblance"), and reduction. From this operation we obtain the 2^n schema and forms of repetition.

(5) Equivalence, which is one of the conditions for any living language (Powers 1976), e.g., equivalence between categories of harmonic degrees; means of emphasizing musical events; variations in Western music; and different realizations of the same melody type, mainly in non-Western music.

Let us remember that an operation is considered a symmetrical activity as long as its result does not overstep the limit of the original system, and that this system (which we consider a schema) may be of various hierarchic levels.

One of the most significant systems in Western tonal music is the diatonic system (as represented by the white keys of the piano), consisting of seven notes chosen from twelve notes. An operation may result in staying within the limits of the system, or in overstepping to the larger system of twelve, as in the dodecaphonic system. For instance, the operation of "shift" or "mirror" within the parameter of pitch may be (a) "diatonic" (i.e., not exact, but remaining within the 7-note schema, that is selected from 12 notes), or (b) "real" (i.e., exact, while the intervals are chosen from the 12-note system). For example, the real mirror of an ascending major chord is a descending minor chord, and that of a major tetrachord is a Phrygian tetrachord.

Tonal music usually has diatonic operations. Bach's music (the duet BWV 803), however, is unusual in that we also find real operations that lead to a sense of bitonality (aside from the real answers to the subjects of fugues under certain conditions). The "real" operation indicates abandoning the 7-note system. The larger the system is – according to information theory – the less severe the restrictions are, and the less "directional" or more unexpected the composition is.

B. Interference of an Extra-Musical Factor: The Aesthetic Ideal

In different cultures and epochs we may find different manifestations of symmetry that contribute to style according to the aesthetic ideal. One may argue that the choice of the kind of symmetry – whether made consciously or unconsciously – reflects that ideal. It should be noted, that the two extreme situations – total symmetry (e.g., Pierre Boulez's *Structures I*, 1952) and a complete lack of symmetry – are meaningless. In reality we find intermediate stages reflecting the ideal.

The following are some of the variables that characterize the aesthetic ideal and are relevant to symmetry: overall structure as opposed to concentration on the momentary event; clear expression as opposed to a blurred one; certainty as opposed to uncertainty, and, in general various kinds of complexity and directionality (Cohen 1994). The concept of directionality relates to the importance of the sequence of events on the time axis, each event evolving from its predecessor: when there is no hierarchy in the sequence – there is no directionality. Directionality may refer to various levels of musical organization, and it represents the "logical" progression of music, enabling us to predict the succession of musical events.[11] Two other important variables are the connection and non-connection to the extra-musical world; and a sense of calm or excitement that represents (according to Curt Sachs [1946]), an ideal of "ethos" and "pathos," which alternate cyclically in the West.

Despite the risk of generalization, we can draw a clear line between the ideals of Western tonal music and those of non-Western music, as well as between different periods in music history of the West. In non-Western cultures, where music has changed less in the course of history than in the West, music tends to be an integral part of life. Music in non-Western cultures does not have a closed, independent existence with overall directionality and complexity and a clear beginning and ending; instead, there is greater emphasis on the momentary event.

How does symmetry contribute to the representation of the aesthetic ideal? We shall demonstrate here merely a few specific realizations of some of the categories of operations; namely, their contribution to clear/unclear directionality and to a sensation of calm/excitement. These operations may increase the degree of clarity or obscurity, calmness or excitement of the musical expression. For example, contrast may be obtained by means of a descent following a small ascent, in which case it causes balance, or by a sudden change between extremes of pitch, intensity, tempo, or density, in which case the result is tension. As another example, the operation of contrast can be realized by the difference between a concave and a convex curve. A convex curve (i.e., a descent following a melodic ascent) contributes to clear directionality, whereas the opposite type of symmetry – a concave curve (i.e., ascent following a melodic descent) – causes excitement and a lack of directionality. The difference between these two kinds of symmetry is that the ends of the concave curve extend to infinity, and therefore we have no certainty as to the continuation of the musical events, whereas the convex curve provides for maximum predictability.

Similarly, operations may lead to nonconcurrence of various sorts between parameters, units, events or schemata – thereby increasing complexity and uncertainty, i.e., reducing directionality. As a result, we find different kinds of symmetry in different styles. For example, we may find convex curves in styles that strive toward clear directionality, and vice versa.

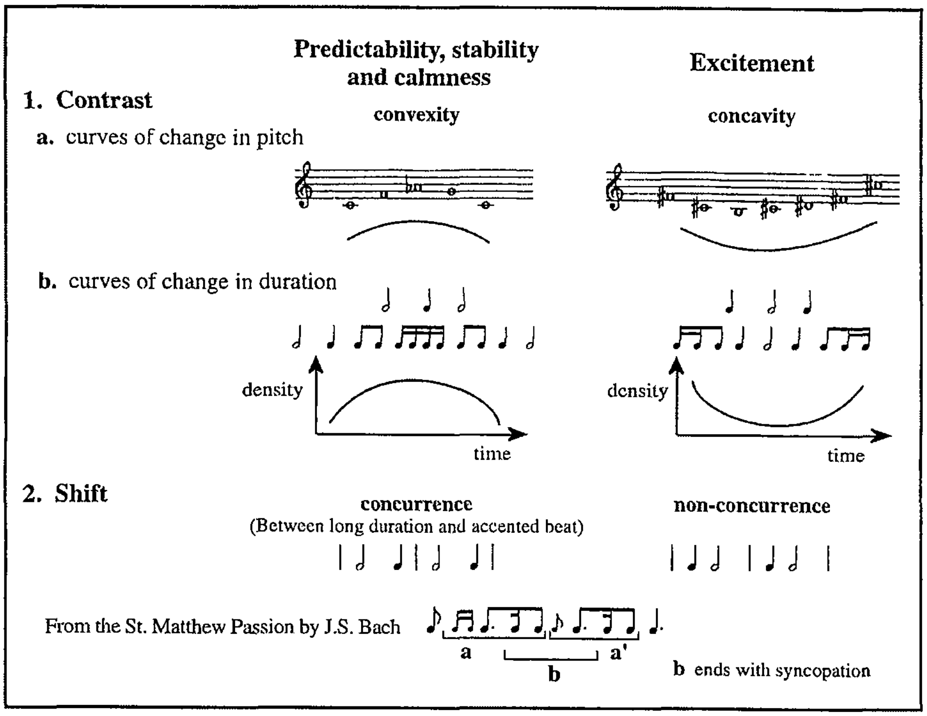


Figure 2: Symmetry causing both Calmness and Excitement: 1) Derived by the operation of contrast; 2) Derived by the operation of shift. The letter *a* at the bottom of the figure represents a characteristic rhythmic motive, repeated in variance *(a-a')* at the opening of the the aria "Erbarme dich" (mm 1-2); The letter *b* represents another characteristic motive, starting as upbeat to m. 3, that may be considered as a shift from *a* Because of the *appoggiatura* on the beat, *b* ends with syncopation (i e , nonconcurrence) and causes excitement, in contrast to *a*.

II KINDS OF SYMMETRY IN VARIOUS ERAS IN THE WEST

In our discussion of symmetry in various eras in Western music, we will consider the constraints of the stylistic ideal and the cognitive constraints that we assume intervene in the (unconscious) selection of specific types of symmetry.[12] The concept of an era is flexible, of course, since the chronological borders and styles of eras often overlap, and because each of them can be subdivided. Here, however, we present a bird's-eye view of the stylistic landscape in accordance with the standard division of music history. For the sake of comparison, we also take a look at non-Western musical cultures.

Our presentation follows the three domains of symmetry discussed above: the principles of (1) raw material; (2) compositional rules; (3) forms.

C. Symmetry in Musical Raw Material in the West since the Seventeenth Century

C1. Binary Contrasts

Our point of departure for comparing symmetry in different kinds of raw material is Western tonal music, in which the ideal calling for overall directionality and complexity reached its peak, and binary contrast assisted to achieve this ideal. Indeed, only in the raw material of Western tonal music, beginning in the seventeenth century, do we find binary contrast. Binary contrast is manifested in the "building blocks"in various parameters: beats (accented/unaccented; only one accented beat per cycle); meters (duple/triple); intervals of a second (major/minor); and scales (major/minor). Binarity, which means a minimum number (=2) of possibilities for the different building blocks in each parameter, makes it possible to get numerous complex forms of organization on high levels. Thus, for example, the paucity of scales (major-minor only) enables their multifarious realizations in different tonal centers; the system as a whole generates the schemata of the chords and of the laws of harmony that characterize Western music, and allows for an overall and complex directionality; and the limited choice of meters enables hypermetric organizations, etc.

Let us stress that the very possibility of obtaining multiple forms of organization with different styles accompanies the possibility of significant changes in the styles of different eras and composers; these changes are themselves an important element of the Western ideal.

Before the seventeenth century, there were more than two scales (modes) in Western music (with virtually no modulation); these modes were obtained through shift operations in the single diatonic scale system of seven notes (which itself constituted a specific hierarchical selection from the pool of twelve notes in an octave).

Until the seventeenth century meters, too, were not simple and did not conform to the principle of binarity: there were more than two kinds of meter and more than two kinds of beat, and in certain cases (in the rhythmic modes, for instance), meter and rhythm were not clearly differentiated. All these basic qualities are even more prominent in non-Western music. We find here the opposite of what we see in Western tonal music: multiple relationships on the immediate level (e.g., complex metric patterns; more than two kinds of beats and more than two kinds of seconds; and numerous scales, which are derived from an abundance of scale systems). This multiplicity makes it possible to increase momentary complexity and prevents overall directionality with complexity, as well as marked changes in style.[13] Therefore, the symmetry of the raw material, the operations, and the forms have less impact on the music of non-Western cultures than on Western music.

C2. Optimal Quantities and Minimal Asymmetry in the Scale System in the West

As stated above, the single Western system with seven named notes constitutes a specific hierarchical selection from the pool of twelve notes in the octave. This system is unique in many ways, and it allows for overarching organization with clear directionality and complexity (something that is found only in Western tonal music). In addition to being a single system,[14] it also constitutes a kind of optimum regarding the number of different intervals. Certain non-Western cultures have, on the one hand, a not-well-defined multitude of intervals and scales (groups of seven), which are obtained from pools larger than twelve, and on the other hand, a small number of intervals or systems with fewer than seven notes – most often five notes (pentatonic systems), obtained from a pool of seven or fewer. In effect, in both of these extreme situations the number of intervals used for musical organization is quite limited; consequently one can hardly speak of a system of intervals. In accordance with the ideal, neither extreme allows for complex overall organization.

For our purposes, it should be stressed that the Western system, like biological systems, has minimal asymmetry, which is considered an essential condition for obtaining complex systems (Atlan 1981, 1987). Outside the West only a few systems maintain minimal asymmetry (Figure 3). Interestingly, it is found also in the rhythmic pattern that is fundamental to African polyrhythmics (Arom 1998).

General:

A A B A A A B A A B . . . A B A A B A B . . .
X X X X X X

In scale systems:

In the West (in units of half-tones): 2, 2, 1 2 2, 2, 1 2, 2, 1 . . .

In Chinesc pentatonics: 2, 3 2 2, 3 . . .

In Japanese pentatonics: 4, 1 2 4, 1 . . .

In ancient India (in *šruti* units): 3, 4, 2 4 3, 4, 2 . . .

In the *māqamāt* (rare examples)

Hejāz-qār : 1, 3, 1 2 1, 3, 1 . . .

Rast (in units of quarter tones): 4, 3, 3 4 4, 3, 3 . . .

In the basic rhythmic paradigm for the African polyrhythm:

♩. ♩ ♩. ♩. ♩
3, 2 3 3, 2 3, 2 . . .

Figure 3: Minimal Asymmetry ın Cyclıcal Systems in Various Musical Cultures
3.1 (General) - Two abstract examples of minimal deviation from the symmetrical pattern X ın cyclical systems; 3 2 and 3 3 (in scales and rhythm) - Selected concrete examples of systems in regard to pitches and ıntervals (scales), and durations (rhythm)

In the West, the specific raw material, characterized by unique features such as minimum asymmetry, binarity, optimal quantity of elements, etc., existed from the seventeenth to the twentieth century. In the twentieth century, however, many styles discard the learned schemata in the raw material. The breakup of the system of seven, together with most of the hierarchical schemata derived from it, demands that the composer himself now determine his own schemata of the raw material, and in addition rely (in part or exclusively) on parameters other than the interval, e.g., on texture and timbre. The Western pool of 12 thus contains the maximum amount of well-defined multiplicity.

D. Symmetry in Compositional Rules

D1. The Relationship to the Raw Material

The systems of twelve and seven and the major and minor scales in the West make it possible to create learned schemata on many hierarchical levels. Let us stress once again the most important schemata inherent to the West – chords and the rules of harmony: both contribute decisively to the overall directionality and to the complexity of compositions of large dimensions. Likewise, the system enables a clear distinction between operations that do not cause any deviation from the group of seven ("diatonic operations") and those that expand the diatonic group towards the pool of twelve ("chromatic operations"). This expansion blurs the distinction between the scales and the tonal centers and heightens the listener's sense of uncertainty regarding the continuation of the musical progression. The system of seven and twelve thus supplies another parameter of certainty/uncertainty, which contributes to a sense of tension and relaxation; this sense is fundamental to shaping a piece of music.

D2. The operations

For the most part, the operations act on small units, from the motive to the complete musical phrase, but they may act on larger sections as well. In small segments, numerous operations may function simultaneously in various parameters, whereas in large segments there is rarely more than one operation, and it generally acts on only one parameter. Extensive use of operations in Western music began in the seventeenth century, when the major and minor scales, harmony, and large forms with complex superstructure took shape. One typical transformation that began in this era and is virtually unique to the West is modulation on various levels, obtained by shift operations that involve the tonal centers.[15] As stated above, the specific selection of the type of operation and manner of realization is determined in accordance with the ideal and cognitive constraints.

In the following, a short survey of the operations in the various periods of music history is given:

Medieval music may be divided into sacred and secular genres. Even if we consider written music only (9th–15th centuries), we can classify it according to the presence or absence of religious associations, the degree of definiteness of the rhythmic organization, and the existence and complexity of polyphony. The rules of polyphony are still flexible. Symmetry phenomena are rare; they are manifested mainly in the formal organization of secular songs according to the differences and similarities between musical and textual units. This type of melodic organization has generated several fixed schemata (*formes fixes*), most of which are based on an additive combination of two units (*a-b*), which are considered different in various respects. Symmetry is also manifested in a constant repetition of melodic and/or rhythmic patterns (motet and isorhythmic motet), and in retrograde motion and voice exchange, as in the case of Machaut's rondeau *"Ma fin est mon commencement."* In monophonic religious music (Gregorian chant), symmetry is manifested mainly in the general convex shape of the melodic curve.

In the late Renaissance, especially in the vocal music of Palestrina, whose ideal is a "calm flow" and whose directional unit is the musical phrase that corresponds to the verbal phrase, the operations are extremely limited. The symmetrical schema of the convex curve – which allows for predictability regarding the continuation of the line and meets the conditions of calm from numerous standpoints – stands out on various levels and parameters. Curves which cause tension and do not enable predictability – concave, zigzag, and flat curves – are not found in this style. Another schema absent in it is the 2^n schema which, although contributing to directionality, also implies division, an undesirable stylistic feature in the religious vocal music of the Renaissance.

The convex curve in the Renaissance is manifested on various levels, from the tiny unit of a combination of harmonic intervals according to the pattern of consonance-dissonance-consonance, to the immediate level involving the succession of melodic intervals or durations, and finally to convexity on the phrase level, and even on the entire structural level (in which case the convex structure is extremely weak). In Palestrina's music the operations of shift and sudden contrast are virtually absent. Contrasts are limited, on the whole, and appear mainly on the immediate level, such as change of direction, as well as in texture, e.g., homophony as opposed to polyphony. We may also find antiphonal echo (*cori spezzati* – in a sense, a stereophonic contrast), and of course the gradual contrast of the convex curve. As to the well-known contrast consonant/dissonant, it should be noted that the main purpose of the dissonance is to highlight the consonance (Zarlino 1588),[16] and its occurrence underlies certain restrictions: it may occur only gradually on an accented beat, and should be prepared by a suspension in one of the voice parts.

In the Baroque Era (the seventeenth and eighteenth centuries), many rules of composition as defined by Renaissance theorists and many balances are deliberately violated, as it were. Therefore we may find on one hand lack of change in many parameters, stretching over long durations (a feature not found in Renaissance music), and on the other hand conspicuous changes such as large skips and changes derived by the operation of contrast. Indeed, this is the era of the concerto, based on the contrast between soloist (or soloists) and orchestra, in which the concerto principle is superimposed on a great part of musical forms (Bukofzer 1947). In addition, the accented dissonance occurring simultaneously in two voice parts, the melodic sequence (i.e., the operation of shift), and contrasts between large melodic skips and stepwise motion are frequent. Another salient feature is that the operations are applied to entire or partial themes (mainly in the fugue), which are being repeated many times. These common operations are inversion (a diatonic contrast regarding the pitch parameter); expansion and contraction (with respect to the parameter of duration); and shift along the time axis (*stretto*). In Bach's works, overt and latent operations in texturally uniform musical elements are especially abundant. For example, almost the entire D-Sharp Minor Fugue from Book II of *The Well-Tempered Clavier* may be regarded as being derived through operations of parts of the first theme, in addition to operations on the theme as a whole. Moreover, Bach extends the manifestations of the operations maximally to various levels of musical organization and even to real operations.

This multiplicity of operations with concurrent or nonconcurrent relationships results in maximum complexity with respect to a minimum of different elements.

In the Classical era, the most symmetrical and directional overall form – "sonata form" – developed. Here we find many changes in texture, which help to clearly shape the parts of the form: each theme in sonata form ("first subject", "bridge," "second subject", "closing theme") may have a different texture, in contrast to the Baroque era, where a single texture often dominates an entire composition or movement. Unlike in the Baroque, in the Classical era operations generally act on parts of the theme rather than on the whole. Particularly salient is the operation of contrast, which appears distinctly, but in various forms. For example, there exist contrasts in the learned schemata (e.g., a chord as opposed to a melodic second), or a binary contrast between the two themes of sonata form in respect to their predetermined tonal centers and also their texture. A clear contrast also exists between the first and second movements of a sonata. Sudden contrast is particularly salient in the works of the so-called "Storm and Stress" composers (mainly C.P.E. Bach) and of Beethoven, where the ideal is greater excitement regarding both the quantity of contrasts and their intensity. Often, the contrast is between two extremes, such as a high register following a very low one, *ff* after *pp*, or a large ambitus as opposed to a restricted one. (Moreover, just being in an extreme range, which is like a deviation from the norm, elicits excitement.)

The following example (Figure 4) demonstrates multiple operations, mainly contrasts, occurring within four measures.

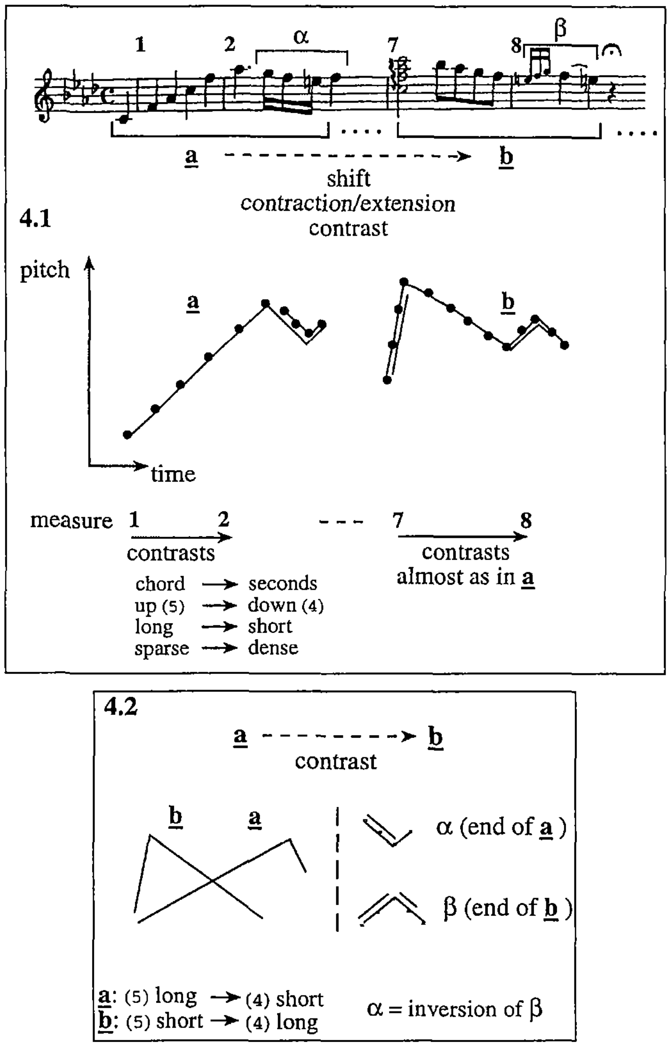


Figure 4: Symmetrical Links within and between Motives a and b in the Opening of Beethoven's First Piano Sonata (Op. 2, No 1)
4.1 Contrasts within each of the units a and b;
4 2 Contrasts between a and b

Side by side with the operation of contrast, a salient phenomenon in the music of the Classical era is the symmetrical schema of 2^n. This schema (which can be regarded as being obtained from the operations of fusion and segregation, or augmentation by means of multiplication) is common in the so-called periodical phrases, and allows for clear directionality with maximum division into hierarchical levels. Adherence to it or deviation from it is an important characteristic of the structure. In sonata form, deviations from 2^n are common in the first theme, but the second theme is almost always based clearly on 2^n.

Deviations from 2^n are mainly of four types:

(1) Division of a unit with 2^n measures into more than two, e.g., division of eight into 2+3+3 (in works by Bach, Haydn, and others).

(2) Division into two, where the units being divided are not 2^n (e.g., 5+5; 3+3). These deviations can be considered as the result of addition or subtraction (i.e., the operation of augmentation or diminution).

(3) An extreme case: Division into three of a unit that is not 2^n, e.g., 3+3+3 (found, for instance, in Scarlatti).

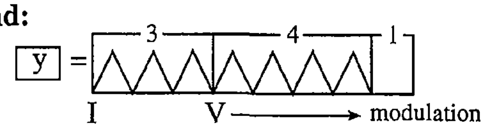
(4) A lack of clarity in the division into units, because of nonconcurrence.

An interesting example of the first kind of consistent and clear deviation from 2^n is found in the first movement of Piano Sonata No. 25 in G Major, Opus 79, by Beethoven (Figure 5). The deviation is manifested in the internal division of a unit (*y* in the figure) of eight measures into three sub-units with 3+4+1 measures. The idea of deviation from 2^n within a unit of eight measures is already suggested by the ambiguous division of the first theme (owing to different types of nonconcurrence), as opposed to the obvious deviation of unit *y*. The consistent deviation to 3+4+1 governs most of the development section (it occurs five times), and can therefore be viewed as a specific schema (division of 2^n) that is unique to the piece.

An example of the second and third types of deviation from 2^n is found in the first movement of Mozart's Piano Sonata in C Major, K. 309 (Figure 6). The first theme consists of twenty measures which are broken down into 6+14; the 14 is divided into 7+7; the 7 is divided into 5+2; the 6 is divided into 3+3. The bridge also fits into this category, whereas the second theme definitely represents the symmetrical schema 2^n.

	Theme	No. of measures	Division	Concurrence with 2^n
A	First	8	Various interpretations	? (one possibility : $1 + 2^{+} + 2^{-} + 3$)
	Bridge	16	4 x 4	#
	Second	16	4 x 4	#
	Closing	14	2 + 2 + 4 + 4 + 2	
	Coda	4	2 + 2	#
B	a	8	?	?
	$\boxed{y_1}$	8 (E major)	3 + 4 + 1	–
	$\boxed{y_2}$	8 (C major)	3 + 4 + 1	–
	b	8	4 + 2 + 2	#
	$\boxed{y_3}$	8 (C minor)	3 + 4 + 1	–
	$\boxed{y_4}$	8 (E^b major)	3 + 4 + 1	–
	c	13 = 12 + 1 =	4 + 2 + 2+ 4 + 1	
	$\boxed{y_5}$	8 (D major)	3 + 4 + 1	–
	d	4	2 + 2	#

Legend:



△ = broken chord

\# = full concurrece

Figure 5: 2^n in the First Movement of Beethoven's Piano Sonata No 25 Op. 79: Adherence and Deviation. Maximal concurrence with 2^n appears in the Exposition (A); consistent deviation from it appears in the Development section (B) in units y (y1-y5). Letters a-d represent various units between the y's; 2+ and 2- (top right in the figure) mean more and less than two measures; y consists of broken chords, and the internal division is regulated by harmony.

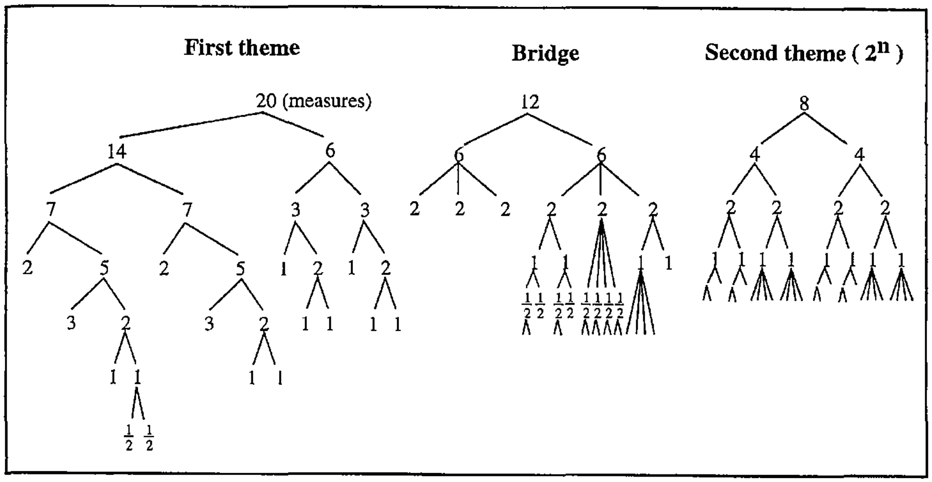


Figure 6: Division of Units into 2, but not 2^n, in the First Movement of Mozart's Piano Sonata KV 309. Only the second subject adheres to 2^n

The Romantic Era witnessed a gradual destruction of the tonal system, achieved in various ways: by the expansion of the schemata on various levels – especially the significant extension of the operations of shift and equivalence; the reduction of hierarchy within and between the schemata; the transgression of the norm, or a sudden break of a directional schema. In practice, we may find various ways of blurring the dichotomy between the similar and the different: between major and minor scales (also by the use of modes); between chords, tonal centers, or progressive and regressive harmonic steps; between sonata movements and sections within a movement (for example, between the exposition and the development section, the latter principle being superimposed also on the exposition), etc. Within these procedures, each composer chooses his own personal way of loosening the system without causing its total collapse. A few examples may serve to illustrate this point:

One of the stylistic traits of Schubert, especially in his late works, is a sudden deviation from the expected by the use of a classical harmonic schema, where the dominant degree (i.e., the most directional chord of the schema) is not led into the tonic, but is instead broken by a partial or full chromatic shift and succeeded by a completely strange chord (for instance, in *"Der Wegweiser"* from the *Winterreise,* the dominant of C – G-B-D – leads into G^b-B^b-E^b, i.e., into an E^b minor chord). This is a significant expansion of the false cadence, and the result is complete surprise. Here and elsewhere (e.g., in the *F* minor piano fantasy for four hands) we may also find different enharmonic interpretations of the diminished seventh chord, or of the augmented triad.[17]

Chopin adheres very often to the 2^n schema, which helps him to keep the general notion of tonality in spite of the fact that the intensive chromatic shifts threaten to endanger it.[18] A primary experiment has shown that without the 2^n schema the listener tends to lose the sense of tonality. Brahms often inflects the sixth note of the major scale, and the sixth degree as a whole, thus shifting suddenly to a tonal center lying a major third beneath the tonic, and also creating a chain of equi-distant major thirds. For example, in Brahms's Rhapsody No 1, Op. 79 in B Minor, we find the major scale with a minor sixth in mm. 22-29; modulations via the inflected sixth degree appear in mm. 1-43, where the tonal centers are F# - D - B^b - G^b (=F#). Interestingly, the piece also opens with a descending line consisting of these notes. Unlike Schubert, who usually opens a work on the tonic, in Brahms's Rhapsody the tonic (B) appears for the first time only in m. 89!

In the Twentieth Century, as the learned schemata became less and less important, the planned natural schemata became more prominent. Despite the difference between tonal and atonal systems and their realizations in specific pieces, we find the same categories of operations expressing various forms of symmetry in all of them.

A well-known, almost classic, example is the "dodecaphonic system" formulated in the 1920s by Arnold Schoenberg. In dodecaphonic music, which is based exclusively on the pool of twelve notes, the learned tonal schemata are replaced by the so-called row schema, which is manifested in a specific one-time arrangement of the twelve notes in the octave. Although it does not seem possible to refer to schemata, since each piece has its own row, the theorists (Babbit 1960) point to categories of rows in terms of the intervals and the internal organization within the row, which represent phenomena of symmetry (Perle 1972). In rows existing in Schoenberg's works, for example, the collection of the first six notes in the row is an inversion, in a sense, of the second collection of six notes; in Webern's music, the row is divided into even smaller units, which are obtained from each other through the operations of contrast and shift.[19] The row schema is realized in various ways, not only in terms of parameters other than pitch, but even with respect to the pitch class. Realizations of the pitch class can thus be considered an operation of equivalence, and a division of the row into sub-units is an operation of segregation; the other operations involving the parameter of pitch, and that are discussed explicitly, are contrast ("inversion" with respect to the pitch axis and "retrograde" with respect to the time axis), and the precise shift (transposition). The method was also expanded to encompass the other parameters and the links between the rows in the parameters (total serialism).

In recent decades we find many musical styles that relate to the parameter of pitch (in the pool of twelve) without clear tonality. In some of them, the compositional rules are based on set theory (e.g., Vieru 1993; Conner 1994). From our standpoint, the concepts used in set-theory – "inclusion", "cutting", "difference", and "completion" – relate to the operations of fusion and division. The tonal schemata are replaced by various sets of notes, each of which is realized in various ways by means of the different operations. The computer is gaining increasing use in all stages of composition – e.g., selection of the raw material, various parameters, the compositional rules, and the musical piece itself – and much of the work done by the computer consists of performing operations (which are still included among our five basic categories). When discussing symmetry in twentieth century music, one cannot ignore the music of Béla Bartók, in which manyfold manifestations of symmetry reached a peak.

E. Symmetry in Forms

"Form" is defined in the *New Harvard Dictionary* (Randel 1986) as "The shape of a musical composition as defined by all of its pitches, rhythms, dynamics, and timbres." This definition is shared by many authors writing on form (e.g., Leichtentritt 1951), but according to our definition form means the organization of units mainly with regard to the difference and similarity between them in various parameters, even without taking into account the nature of the unit (be it melodic, harmonic, rhythmic, etc.).[20] The very distinction according to similarity and difference is a basic cognitive activity (Tversky 1977); it may be perceived as a schema, and considered as derived from the extended operations of fusion and segregation on various levels.[21]

An initial categorization of form relates to (a): the clarity of the distinction between similar and different; (b) the degree of adherence to a symmetrical organization; and (c) the levels of the symmetrical phenomena.

(a) Regarding the degree of distinction between similar and different, the repetition *a-a* (referring to segments of varying size) may appear in identical manner – with precise repetitions designated by a repetition sign (//: :\//) – or with small or large changes, so that we would not know whether to define the section as "repetition with changes" (*a-a'*) or as a different section, professing a few points of similarity with the previous one.[22] This problem gets even more involved when we consider that different cultures vary in their definition of the similar and the different. On the other hand, different sections (*a-b*) usually exhibit some similarity (even by means of the operation of contrast, or another principle). Thus, the difference between *a* and *b* may be expressed in several parameters or in only one (as, for example, in works by Schubert in *a-b-a* form, where *a* and *b* may differ mainly in terms of the mode – major as opposed to minor). In the *a-b-a* form, the opening and closing *a*'s may profess varying degrees of similarity: they may be identical (as in the *aria da capo),* similar though not identical, with a fixed change (as in sonata form), or similar with no fixed change.

(b) As to the adherence to symmetrical organization, the most common schemata in the forms of Western tonal music are *a-a* (*a-a*'), and *a-b-a.* A single repetition (exact or not), that forms the basis of 2^n, always increases directionality (for examples of adherence to symmetry and deviations from it see Figures 5 and 6 above.). On the other hand, multiple repetitions based on the additive principle reduce the overall directionality of the collection of *a*'s. Multiple repetitions may occur in relation to units on the immediate level, or in a schema or schemata that are inherent in *a*, e.g., a harmonic pattern (common in variation form), the bass line (*passacaglia*), or the "melody type" in improvisational music in non-Western cultures. In the latter the repetition always entails some change. However, there is also a precise structure *a-a-a...* found especially in polyrhythmic African music, which lacks all directionality (in accordance with its aesthetic ideal).

Symmetries may also occur on the overall level, based on the expansion of the principle of *a-b-a*. The latter represents retrograde motion in respect to the location of entire sections, but not regarding their musical essence: the closing *a* is a repeat, and not a retrograde of the opening *a*.

Common augmentations of *a-b-a* are *//:a://:b://* or *//:a://:a-b://*, where the *b* section includes in its closure a part of the *a*. Another possibility of augmentation is *a-b-c-d ... d-c-b-a* (as in Bach's *St. Matthew Passion* with respect to the texture of the various sections; in Brahms's *German Requiem,* where there is also an affinity between the themes; or in Penderecki's *Seven Gates of Jerusalem*). From the standpoint of form, this type of organization indicates inversion (*a-b . . . b-a*), but similar sections are repeated and not inverted. The question of whether the listener is able to perceive the symmetry of such large-scale compositions as Bach's *St. Matthew Passion* must remain open. An extreme case of a loose *a-b-a* form is when the opening thematic material *a* is repeated at the end of a composition, resulting in an auditory experience of "closing the circle", a frequent procedure in multimovement compositions of the nineteenth-century.

(c) As regards the level of symmetrical phenomena, when a relatively long section in tonal music is repeated *(a-a)*, the repetition – owing to the constraints of human memory – is usually without operations and relates to the entire section. The operations mainly refer to motives or themes on the immediate level (dependent on style); but also cases of repetitions with an operation on a higher, and even on the overall level may also be found.

An extreme case is the E minor Fugue in Book I of *The Well-Tempered Clavier* by Bach (Figure 7). Here the fugue falls into two equal parts (A and B in the Figure), the second one being derived from the first by the operation of contrast (a precise exchange of roles between the two hands). A and B are continuously divided into two similar units (A1, A2; B1, B2): each consists of eight measures + addition (two measures in A1, one measure in A2, and likewise in B), and each unit of eight measures is divided into two: the first four measures contain the main subject twice, and the second four – an episode. Each of the units of four measures is split into two, and the second episode – into four. Likewise, the coda is divided into two, as are the additional measures (19 and 38) that close parts A and B with parallel octaves. All in all, the division into two appears here on eight levels, including the overall level. In most cases, this division is made by operations (despite the rare case of division into two without operation, found in the above-mentioned measures with parallel octaves). A selection of various manifestations of these operations is summed-up in the following:

(1) Contrast: Between A and B; between the themes in the four measures that open the sub-units A1, A2, B1, B2; within episode 2 – the fourfold voice exchange between the upper and the lower part; between rising and falling sixths; between ascending broken chords (in most cases) and descending ones (at the beginning of the coda); between a chord and a second; between polyphony and homophony (up to unison); between melodic motives, which we shall not specify here.

(2) Shift: Between the themes; between the half episodes; between motives.

(3) Augmentation/diminution: In the motion of the triads (in eighths or sixteenth notes); the curtailment of the main theme in the coda; in the interval of the sixth – whether gapped or stepwise.

(4) Fusion/segregation: The episodes may be conceived as derived from various combinations of elements from the main theme and the countersubject (e.g., the triad, the sixth interval, etc.)

(5) Equivalence: The existence of different episodes, serving as intermediate units that cause relief from polyphonic rigor.

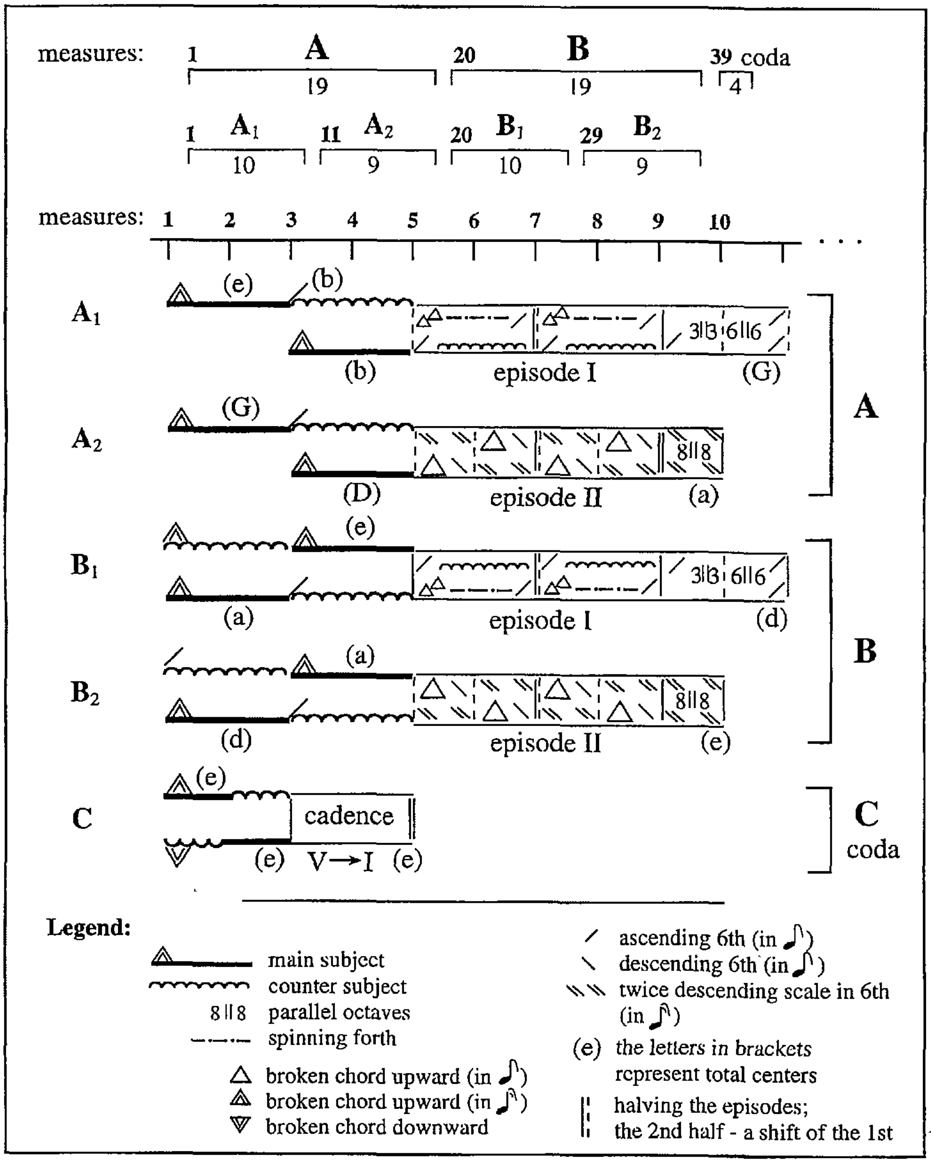


Figure 7: A Schematic Outline of the Symmetrical Two-Part Fugue in E Minor from Bach's *Well-Tempered Clavier I*· An Extreme Case of Complex Symmetrical Relations on Various Levels

It should be noted, however, that the picture gets more involved once one analyses the details of the themes and discovers many non-concurrences and uncertainty as to the boundaries of the sub-units.

Essentially, the fugue form is based on a chain of many repetitions, with operations (from all five categories) applied to entire themes. However, the overall form of the fugue is flexible, even if the general formula of *a-b-a* is obtained with respect to the tonal centers in each fugue (for another analysis of this fugue, see Werker 1922).

Another example of overall symmetrical organization is found in Bach's "Goldberg Variations" where every third variation is a canon in a gradually ascending interval (i.e., the operation of augmentation) – in addition to various other operations, such as the mirror in Variation 15 (a canon in the fifth).

All these observations contribute to the distinction between small and large-scale forms, simple and complex, and directional and non-directional ones. These distinctions follow the stylistic ideal of the various periods of music history.

F. A Summary of the Symmetrical Phenomena in Six Eras of Western Music

We attempt to summarize here both aspects of the subject: on the one hand, the main characteristics of symmetry, and on the other hand, the characteristics of styles in the different eras (guided by different ideals) by means of the symmetry variables.

F1. The Variables of Symmetry Relevant to the Relationship with the Stylistic Ideal:

i. The basic tonal system which serves as raw material from which the various schemata are derived: for example, the scale system of seven (with minimal asymmetry), the system of twelve (with maximum equality and without hierarchy), or both, distinguishable from each other to varying degrees.

ii. Types of schemata in the raw material (scales, chords, rhythmic patterns, etc.): The quantity of types, manifestation of binarity, degree of distinction between schemata in the same category, and the hierarchy in each schema (the hierarchy among the notes of the scale, between chords in harmonic phrases, between beats in a meter, etc.).

iii. The range of occurrence of the various parameters (a characteristic of texture) with regard to register, ambitus, and degree of change, and with attention to the U-function (in the optimal domain, which is a norm, as opposed to deviations in the direction of the extremes).

iv. Concurrence/nonconcurrence regarding the simultaneous occurrence of different schemata or units in various parameters; effects the contribution of symmetry to excitement or calmness.

v. Cases of balance or imbalance (balance may itself be considered part of the ideal), with attention to the range of occurrence: A balance between the extremes is different from that within the optimal range.

vi. Types of curves of change (another characteristic of texture that contributes to predictability or uncertainty). The prominent representative of the first group is the convex curve; the definitive representatives of the second group are the concave, zigzag, and flat-line curves.

vii. Reference to the directional symmetrical schema 2^n.

viii. Types of structures and forms on different levels in terms of symmetry, types of directionality and complexity, and the quantity and range of options for various realizations (in non-Western cultures we find forms that restrict the range of realizations; while sonata form, for example, may be realized in various styles).

ix. Operations – manifestation of symmetry in the compositional rules (one of the conditions for a complex, directional super-structures): Quantity, kind and function.

F2. A Summary of Historical Eras in Western Music According to the variables of Symmetry

Early Middle Ages (9th –13th centuries): monophonic music (Gregorian chant)

Ideal: Simple, calm; maximum adherence to the text and to the religious context.

i. Basic tonal system: The diatonic system (of seven), with minimal asymmetry.

ii. Schemata in the raw material: Scales (of modes – more than two); the intervals are limited in size and are all melodic; rhythmically and metrically undefined.

iii. Range of occurrence: In the optimal range of the vocal diapason; the musical instruments are technically not well developed.

iv. Concurrence/nonconcurrence: not relevant to the simple system.

v. Balance: Exists mainly through the melodic curve, particularly for the parameter of pitch.

vi. Curves of change: Convex curve. A paradigmatic example of symmetry is found in psalmody, where the second half of the verse is the opposite of the first.

vii. 2^n: None.

viii. Structures and Forms: Simple and short; dependent on the text and the liturgical structure.

ix. Operations: Few and extremely simple. Though there are some repetitions, there are mainly contrasts between ascending/descending tones, solo/chorus, syllabic/melismatic, step/skip.

Late Renaissance: religious vocal music (16th century)

Ideal: Calm and balance; directionality chiefly regarding the unit of the verse, with full concurrence between the musical and textual verses.

i. Basic tonal system: Mainly a single system – the diatonic system, with few chromatic notes.

ii. Schemata in the raw material: scales (modes – more than two); chords (with no hierarchy between strong and weak progressions); melodic and harmonic intervals that are categorized meticulously and are subject to strong restrictions on separate appearances in various contexts; consonance-dissonance; homophonic or polyphonic texture; structures depend largely on the text.

iii. Range of occurrence: Maximum adherence to the optimal range of change, ambitus, and register in all relevant parameters (durations and intervals).

iv. Concurrence/nonconcurrence: Little nonconcurrence.

v. Balance: Maximum in the optimal range, in a single melodic line, and between melodic lines.

vi. Curves of change: Almost entirely convex for the parameters of pitch and duration on the immediate and phrase levels.

vii. 2^n: Hardly exists at all in vocal music, but is found in dance music.

viii. Structures and Forms: Simple, mostly text-dependent; in a latent manner, using proportions of the durations of movements and sections, as in visual art (e.g., golden section or perfect symmetry manifested in a convex curve).

ix. Operations: Few, mostly of the category of "soft" contrast regarding the parameters of pitch (e.g., ascending/descending, consonance/dissonance, step/skip) and duration; appear mainly on the immediate level and contribute to calm.

Baroque (17th–18th centuries)

Ideal: Multiple styles; emphasis on excitement; unclear directionality with maximum complexity ("an oddly-shaped pearl"; Pluche 1770, in Palisca 1968); related to the religious and theatrical framework.

i. Basic tonal system consists of the two fundamental Western systems – the diatonic (with seven notes) and the full chromatic system (with twelve notes).

ii. Schemata in the raw material: Scales (chiefly major/minor); harmony (with a noticeable preference for strong steps, but no strong hierarchy between degrees); expansion of the complexity of the polyphonic texture, homophony, and the ranges of overlap between them. A prominent example of the lack of harmonic hierarchy (equality between progressions and harmonic degrees) can be seen in the complex chains composed of sequential links (cf. David and Mendel 1946, pp. 394-398). The best known of these is the sequence of descending fifths.

iii. Range of occurrence: Numerous deviations from the optimum; notable polarity of both lack of change and salient change. For example, there may be a polar ambitus and lack of balance between instrumental groups (attested to by the development of the concerto), as well as sudden changes, as in recitative music or certain fantasies (e.g., Bach's "Chromatic Fantasy"). In contrast, we may find a lack of change from various standpoints, e.g., on the immediate level (sequence, multiple repetitions of a motive or rhythm), on the level of the movement (uniformity of texture), and on the level of the overall piece (lack of change in tonality and duration of the movements).

iv. Concurrence/nonconcurrence: Extensive nonconcurrence between various parameters (especially salient in Bach's works).

v. Balance: Numerous imbalances due to adherence to the ends of the U-function in many parameters (cf. Hargreaves 1986).

vi. Curves of change: All sorts and on all levels; in Bach's works we find – in addition to convex curves – concave curves and zigzags from various standpoints, both on the immediate level and in the relationship between movements in a complex piece, (e.g., concave and zigzag curves for the textures and lengths of movements in Bach's *St. Matthew Passion*).

vii. 2^n: Appears at times, alongside a flexible schema with three links and various expansions of it.

viii. Structures and Forms: Numerous types of asymmetrical and symmetrical structures with unclear directionality. Some uncommon but interesting symmetrical structures in Bach's music: *a-b-a* in a fugue (e.g., Duet in F Major, BWV 803; in the first movement of the Violin Concerto in E Major, BWV 1042); *a-a* in a fugue (the E minor Fugue from Book I of the *Well-Tempered Clavier*, where the second *a* is obtained from the first by an operation, and the two are equal in size (see Figure 7 above); the second "Kyrie" from the *B Minor Mass*). There are also symmetrical structures that are expansions of *a-b-a* or retrograde, such as the overall structure of the *St. Matthew Passion.* Such symmetry with respect to texture is found, for example, in the Credo of the B Minor Mass and in Cantata BWV 4; with respect to tonal organization it is found in Cantata BWV 78. In Bach's compositions we may also find specific, one-time superstructures, such as in the *Goldberg Variations* and *The Art of the Fugue*.

ix. Operations: Numerous operations of all types; may act on entire sections.

Classical Era (18th century)

Ideal: Clarity; clear directionality on various levels ("polished diamond"); the peak of autonomous music, separated from the extra-musical world (with exceptions, such as opera).

i. Basic tonal system: The two systems of 7 and 12 notes, with maximum separation between them (the diatonic and chromatic systems).

ii. Schemata in the raw material: Clear binary separation between the major and minor scales; maximum hierarchy among the harmonic schemata with respect to the degrees and the harmonic progressions; maximum separation between meter and rhythm; mainly simple meters (duple or triple); schemata highlighted by means of textural changes.

iii. Range of occurrence: Both adherence to the optimum and deviations from it (particularly noticeable in Beethoven's works); an optimum of well-defined categories of musical instruments to highlight the structure. For the first time meticulous attention is paid to the orchestration.

iv. Concurrence/nonconcurrence: Multiple cases of concurrence in various parameters and on various levels, and in general between natural and learned schemata. Therefore, cases of nonconcurrence are the more salient (Beethoven).

v. Balance: Prevalent, but may occur at the extremes, which are moving away from the optimum (Beethoven).

vi. Curves of change: Limited in comparison with the Baroque era.

vii. 2^n: Serves as a point of departure.

viii. Structures and Forms: Mainly sonata form, in which symmetry is salient on various levels, even in the order of movements producing *a-b-a* in terms of the tonal centers.[23]

ix. Operations: Numerous manifestations in all types of operations, in all parameters, and on all levels; particularly prominent is the operation of contrast, which occurs suddenly and not gradually. All the operations contribute to logical structure and complexity.

Romanticism (19th century)

Ideal: Excitement; unclear directionality; a close relationship with the extra-musical world

i. Basic tonal system: Blurring of the systems of seven and twelve.

ii. Schemata in the raw material: As in the Classical era, but with a blurring of meters, of major/minor scales, of harmonic degrees, of strong and weak harmonic progressions, of the identity of chords, and of the close ties between harmony and melody.

iii. Range of occurrence: Adherence to extreme deviations from the norm (in two directions: that of "more" and that of "less") in all the parameters and in respect to all kinds of occurrences – such as amount of change (no change/ large change) ambitus, durations of the pieces and movements, etc.; a multitude of musical instruments with no well-defined categories, which emphasize the parameter of timbre rather than of structure.

iv. Concurrence/nonconcurrence: Numerous cases of nonconcurrence of all sorts.

v. Balance: Numerous violations.

vi. Curves of change: Numerous manifestations. Especially salient are the concave, zigzag, and flat curves, which produce excitement on all levels.

vii. 2^n: On the one hand, numerous exceptions; on the other hand, adherence, with deviations in the other parameters (salient in Chopin's works).

viii. Structures and Forms: Numerous types of symmetrical and asymmetrical structures in small and large-scale compositions and with clear and unclear directionality; *a-b-a* form prevails.

The functions of the components of sonata form (exposition – development – recapitulation) are blurred by various means (e.g., by adopting the development principle in the exposition).

ix. Operations: As in the Classical era, but with an expansion of the operations of shift and of equivalence, and with manyfold manifestations of operations contributing to states of excitement.

20^{th} century

Ideal: Multiple ideals, including relationships with the extramusical world. Prominent ideals are non-directionality, frozen time (= "space"), and the use of many non-Western principles.

i. Basic system: With respect to pitch, mainly the system of twelve (dodecaphony), but also other divisions of the octave. Timbre, texture, and interim states between them are of prime importance (Cohen—Dubnov 1997); by their very nature they arouse extramusical associations and contribute to a focus on momentary events.

ii. Schemata in the raw material: Usually a group of compositions does not share any specific, predetermined learned schemata, but there are schema types, such as serial rows (in dodecaphonic music), types of pitch sets, types of modes (Messiaen), and types of textures and timbres. The composer may or may not adhere to schema types.

iii. Range of occurrence: This factor, which is the main characteristic of texture, is of the greatest importance and appears in an enormous variety of ways in all parameters.

iv. Concurrence/nonconcurrence: All sorts, in all parameters; not always relevant.

v. Balance: The balance is generally broken in various ways.

vi. Curves of change: Of all sorts.

vii. 2^n: Sometimes exists, sometimes is irrelevant.

viii. Structures and Forms: Owing to the not well-defined concepts of units and forms, predetermined structures are rare. There are numerous plans for all parameters, with varying degrees of complexity and directionality, including symmetrical structures (e.g., the golden section in Bartók's works). The mathematical organization of a musical composition, especially in computerized music, has reached the peak of complexity, but we do not as yet know how far it is significant for our capacity as listeners.

ix. Operations: The most important factor in organization, replacing the learned schemata of tonal music; especially prominent in computerized music. The operations occur with respect to all possible parameters. Interestingly, one can demarcate them by means of the same five basic cognitive categories mentioned throughout this paper (contrast, shift, expansion/contraction, fusion/division, equivalence).

CONCLUSION

As we have seen, all music has some sort of symmetry, which can be regarded as a specific case of symmetry in human life and natural phenomena. Any characterization of structural organization must relate at least in part to the presence or absence of various forms of symmetry. Moreover, as we tried to show, the various manifestations of symmetry reflect aesthetic ideals that take shape against the backdrop of numerous extramusical factors.

Clearly, the ideal is what determines (consciously or unconsciously) the selection of types of symmetry (and not vice versa). However, because the selection takes cognitive constraints into consideration, there is also an opposite relationship according to which we can draw conclusions about the ideal that guided the selection of symmetry, by observation of the laws of symmetry in a particular style.

In this study we have attempted to classify the realizations of symmetry in terms of its reciprocal relationship with the stylistic ideal. We have done so by means of theoretical considerations and analysis of examples of various styles, governed by different ideals.

We believe that the present subject deserves further study, to investigate in more depth the nature of symmetry in the music of various eras in the West, and to extend such in-depth research into non-Western musical cultures and the inter-relationship with other art forms.

ENDNOTES

[1] In some specific cases symmetrical structure, expressed mainly by the number of measures in parts of a composition, is so exact and complex that it points to a careful and conscious planning by the composer (Powel 1979; Guletsky 1995)

[2] We exclude here Baroque musical theoretical writings applying principles, concepts and terms from the discipline of rhetoric, especially those categorized as "Figures of melodic repetition" (Buelow 1980).

[3] For a recent similar definition of symmetry see Kempf 1996.

[4] The frequency ratios of the intervals in the major triad are 4: 5: 6, and in the minor – 1/4 : 1/5 : 1/6.

[5] An interesting case is that of Schillinger (1948), who tried to formulate an abundance of artificial scales and rhythms that should serve as foundations of music The very concept of ISIS is, of course, a notable expression of this comprehensive notion

[6] Schemata based on the vertical parameter (pitch) require the invention of instruments, in order to enable exact measurements. Let us stress that, although these schemata are learned ones, they are not arbitrary but derive from the aesthetic ideal and from cognitive constraints. Some of them have even been explained and formulated mathematically (see note 14 below).

[7] Excitement may be caused, for instance, by a sudden change in contrast to a gradual one, by deviation from the norm, and other natural phenomena pertaining to the parameter of texture. Let us emphasize again, that we believe these to be universal laws, manifested even in birdcalls (Cohen 1983). Moreover, some of these principles, as well as others not specified above (e.g., rareness, ambiguity, etc.), may be applied not only to

basic parameters, but also to organizations on different levels, namely, learned and natural schemata. From the vast literature on that subject let us mention Bolinger 1972; Clynes 1982; Howel, Cross and West 1985; McAdams 1987; Sloboda 1991; Gabrielson and Juslin 1996; Cohen and Inbar 2001.

[8] An illuminating description of 2^n from the eighteenth century is given by Comte de la Cépéde (1785; in le Huray and Day, 1981, p. 181):

> Symmetry [...] requires that the corresponding sections of a composition shall have the same structure and the same number of components; it is essential at least that the music is composed in such a way that the ear can easily connect and divide phrases and thus discover parallel arrangements and groupings of sections. These goals have only been attained [..] when the length of phrases has been based on the number two and its multiples. It is agreed therefore that all phrases must be constructed of two, four, eight or sixteen measures and so on. The ear is easily able then to divide the pieces into equal phrases of four or two measures, or to build four bar phrases from successions of two, on the assumption that the ear wishes to be guided by the laws of song or of symmetry.

[9] This concept (Cohen 1971; Cohen and Wagner 2000) was indirectly referred to by many scholars, who used a different terminology, such as non-congruence, ambiguity (Cooper and Meyer 1960), concinnity (LaRue 1970), conflict (Schachter 1970)

[10] It should be noted, that most of the discussions on transformation (e.g., Reti 1951; Real 1970; Rosen 1980, Cone 1987; Kamien 1990; Appel 1992; Conner 1994) do not associate it with symmetry, while only few scholars refer to transformation as a manifestation of symmetry (e.g., Solomon 1973; Wilson 1986; Cohen 1996). Other scholars (e.g., Lendvai 1977) discuss symmetry, but without reference to transformation. The close tie between the two was detected only recently.

[11] The concept of directionality in its general significance and without sub-divisions appears in the theoretical literature under different terms, such as progression, processive forms, flow, goal, and approaching.

[12] It should be emphasized, that artificial symmetrical organization that ignores psychoacoustic and cognitive constraints loses its meaning for the listener. Such attempts have been made in modern times (e g., by Boulez, see above), and the auditory result was complete chaos.

[13] This idea – the von Forster Theory – was also expressed mathematically (Koppel et al., 1987; Shanon and Atlan 1990).

[14] The diatonic system may be obtained from the "cyclic generator" (octave and fifth), and the interval system derived from it meets the conditions of maximum coherence and efficiency. The priority of the Western system has been proved also by its comparison with hypothetical systems (Balzano 1980; Gouldin 1983; Agmon 1989).

[15] Modulations are also conspicuous in Arab music, but only on the immediate level, and especially between different maqāmāt rather than between the same maqāmāt with different tonal centers.

[16] In Book III, chapter 27 of his *Instituzioni armonichi,* Zarlino states:

> And although every composition, every counterpoint, and in a word every harmony is made up primarily and principally of consonances, dissonances are used secondarily and incidentally for the sake of greater beauty and elegance. (Strunk 1950, 231-232.)

[17] For the "melting" of the dominant, see "Der Wegweiser", mm. 12-13, 34-37, 47-48. Enharmonic interpretations of the diminished seventh chords, leading to an equi-distant chain of tonal centers in minor thirds appear in the same song, mm. 57-67. Both devices illustrate Schubert's subtle textual interpretation: the road leads to a place from which nobody has ever returned, and where there is no significance of directionality of any sort.

Enharmonic interpretations of the augmented triad, creating a series of tonal centers in chains of descending major thirds may be found in the F Minor piano fantasy for four hands, mm 65- 91.

The tonal centers (all in minor) are: F - D^b - A - F.

[18] Good examples are Chopin's Prelude No. 4 in E Minor, the Mazurka in A Minor, Op. 17, No. 4, and many other mazurkas.

[19] For a detailed discussion of a paradigmatic example, Webern's Concerto for nine instruments, Op. 24, see Bailey 1996.

[20] Many music theorists also believe that musical form and content cannot be separated (e.g., Randel 1986). We believe that the two can be separated, although musical contents often influences the form. Thus, for

example, tonal centers are decisive in shaping sonata form, whose overall design is *a-b-a*. The degree to which content intervenes in form may be considered one of the characteristics of style.

[21] Other variables of formal organization are conspicuous/not conspicuous, and frequent/rare, and these, together with different/similar, effect the reaction of our brain waves, as attested by ERP (Event-Related Brain Potential) research (Cohen—Granot 1995).

[22] The repetition may be imprecise (*a-a'*), where *a* and *a'* differ only in their closure; examples are found in the "periodic phrase" of the Classical era. Repetition may also occur with the addition of embellishments (augmentation), or even with an operation of any kind (although this is uncommon)

[23] An interesting example of latent symmetry in harmonic progressions can be found in the first movement of Piano Sonata No. 42, Hob XVI/27 in G Major by Haydn, and in the development section of the first movement of Piano Sonata No. 5 Op. 10 in C Minor by Beethoven. The tonal centers of the modulations are arranged here in a two-way symmetrical series, where the retrograde part is extremely short:

Haydn: G - D - a – e // e - a - D - G

Beethoven: c - f - b^b - D^b // D^b - b^b - f - c

Another example is obtained when a harmonic schema composed of a chain of sequential links appears also in retrograde Thus, for instance, the retrograde schema of the "Pachelbel Canon",

I - V - VI - V - VI - III - IV - I, is I - IV - III - VI - V - VI - V - I,

and both are found in Beethoven's *Egmont* Overture.

REFERENCES

Agmon, E. (1989) A mathematical model of the diatonic system. *Journal of Music Theory* 33(1), 1–25.

Appel, R (1993) A cognitive model for the procedural aspects of the creative process in music with emphasis on transformation processes in Western music, Ph.D. dissertation, University of Haifa.

Arom, S. (1998) Description and modelization of traditional music: the dialectic between indigenous nomenclature and structural analysis In Braun, J. and Sharvit, U., eds., *Studies in Socio-Musical Sciences*, Ramat-Gan: Bar-Ilan University Press.

Atlan, H. (1981) Hierarchical self-organization in living systems: Noise and meaning. In Zeleny, M., ed, *Autopoiesis: A Theory of the Living Organization*. North Holland

Atlan, H (1987) Self creation of meaning, *Physica Scripta* 36, 563–576.

Avital, T. (1996) The connectivity principle of art, *Symmetry: Culture and Science* 7(1), 27-50

Avital, T. (1998) Mindprints: the structural shadows of mind reality?, *Symmetry: Culture and Science* 9(1), 47-76.

Babbit, M. (1960) Twelve-tone invariants as compositional determinants, *The Musical Quarterly* 46, 246–259

Bailey, K. (1996) Symmetry as Nemesis: Webern and the first movement of the concerto, opus 24, *Journal of Music Theory* 40, 245-310.

Balzano, G. J. (1980) The group-theoretic description of 12–fold and microtonal pitch systems, *Computer Music Journal* 4(4), 66–84.

Bolinger, D., ed. (1972) *Intonation: Selected readings*, Penguin Education.

Buelow, G J. (1980) Rhetoric and music, *The New Grove Dictionary of Music and Musicians* 15, 791-803.

Bukofzer, M. (1947) *Music in the Baroque era*, New York. Norton.

Clynes, M , ed (1982) *Music, mind and brain*, New York: Plenum Press.

Cohen, D. (1971) Palestrina counterpoint – a musical expression of unexcited speech, *Journal of Music Theory* 15, 68–111.

Cohen, D. (1983) Birdcalls and the rules of Palestrina counterpoint· towards the discovery of universal qualities in vocal expression, *Israel Studies in Musicology* 3, 96–123.

Cohen, D. (1994) Directionality and complexity in music, *Musikometrica* 6, 27–77.

Cohen, D (1996) Aspects of symmetry in music, *Proceedings of the 2nd International Conference of ISIS*, Washington.

Cohen, D. and Dubnov, S (1997) Gestalt principles in musical texture. In Leman, M., ed., *Music, Gestalt and Computing: Studies in Systematic and Cognitive Musicology*, Berlin: Springer-Verlag.

Cohen, D. and Granot, R. (1995) Constant and variable influences on stages of musical activities: Research based on experiments using behavioral and electrophysiological indices, *Journal of New Music Research* 24, 197–229.

Cohen, D and Inbar, E (2001) Musical imagery as related to schemata of emotional expression in music and on the prosodic level of speech, in Godøy, R and Jørgensen, H., eds., *Musical Imagery*, pp 91–113.

Cohen, D. and Wagner, N. (2000) Concurrence and non-concurrence between learned and natural schemata. *New Music Research* 29(1), 23–36.

Cone, E. T. (1987) On derivation, syntax and rhetoric, *Music Analysis* 6(3).

Conner, T. A. (1994) Techniques for manipulating the internal structure of mosaics, in *Theory Only* 12(7–8), 15–34.

Cooper, G. and Meyer, L. B. (1960) *The rhythmic structure of music*, Chicago: University of Chicago Press

Danielou, A (1943) *Introduction to the study of musical scales*, London: The Indian Society

David, H. T., and Mendel, A., eds (1946) *The Bach reader*, London: J. M. Dent.

Gabrielsson, A. and Juslin, P. (1996) Emotional expression in music performance: Between the performer's intention and the listeners' experience, *Psychology of Music* 24(1).

Gauldin, R. (1983) The cycle-7 complex: Relations of diatonic set theory to the evolution of ancient tonal system, *Music Theory Spectrum* 5, 39–55.

Guletsky, I. (1995) Proportion in Palestrina's masses, in *Proceedings of the Third International Conference of Study of "Palestrina e L'Europa"*, Palestrina, October 1994.

Hargreaves, D. J. (1986) *The developmental psychology of music*, Cambridge University Press

Howell, P., Cross, J. and West, R., eds. (1985) *Musical structure and cognition*, London and New York. Academic Press.

Huron, D. (1996) On the kinematics of melodic contour: deceleration, declination and arch-trajectories in vocal phrases, in *Proceedings of the 4th ICMPC*, Montreal.

Kamien, R (1990) The slow introduction of Mozart Symphony no 38 in D, K. 504 ('Prague'): A possible model for the slow introduction of Beethoven's Symphony no. 2 in D, op. 36, *Israel Studies in Musicology* 5, 113–131.

Kempf, D (1996) What is symmetry in music?, *International Review of the Aesthetics and Sociology of Music* 27, 155-165.

Koch, H. C (1983, 1782) *Introductory essay on composition: The mechanical rules of melody, sections 3 and 4* (translated from German by Baker, N. K), New Haven: Yale University Press.

Koppel, M., Atlan, H. and Dupoy, J. P. (1987) Triviality and alienation in systems: Proof of Von Forster's conjecture, *International Journal of General Systems*.

LaRue, J. (1973) *Guidelines for style analysis*, New York: Norton.

Le Huray, P., and Day, J., eds. (1981) *Music and aesthetics in the eighteenth and early-nineteenth* centuries, Cambridge University Press.

Leichtentritt, H. (1951) *Musical form*, Cambridge, Mass.: Harvard University Press.

Lendvai, E. (1972) Duality and synthesis in the music of Béla Bartók, in Kepes, G., ed., *Modules, Symmetry, Proportion*, 174–193, London: Studio Vista

Leyton, M. (1992) *Symmetry, causality, mind*, Cambridge, Mass.. MIT Press.

McAdams, S., ed. (1987) *Music and psychology: A mutual regard*, London and New York Harwood Academic Publishers.

Nettl, B. (1964) *Theory and method in ethnomusicology*, London and New York. Free Press of Glencoe, Collier-MacMillan.

Oppenheim, D U. (1989) DMix: An environment for composition, in *The Proceedings of International Computer Music Conference*, Columbus, Ohio.

Palisca, C. V. (1968) *Baroque music,* Englewood Cliffs, New Jersey: Prentice-Hall.

Perle, G (1974) *Serial composition and atonality*, Berkeley· University of California Press.

Powel, N W. (1979) Fibonacci and the golden mean: Rabbits, rumbas, and rondeaux, *Journal of Music Theory* 23(2), 227–273.

Powers, H (1976) The structure of musical meaning: A view from Banaras, *Perspectives of New Music* 14(2) and 15, 308–334.

Rameau, J. P. (1722) *Traité de l'harmonie*, Paris: Ballard, English translation (1965) New York: Broude Brothers.

Randel, D.M, ed. (1986) *The New Harvard Dictionary of Music*, Cambridge, Mass.: Harvard University Press.

Ratner, L (1980) *Classic Music: Expression, form, and style,* New York: Schirmer Books.

Reale, P V. (1970) The process of multivalent thematic transformation, Ph.D dissertation, University of Virginia.

Reti, R (1951) *The thematic process in music,* New York: MacMillan.

Rosen, C., Influence, plagiarism and inspiration, *Nineteenth Century Music* 4(2), 87–100.

Sachs, C. (1946) *The commonwealth of art,* New York: Norton.

Schachter, C (1976) Rhythm and linear analysis, *The Music Forum* 4, 281–334

Schillinger, J (1948) *The mathematical basis of arts*, New York: Philosophical Library.

Shanon, B. and Atlan, H. (1990) Von Forster's theory: Semantic application, *New Ideas in Psychology* 9, 81-90.

Sloboda, J. (1991) Music structure and emotional response: Some empirical findings, *Psychology of Music* 19(2), 110-120.

Solomon, L. (1973) New symmetric transformations, *Perspective of New Music* 11/12, 257-264.

Strunk, O. (1950) *Source readings in music history*, New York: Norton.

Tversky, A (1977) Features of similarity, *Psychological Review* 84(4), 327–352.

Vieru, A. (1993) *The book of modes*, Bucharest: Editura muzicola A uniunii compozitorilor ci muzicologilor din Romania

Werker, W (1922) *Studien uber die Symmetrie im Bau der Fugen*, pp. 268–280, Leipzig.

Weyl, H. (1952) *Symmetry*, Princeton, N.J.: Princeton University Press.

Wilkinson, D. (1988) Symmetry in art and nature, *Queen's Quarterly* 95(1), 107–132.

Wilson, D. (1986) Symmetry and its love-hate role in music, *Comp. & Maths* 128(1/2), 101–112.

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SYMMETRY: SCIENCE AND ART

THE EMERGENCE OF SYMMETRY CONCEPTS BY THE WAY OF THE STUDY OF CRYSTALS (1600-1900)

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INTRODUCTION

The word symmetry has, in the dictionaries, two types of definition. The first corresponds to the ideas of "harmony" or "just or due proportion." In the beginning of the period that we are studying, this was the only meaning of the word and it was used by artists. The second meaning corresponds to set of isometries that are geometrical motions such as rotations, translations and reflexions. This last mathematical meaning was issued from the first one during these three hundred years. The most important period for this transformation was the beginning of the 19th century. The choice of 1900 to close our historical description is due to two important scientific results published between 1890 and 1900: E. Fedorov (1890) and A. Schoenflies (1891) published almost independently the list of the 230 "space groups of symmetry" which are a model of formalization of symmetry properties of objects in our three dimensional space. Then in 1894, P. Curie introduced symmetry in physics in a famous paper. The 20th century would then be the century of the use of symmetry concept in many branches of modern science.

During three centuries, one of the essential ways which allowed the transformation from the original sense of harmony to that of set of certain geometrical transformations was the study of crystals.

1 OBSERVATION OF CRYSTALS

Among many curious features presented by nature, crystals with their planar faces limited by rectilinear edges, held the scientists' attention. Some of them which will be presented here attempted to establish a relation between the crystal shape and a hypothetical microscopic model of solid matter: shapes and organization of minute particles could explain the observed macroscopic shape of crystals characterized by certain regularities.

This was not the only question formulated by these scientists, but it was a central one. It was also common with the scientists who developed theories of structure of matter during 17th, 18th and 19th century from which theory of symmetry was an essential component, as will be seen subsequently.

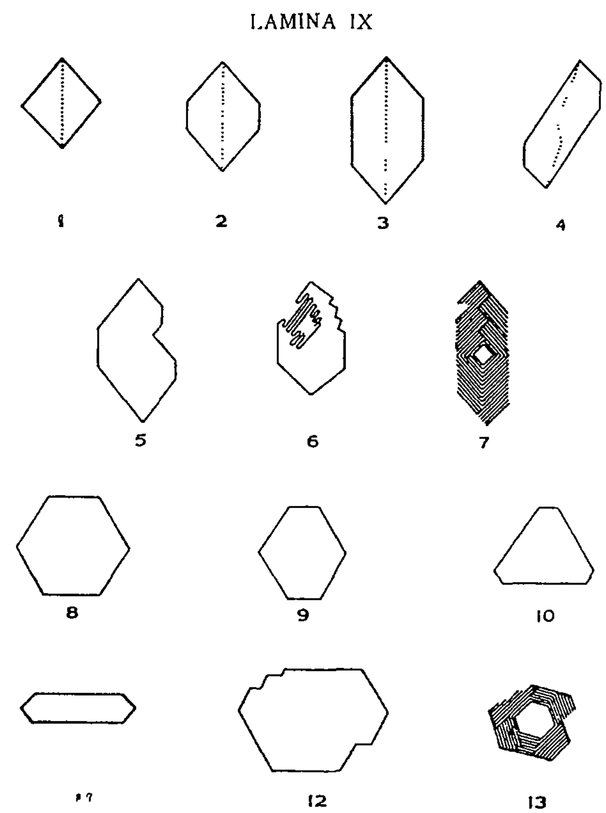

Figure 1: Plate *n*° IX of the Prodomus by N. Steno

The first usually cited scientist is Johann Kepler. In his small book *Strena Seu de Nive Sexangula* (1611), he presented observations of snowflakes with their characteristic angle of sixty degrees between rods that come from a common center. He studied the different possible packing of minute spheres of ice in an attempt to explain the hexagonal regularity observed on snowflakes. He did not succeed in explaining the shapes of snowflakes on a purely geometric basis. But its intellectual process would prove fruitful in the future.

In 1669 Nicolas Steno published the summary of an ambitious opus in which, among other things, he drew sections of quartz crystals cut up into different directions, and he measured the angles of the polygons of these figures. He pointed out the constancy of these angles in the case of quartz (Figure 1).

In the same year, 1669, Erasmus Bartholinus published a paper dedicated to Iceland spar: some beautiful crystals of calcite had been brought back from this island the year before. He discovered the double refraction of light and named the two types of rays ordinary and extraordinary. First, he observed the rhombohedral shape of these crystals and also that cleavage retained this form. He characterized this geometrical form by measurements of ordinary and dihedral angles.

Christiaan Huyghens took up again this study in his Traité de la Lumière published in 1690. The double refraction of Iceland spar he analyzed and explained allowed him to propose his theory of light as a vibration. He described more precisely the rhombohedron shape of these crystals and proposed a model of *ellipsoïdal* particles as constituants (Figure 2). Then he could explain the external shape and cleavage properties. This very deep intuition had been understood and supported only 150 years later.

These examples, taken out among many others as R. Hooke, D. Guglielmini, show us the process and the results obtained on these subjects in the beginning of the 18^{th} century: geometrical regularities (such as angles of 60°, 90°) were observed on minerals. However attempts to explain these regularities from microscopic theories of structure of solid matter failed.

Nevertheless, from our point of view, i.e., the progressive elaboration of symmetry concepts, the regularities observed on the shape of different crystals constitute an experimental fact which will be used later in the definition of "homologues" or "identical" parts of crystal as edges or vertices. And then, it constitutes an important step in this history.

sels, et de celle du sucre, l'on trouve d'autres angles solides, avec des surfaces parfaitament plattes. La neige menue tombe presque tousjours formée en petites estoiles à 6 pointes, et quelques fois en hexagones dont les costez sont droits. Et j'ay souvent observé, au dedans de l'eau qui commence à se geler, une maniere de feuilles

plattes et deliées de glace, dont la raye du milieu jette des branches inclinées d'un angle de 60 degrez. Toutes ces choses meritent d'estre recherchées soigneusement, pour reconnoitre comment et par quel artifice la nature y opere. Mais ce n'est pas maintenant mon dessein de traiter entierement cette matiere. Il semble qu'en general la regularité, qui se trouve dans ces productions, vient de l'arrangemet des petites particules invisibles et égales dont elles sont composées. Et pour venir à nostre Cristal d'Islande, je dis que s'il y avoit une piramide comme ABCD, composée de petits corpuscules ronds, non pas spheriques, mais spheroides plats, tels que se feroient par la conversion de cette ellipse GH sur son petit diametre EF; dont la proportion au grand est fort prés celle de 1 à la racine quarrée de 8. Je dis donc que l'angle solide de la pointe D, seroit égal à l'angle obtus et equilateral de ce Cristal. Je dis de plus, si

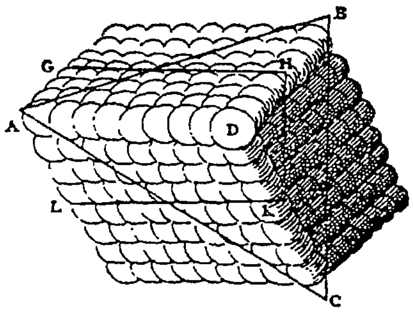

Figure 2: Extract from the Traité de la Lumière by C. Huyghens

Throughout the 18th century, almost up to its end, the type of questioning of Kepler or Huyghens upon crystals was given up. Using our categories, one can say that Physics and Natural sciences were split off. Naturalists who studied minerals were interested essentially in the questions of their classification and of their origin. As regards classification, it seemed to many people that the external shape of crystals was not a relevant criterion: one mineral as calcite for example may take on various shapes; different minerals appear to have the same shape, the cubic one, for example, as rock salt and pyrites. Buffon can be held up as an example of this position.

In contrast, C. Linneaus reproduced the shapes of about forty minerals in the plates of his Systema Naturae (1768). It demonstrates he thought that shape could be an interesting feature. Thus, he showed the way to J. B. Romé de Lisle who accomplished a decisive step in the observation of the geometrical regularities of the shapes of crystals.

2 T. BERGMAN, J-B ROME DE LISLE, R-J HAÜY. BIRTH OF CRYSTALLOGRAPHY

The word crystallography was introduced by M. Cappeller in the beginning of the 18^{th} century. But the crystallographic science was made possible only when a microscopic theory of structure was postulated the predictions of which could be compared with experiments. This foundation can be attributed to R-J Haüy. However, the role of two precursors must be described before.

J-B Romé de Lisle established the catalogues of several mineralogic collections and then wrote *Essai de Cristallographie* (1772) which was revised and published under the title *Cristallographie* in 1783. The number of minerals described in this last book reaches 400. Two important ideas must be pointed out. The first one is the process thanks to which a crystal "habit" can be described: a "primitive form" is truncated on its edges or on its vertices by little planes in order to obtain the observed crystal shape. This process is seen as an intellectual operation and not as an operation of Nature. Classification of crystals based on the symmetry of their primitive form thus became possible. The attempts of Romé de Lisle to classify crystals in this way show some errors. The second essential point in included in the first law of crystallography, named the law of constancy of interfacial angles. It appears in his last book. This law extends the observations of Steno or Huyghens made on certain crystal species. It says that dihedral angles between crystal faces are the only pertinent parameters allowing a quantitative description of the external forms of a crystalline species. Areas or shapes of crystal faces are not such parameters.

One can remember that A. Carangeot, Romé's assistant who had to make crystal models drew the first *goniometer* and observed the equality of dihedral angles measured on real crystals. Then Romé verified this fact and could publish the law under a generalized form. It can be noticed than Romé did not want to build a theory, because he thought that it would be too hypothetical at this time.

Torbem Bergman, a Swedish mineralogist and chemist, explored the idea of determining the primitive form of crystals, using cleavage. He could prove than one habit of calcite, the scalenohedron, was then related to another, the obtuse rhombohedron. But, his attempts to generalize this approach failed off and he abandoned this research.

All the credit of the establishment of an ambitious and general theory of crystal structure is due to R-J. Haüy which is thus named "the father of crystallography."

The bases of his theory were laid out as early as 1781, 1782; a first synthesis was published in 1784 and until his death in 1822, this theory did not evolve in a significant manner. The easy cleavages of a crystal of any shape allow to obtain the primitive form of its nucleus, a polyhedral form which corresponds to that of the chemical unit of this crystal, named *molécule intégrante*. These building blocks of the crystal present the same polyhedral shape but they are not visible to the naked eye, due to their small size. Juxtaposition of these small units permits to obtain the real macroscopic form of the crystal: the crystal faces are smooth or stepped according to their own "law of decrement" which can be characterized by integer numbers, characteristic of each face.

This model was illustrated by figures as that of Figure 3, which played an important role in the spreading and understanding of this theoretical model. The important point is the possibility of comparison between the experiment – precise geometric characterization of crystal shape – and the theory – hypothetical form of the polyhedron of the *molécule intégrante*.

At the origin of substantial progress in physics, mathematics and chemistry, this theory suffered from some shortcomings. Two of these will be enunciated here: the possibility for the primitive form to be an octahedron or a tetrahedron leads to difficulties in order to fill space. Then, the theory was complicated by the introduction of the *molécules soustractives*. The consistency of the theory was significantly reduced. A second difficulty arose with the identity claimed between geometrical unit and chemical unit. Phenomena such as polymorphism and isomorphism recognized during Haüy's life were inconsistent with this element of the theory.

Nevertheless, Haüy introduced in the study of crystals the relevant part of mathematics – geometry in three-dimensional space – which will be used as the language and the frame to test structural theories. It was the law of constancy of dihedral angles which enabled to submit hypothetical geometric constructions to critical test. Mathematical concepts of symmetry were born from this experience.

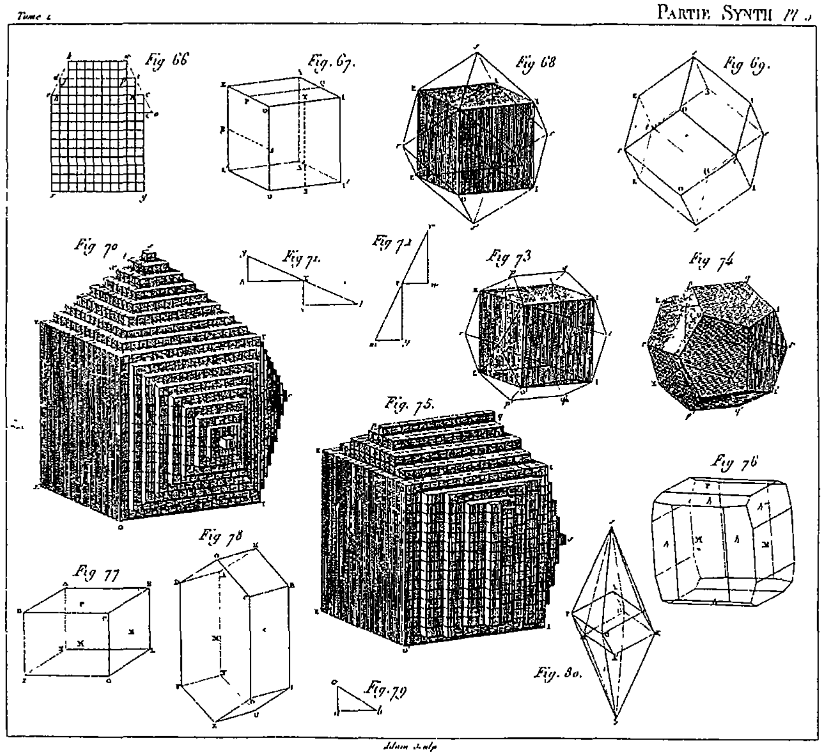

Figure 3: Plate *n*° 5 of the Atlas of the *Traité de Cristallographie* (1822) of R-J Hauy

3 R-J HAÜY – FROM IMPLICIT TO EXPLICIT SYMMETRY

The scientific production of Haüy consists in some books and more than 100 articles, most of them corresponding to the description of mineral species. One can follow the evolution regarding the general concept of symmetry. From the beginning of the nineties, remarks such as the following ones can be read (Haüy, 1796a):

"Le calcul théorique fait voir, de plus, que chaque rhombe c d e q, g o n y, *&c. est semblable au rhombe primitif du spath calcaire, et que l'angle obtus* e g y *de chaque rapèze est égal à chacun des angles* e d L, 9 o I, *&c. de l'octogone voisin, c'est-a-dire, qu'il est de 116^d 33' 55". On a pu remarquer encore dans ce qui précède, que l'inclinaison des mêmes trapèzes sur les pans adjacens était égale à celle que les pans gardent entre eux, c'est-à-dire de 120^d. Toute la théorie est pleine de ces analogies et de ces propriétés géométriques, qui répandent une sorte d'harmonie dans les résultats des lois auxquelles est soumise la structure des cristaux."* (Figure 4).

The last sentence could be translated as follows:

"All the theory is full of these analogies and of these geometrical properties which spread a sort of harmony into the results of the laws which the structure of crystal obeys."

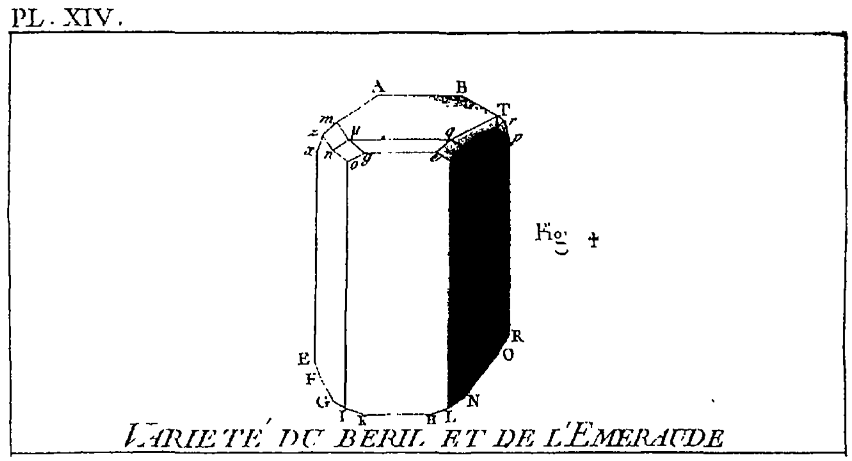

Figure 4: Figure of the note on crystallisation of emerald (Haüy 1796a)

In a paper published slightly later, Haüy (1796b) used properties of symmetry without any precise characterization. In order to establish a nomenclature of vertices, edges and faces of polyhedron representing crystals, he named these elements of figure by letters (Figure 5). He took into account the proper symmetry of each polyhedron and then used a minimal number of different letters. In the description of the homology between the edges (or vertices or faces) of regular polyhedra, one can see an implicit knowledge of their characteristic symmetry (with several errors).

This implicit use of symmetry rules would continue up to the publication in 1815 of a paper entitled *"Mémoire sur une loi de cristallisation appelée Loi de Symétrie"* (Memoir on one law of crystallisation named Law of Symmetry). In this paper, Haüy related the number and the position of the faces observed on the external form of crystals to the symmetry of the hypothetical nucleus, the *molécule intégrante*. This last symmetry is considered obvious and does not need to be described: in a cube, all the vertices are equivalent (Haüy writes identical), and so are the edges and the faces. In a square-based prism, only 4 faces are equivalent and the 2 others only between themselves, etc.

It must be emphasized that the word symmetry here possesses the new acception (set of isometries), but that the tools necessary to describe these isometries do not exist. Nevertheless, the concept of the homologous parts of a figure has been found out, and it will be used as a pathway to transform this qualitative concept to quantitative ones.

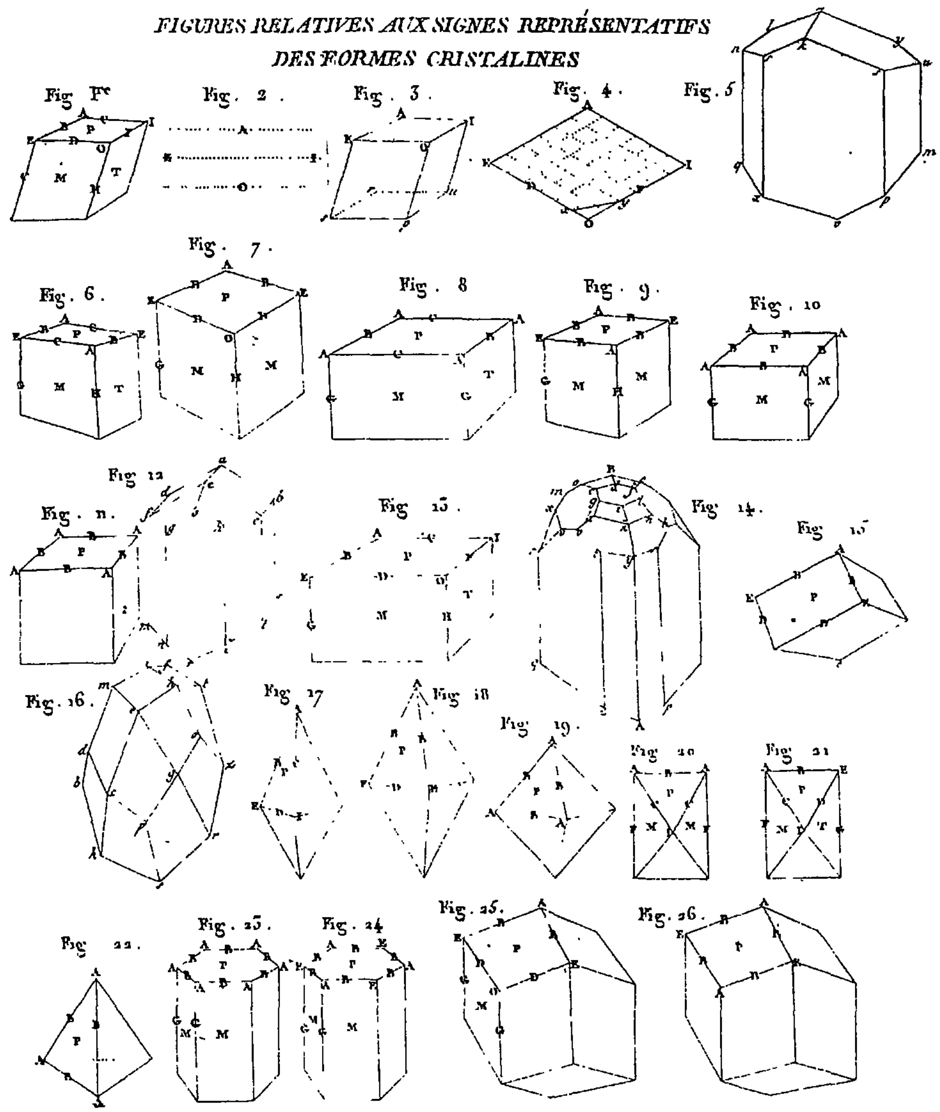

Figure 5: Figure of the *"Exposé d'une méthode simple et facile..."* (Haüy 1796b)

4 FROM S. C. WEISS TO A. BRAVAIS

The great importance of Haüy's Law of Symmetry is due to the connection he establishes between the obvious symmetry of the microscopic nucleus and that of the shapes of real crystals. This would lead to the great development of studies and classifications of the external symmetries of crystals. It will be the work of S. C. Weiss a German crystallographer, a disciple of Werner and of Haüy and that of his pupils, G. Rose, F. E. Neumann, F. Mohs, ... The notion of the crystal system based on symmetry emerged during the period 1815-1830. The description of the 32 crystal classes was published by L. M. Frankenheim in 1826 and independently by J. F. C. Hessel in 1830. Neither had any influence on the science of the time.

The 32 crystal classes correspond to the combinations of the following symmetry elements: the rotation axis of order 2, 3, 4 and 6, the inversion center and the mirror plane. The existence of these classes is proved by the possible coexistence of one of these symmetry elements with one or more others.

This type of purely mathematical analysis must enable the crystallographer to divide real crystals among the 32 classes, each crystal corresponding to a unique class.

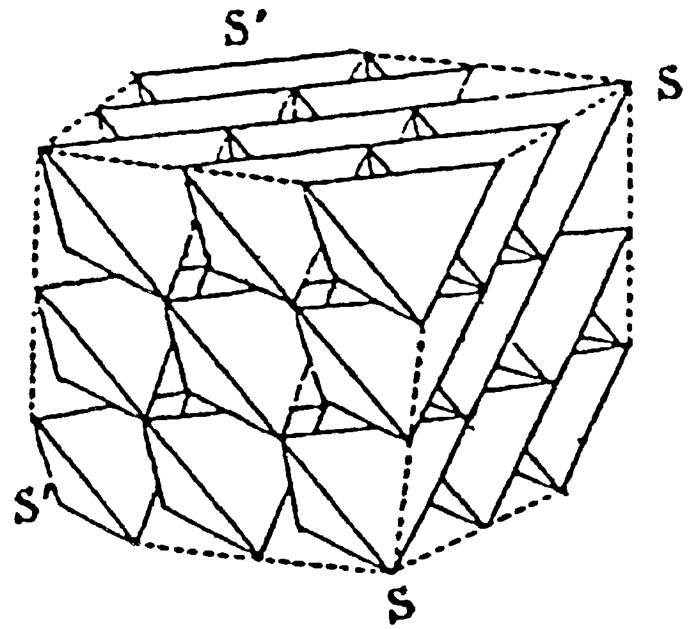

Figure 6: Model proposed by G. Delafosse for the boracite crystal (Delafosse 1843)

Another way conducted A. Bravais to the same result. As Frankenheim and other scientists (Wollaston, Seeber, for example), G. Delafosse wanted to improve the microscopic structural model of crystalline matter proposed by Haüy. He assumed that the building blocks of matter, each separate from one another (and not contiguously joined as in Haüy's model) consisted into polyhedral molecules similarly oriented at the nodes of a three dimensional lattice. Figure 6 gives an example of tetrahedral molecules situated on a cubic lattice. One more time, this type of representation would substantially help the diffusion of the model which was revealed as fruitful. L. Pasteur and A. Bravais used this model for their own discoveries. It may be noted that Delafosse does not say that molecules are polyhedra, but they do possess the symmetry of a polyhedron.

In order to analyse completely the symmetry of crystals, from this theoretical point of view, A. Bravais studied separately the 14 possible symmetries of lattices (regular system of points, 1849) and the 23 possible symmetries of polyhedra (1850) (Table 1). In 1834, Frankenheim had stated without demonstration the existence of 15 (which are actually 14) space lattices. Besides, one class of polyhedra was omitted by Bravais in his paper, but it was reintegrated in his Crystallographic Studies of 1851 when he applied his mathematical results to that study of real crystals where he gave out the existence of the 32 crystal classes (crystal point groups nowadays).

At the crucial stage of our history, it may be useful to do some comments. Bravais was a remarkable mineralogist and crystallographer as well a rigorous mathematician, and later, he analysed the very qualitative model of Delafosse with the relevant geometrical tool. The rigour and clarity of his papers ensured that they were read up and therefore they had a numerous posterity in mathematics as well as in crystallography. The 14 "Bravais lattices", the 23 symmetry classes of polyhedra or the 32 crystallographic point groups are known nowadays as groups in the mathematical sense of this word. They were groups before the existence of groups. One can note that the notion of the symmetry element is present but not that of the symmetry operation. Likewise Bravais studied the coexistence of symmetry elements, their combination but the idea of a composition law of these operations was not yet present.

Known for more than two thousand years, the five Platonic solids fascinated artists and mathematicians for their "regularity", i.e., the equality between faces and between edges and for their limited number. Nowadays, these polyhedra are seen as archetypes of symmetry groups. But virtue of hindsight, it may be considered curious that their essential characteristic, from our point of view, was not underlined before 1850. One can think that experiments carried out on less regular polyhedra that are real crystals have permitted to understand what symmetry is: First, there are degrees into symmetry, and second, the symmetry of an object can be decomposed into elementary operations.

Classification des polyèdres, d'après la nature de leur symétrie.

POLYÈDRE			SYMBOLE DE LA SYMÉTRIE DU POLYÈDRE.	CLASSE du polyèdre
Asymétrique			oL, oC, oP.	1re
Symétrique	dépourvu d'axes		oL, C, oP.	2e
			oL, oC, P.	3e
	pourvu d'un axe principal	d'ordre pair	A^{2q}, oL^2, oC, oP.	4e
			A^{2q}, oL^2, C, Π.	5e
			A^{2q}, qL^2, qL'^2, oC, oP.	6e
			A^{2q}, oL^2, oC, qP, qP'.	7e
			A^{2q}, qL^2, qL'^2, C, Π, qP^2, qP'^2.	8e
			A^{2q}, $2qL^2$, oC, $2q$P.	9e
		d'ordre impair	A^{2q+1}, oL^2, oC, oP.	10e
			A^{2q+1}, oL^2, C, oP.	11e
			A^{2q+1}, oL^2, oC, Π.	12e
			A^{2q+1}, $(2q+1)L^2$, oC, oP.	13e
			A^{2q+1}, oL^2, oC, $(2q+1)$P.	14e
			A^{2q+1}, $(2q+1)L^2$, C, $(2q+1)P^2$.	15e
			A^{2q+1}, $(2q+1)L^2$, oC, Π, $(2q+1)$P	16e
	sphéroédrique	quaterternaire	$4L^3$, $3L^2$, oC, oP.	17e
			$4L^3$, $3L^2$, C, $3P^2$.	18e
			$4L^3$, $3L^2$, oC, 6P.	19e
			$3L^4$, $4L^3$, $6L^2$, oC, oP.	20e
			$3L^4$, $4L^3$, $6L^2$, C, $3P^4$, $6P^2$.	21e
		décemternaire	$6L^5$, $10L^3$, $15L^2$, oC, oP.	22e
			$6L^5$, $10L^3$, $15L^2$, C, $15P^2$.	23e

Dans ce tableau, q est un nombre entier quelconque, positif, et au moins égal à 1.

Table 1: Classification of polyhedra (Bravais 1849)

In the first half of the 19^{th} century, the aim pursued by the scientists who established the basic notions of geometrical symmetry was to obtain a powerful tool in order to progress in the knowledge of the structure of solid matter. After them, some mathematicians would follow the same line as crystallographers in order to develop this new scientific field.

5 THE TWO HUNDRED AND THIRTY SPACE GROUPS OF SYMMETRY

The works of Bravais have inspired numerous scientists. Here are presented four of them, two mathematicians (C. Jordan and A. Schoenflies) and two crystallographers (L. Sohncke and E. Fedorov). This is not an arbitrary choice because all the four of them have produced significant contributions to the field of symmetry.

The interest of mathematicians for the notion of group (elaborated by Galois around 1830) began in the sixties and C. Jordan was one of those who studied and classified groups in order to study this object thoroughly. In 1868 and 1869, he published a paper where he studied the groups' motions which can be finite or infinitely small translations or rotations. Following Bravais, he combined translations and rotations but some motions are not permitted in crystallography. The classification gave one hundred and seventy-four groups. For the first time, symmetry operations were considered and combined. Groups are no more seen as the mere coexistence of symmetry elements. The notion of subgroup appeared under its crystallographic name of hemiedry.

Leonhard Sohncke (1879), a physicist and a crystallographer, took up again this problem of the association of translation groups with orientation groups of polyhedra. He no longer assumed as Delafosse and Bravais, the uniqueness of the orientation in space of molecular polyhedra and he explicitly introduced a mixed symmetry operation, the screw axis which combines a translation and a rotation. Using the method of Jordan, after the rectification of some errors of the paper of this last author, he found out sixty-five crystallographic space groups (sixty-six but two are identical). One can think that for Sohncke, the introduction of improper motions (as reflexion) would involve the existence of two sorts of particles (one "right" and one "left") and that he did not find this situation representative of reality.

The Russian mineralogist E. Fedorov who wanted to resolve the problem of the filling of space by polyhedra followed the way of Sohncke and he demonstrated the existence of two hundred and thirty space groups or possible distributions of identical objects in three dimensional space. A new type of mixed motion was introduced, a translation combined with a reflexion (or glide plane). The publication dates back to 1890.

The mathematician A. Schoenflies, disciple of F. Klein, studied the same problem – classifying the crystallographic groups of motion – along the way outlined by Jordan. Publication of his results (1891) followed by several months that of Fedorov.

CONCLUSION

From the observation of natural crystals with their geometrical regularities as plane faces of polygonal forms, the question encountered by numerous scientists was that of a microscopic hidden order which could explain the obvious external order displayed by crystals. From Kepler to Fedorov, it was the same question.

Throughout this long exploration, physics and chemistry questions were tackled and many of them have been fruitful in these fields. But the pertinent analysis of the geometrical order (observed or theoretical) allowed Haüy and his successors to discover and to understand symmetry, a new type of harmony. Following the meeting of symmetry, the group theory in the second half of the last century, led to an unbelievable scientific fertility.

But even if symmetry was discovered during the nineteenth century, clearly, artists and scientists of antiquity knew something about symmetry as Egyptian pyramids or Platonic solids still show us to this very day.

REFERENCES

Bartholin, E. (1669) *Experimenta Crystallı Islandici dısdıaclastici quibus mıra et ınsolıta refractıo detegıtur*, Daniellis Paullı Hafniae.

Bergman, T. O. (1773) Variae crystallorum formae a spatho orthae, *Nova Acta Regiae Soc. Sci. Ups.* 1, 150-155.

Bravais, A. (1849) Mémoıre sur les polyedres de forme symétrique, *J. Math. M. Liouvılle*, 14, 137 -180.

Bravaıs, A. (1850) Mémoire sur les systèmes formés par des points distribués régulierement sur un plan ou dans l'espace, *J. Ec. Polytechnique*, 19, I-128.

Bravais, A. (1851) Etudes cristallographiques, *J. Ec Polytechnique*, 20, I01-276

Burckhardt, J. J. (1988) *Die Symmetrie der Kristalle*, Basel: BirkHäuser, Verlag.

Carangeot, A (1783) Goniomètre ou mesure-angle, *Journal de Physique*, 22(1), 193-197.

Collectif (1918) 175ème anniversaire de la naissance d'Hauy, *American Mineralogist*, 3(6), 49-136.

Collectif (1944) 200ème anniversaire de la naissance d'Haüy, *Bull. Soc. Franc Minéralogie*, 67, 1-6.

Curie, P (1894) Sur la symétrie dans les Phénomènes Physiques, symétrie d'un champ électrique et d'un champ magnétique, *Journal de Physique*, 3ème série, 393-415.

Delafosse, G. (1843) Recherches sur la cristallisation considérée sous les rapports physiques et mathématiques, *Mémoire des savants étrangers*, 8, 641-690.

Fedorov, E. S. (1890) *La symétrie des systèmes réguliers de figure (en russe)*, Saint-Pétersbourg. A Yakob.

Frankenheim, M. L (1826) Kristallonomische Aufsatze, *Isis* 19, 497-515, 542-565.

Frankenheim, M L (1835) *Die Lehre von der Cohäsion*, Breslau· August Schulz.

Haüy, R J. (1782) Sur la structure des cristaux de grenat, *Journal de Physique*, 19, 366-370

Haüy, R. J (1782) Sur la structure des spaths calcaires, *Journal de Physique*, 20, 33- 39.

Hauy, R. J. (1784) *Essai d'une théorie sur la structure des cristaux appliquée à plusieurs genres de substances cristallisées*, Paris. Gogué et Née de La Rochelle.

Hauy, R. J. (1796a) Note sur la cristallisation de l'émeraude, *Journal des Mines*, 3ème trimestre, An IV, 19, 74-74.

Haüy, R. J. (1796b) Exposé d'une méthode simple et facile pour représenter les différentes formes cristallines par des signes tres abrégés qui expriment les lois de décroissement auxquelles est soumise la structure, *Journal des Mines*, 23, Thermidor An IV, 15-36.

Hauy, R. J. (1815) Mémoire sur la loi de crystallisation appelée loi de symétrie, *Journal des Mines*, 37, 215-235, 347-369; 38, 5-34, 161-174

Haüy, R. J. (1822) *Traité de Cristallographie* (2 tomes + atlas), Paris: Bachelier et Huzard.

Hessel, J F C. (1831) *Kristallometrie oder Kristallonomie und Kristallographie*, Leipzig. Schwickert.

Huyghens, C. (1690) *Traité de la lumière*, Leiden: Pierre von der Aa.

Jordan, C. (1868) Mémoire sur les groupes de mouvements, *Ann. Mat. Pura Appl* Ser. 2(2), 167-215

Jordan, C. (1869) Mémoire sur les groupes de mouvements, *Ann. Mat Pura Appl* Ser. 2(2), 322-345

Képler, J. (1611) *Strena, seu de nive sexangula*, Francfort. Godfried Taupach.

Lima de Faria, J (1990) *Historical Atlas of Crystallography*, Dordrecht· Kluwer Academic Publishers

Linnaeus, C. von (1768) *Systema naturae*, vol. 3 Laurentii, Stockholm.

Mauskopf, S. H (1976) Crystals and compounds, Molecular Structures and composition in Nineteenth century French Science, *Trans. Am. Philos Soc.*, 66, 1-82

Metzger, H. (1969) *La genèse de la science des cristaux*, Paris: Librairie Scientifique et Technique Albert Blanchard.

Schoenflies, A. M. (1891) *Krystallsysteme und Krystallstruktur*, Leipzig· Teubner.

Sohncke, L. (1879) *Entwicklung einer Theorie der Kristallstruktur*, Leipzig: Teubner.

Steno, N. (1669) *De solido intra solidum naturaliter centento dissertationis prodomus*, Florence.

Weiss, C. S. (1815) *Uebersichtliche Darstellung der verschiedenen natürlichen Abteilungen der Kristallisations-Systeme*, Berlin· Abh. K. Akad. Wiss , p 289-344

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SYMMETRY AS THE LEITMOTIF AT THE FUNDAMENTAL LEVEL IN THE TWENTIETH CENTURY PHYSICS

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Abstract: *Invariance considerations gradually entered physics in the 17th-19th centuries, but it was only in the early 20th century that (geometrical) symmetries became the dominant element. In Einstein's Special Theory of Relativity, the* Poincaré global symmetry group *rules over the* kinematics. *The General Theory of Relativity is "spanned" by two principles,* Equivalence *and* Covariance, *corresponding to local symmetries, a symmetry under* active *Lorentz transformations on local frames and a symmetry under* passive local Diffeomorphisms. *Local symmetries involve transformations, which, when performed at different locations, can be implemented at each such position to an arbitrarily different quantitative extent, thus imposing strains in the medium, known in geometry as* curvature, torsion, *etc. Local symmetries thus involve dynamics. The linkage between Symmetry and* Conservation Laws *was clarified in two theorems due to Emmy Noether (1918).*

Outside of Relativity, symmetries played a minor role between the establishment of Quantum Mechanics in 1925 and our discovery of unitary symmetry *in 1961. Our identification of SU(3) as the key to the hadron spectrum lead to the uncovering of yet another layer in the structure of matter, that of the* quarks *(1962-69). The inobservability of free quarks and the spin-statistics correlations of the quark model led to Quantum Chromodynamics as the basic Strong Interaction. Experimental studies of the* Weak Interactions *(1956-58) showed them to be induced by conserved charges corresponding to local gauge symmetries. Weinberg and Salam reconstructed the parent Electroweak local gauge symmetry, whose spontaneous breakdown around*

.2TeV *yields both electromagnetism and the Weak interactions. All of this was melted in a grand synthesis, the* Standard Model *(1975), a quantum theory of everything but gravity, entirely based on local gauge symmetries and covering the entire range of explored physics* (<1 TeV). *A parallel process of further geometrization has thereby characterized the emergence of symmetry as the dominating factor in Fundamental Physics, in the second half of the century.*

1 THE BACKGROUND

Geometrical considerations were prominent in the Pythagorean doctrines - presented with the enthusiasm generally ascribed to a *cult.* The Pythagorean Philolaus was one of Democritus' teachers and may be responsible for the latter's eventual invocation of geometrical features as the sole characterization of atoms, the latter concept being perhaps inspired by his main other teacher, Leucippus. Plato laid the foundations for a view of space in which this is more than the set of relations between objects and for a rationalized geometrical view of the physical world. It did not catch with Aristotle. From Aristotle to Einstein, geometry was excluded from physics. Newton's contributions were direct and concrete, his metaphysical interests notwithstanding. His First (inherited from Galileo) and Third laws, however, could clearly be enunciated as *invariance principles.* What was missing was the *metatheory.* Huyghens' shortest path for light was another close hit, but it took the philosophical turn of mind of a Leibniz for a metatheory to be conceived. Voltaire distorted the idea and poked fun at "the best of possible worlds" – but Leibniz, whose independent invention of the calculus was published under the title "Of Maxima and Minima", was clearly laying the foundations of our metatheory – the *Action Principle.* Johann Bernoulli first managed to provide Leibniz's ideas with a concrete mould. From him to Euler, to Maupertuis (badly maligned by Voltaire for having "stolen" Leibniz's idea...), to Lagrange, to Hamilton and to Jacobi – and the "best of possible worlds" had somehow become the laziest – the one with *the least action...* The mathematical toolkit was, however, incomplete. *Group theory* still had to be invented by these two tragic teenager figures – Evariste Galois and Hendrik Abel. Gauss and his student Riemann still had to develop *differential geometry,* Darboux brought it closer to algebraization, Felix Klein and Sophus Lie launched the *Erlangen program* in 1872, and the wedding of geometry with group theory was finally consummated by the beginning of the twentieth century. Einstein's two *Theories of Relativity,* his 1905 *Special Theory* and his 1915 *General Theory* suddenly hit physics with this new instrumentation – and physics has never been the same. Emmy Noether (1918) extracted the essence of the application of the new tools in her two theorems.[1]

2. EINSTEIN'S TWO SYMMETRIES – THE GLOBAL AND THE LOCAL – AS MODELS

Einstein's Special Theory – "shocking" as it was - amounted to a kinematical prescription, the invariance of the action (and later of the S-matrix) under the *Poincaré group* (or the *inhomogeneous Lorentz group*). It was first formulated ungeometrically – until Hermann Minkowski, in a 1908 address, brought out the geometrical context.[2] We have known ever since that in the approximation of weak gravity, we exist in a 4-dimensional spacetime with a pseudo-Euclidean metric, the *Minkowski metric* (-1,1,1,1) (time is here $x^0 = ct$, c the velocity of light). The application of the Poincaré group as a symmetry of the kinematics is *global*, in the sense that the same transformation is assumed to hold everywhere, at least in its *passive* implementation, i.e., on the coordinates. In the *active* mode, the initial conditions may restrict the implementation to a single body, assuming the rest of the universe to be empty; for a *field*, however, *active* global transformations also have to be applied everywhere equally. The *action* is required to be *globally Poincaré invariant*, as a result of the geometrical properties of the spacetime manifold in which it exists. Spacetime thus impacts the material action, but the opposite does not hold for global symmetries.

Note also that although it would seem that the direct perception of a symmetry should eliminate the possibility of its not being geometrical, non-geometrical (or *internal*) symmetries may arise indirectly, through the First Noether theorem. One might detect a non-geometrical conservation law, which would then imply the existence of a symmetry. Noether's theorem might then involve a non-spacetime manifold over which the transformations by the relevant Lie group are defined – although this is not necessary, a definition of their action on the fields being all that is really needed. Also, *geometrical* symmetries are not restricted to be spacetime-geometrical, once the arena of physics is enlarged. In quantum theory, for instance, F. London[3] showed in 1928 that the electron wave function should be symmetric under rotations of its complex phase – a geometrical feature of Hilbert space! T. Kaluza and O. Klein[4] had earlier shown how electrical charge could also be derived by adding one dimension to spacetime.

Minkowski's comment is now at the foundation of physical theory, but it was also crucial to the entire evolution of physics in this century, a real turning point. It was thanks to the availability of this algorithm that Einstein could go on and bring about coherence between the new relativistic kinematics and gravity, which in its Newtonian formulation obeys Galilean kinematics. This he did in his General Theory, in which spacetime is a curved Riemannian manifold, but with the *Equivalence Principle* requiring it to be *locally Minkowski-flat*, i.e., to provide for invariance under *local* (homogeneous) Lorentz transformations performed over a local frame (Darboux's *repères mobiles*, "tetrads" or *vierbeins*). The *Principle of Covariance* states that there is no preferred frame or coordinate system; mathematically it imposes invariance under *local diffeomorphisms* of the Riemannian R^4. The wording used in enunciating this principle can be extended to any *passive* symmetry – namely that there is no preferred set of coordinates. Moreover, any *active-* mode symmetry can be reformulated as *the inexistence of a preferred setting for the system*, in terms of the relevant parameter. E. Whittaker has stressed this aspect by describing such symmetry principles as *postulates of impotence*.[5]

A *local* symmetry, such as that of the Lorentz group (in fact, because of Quantum Mechanics, of its universal covering group) $SL(2,C)$, implies the freedom of performing the transformations to different quantitative extents at different locations. The simplest model is a tube-shaped object. It is invariant under 2-dimensional rotations (and under a longitudinal translation, for an infinite tube only). Rotating the entire tube as one piece represents the *global* symmetry, under which the tube is indeed invariant. However, in a tube made of rubber, we might rotate (or twist) one sector only, or two sectors by different angles. The rubbery material can absorb the stresses (*curvature*). We could also at the same time stretch the tube at one or more locations, generating *torsion* stresses. Such a *local* invariance clearly induces stresses in the medium – spacetime in the case of General Relativity, Einstein's theory of Gravity. A *local symmetry therefore produces a dynamical theory, with its field-inducing source given by the Noether-conserved charge-current, as defined by the corresponding global symmetry*. Noether's Second theorem describes this relationship. Here the matter content does impinge on the medium, spacetime. Our example of a rubbery tube, and first foregoing the longitudinal translations, comes under the algebraic-geometry notion of a *fiber bundle manifold*, with the longitudinal dimension as its *base space* and the local circles (along this axis) making up the tube as its *fibers*, each fiber thus sitting over one point of the base space. H. Weyl[6] first successfully introduced such an object in 1929, with Minkowski spacetime as the base-manifold, and the circle carrying the quantum phase of the electron as the fiber – i.e., a phase-carrying circle at each point in spacetime, with the possibility of rotating that phase by different amounts at different spacetime locations - this is Maxwell's electromagnetic theory in its quantum version, QED (quantum electrodynamics). The physics name for a fiber bundle is a *local gauge theory*.[7] Twenty-five years after Weyl, C. N. Yang and R. Mills[8] constructed a generalization of Weyl's gauge theory in which the local symmetry group is nonabelian, i.e., with noncommutative transformations; as an example of such a group, take rotations in 3 dimensions. Put a book on the desk before you, face-upwards. Let x be an axis parallel to the desk length, y to its breadth, z to the vertical direction in the room. Perform a 90° rotation around x, counter-clockwise as seen from the right; the book is now standing, facing you. Now rotate it by 90° clockwise (when looking from above) around z; the book cover is facing the left. Now go back to the initial position and perform the two rotations in the opposite order. First, the 90° z-rotation (clockwise when seen from above); the book is lying parallel to the x axis, face upwards. Now the 90° counter-clockwise rotation (as seen from the right) around x; the book is standing on its side, the front cover facing you, but the rows of writing are in the vertical direction. The results are thus entirely different when the transformations are performed in different ("commuted") orders. The group $SO(3)$ of rotations in 3 dimensions is indeed the smallest noncommutative group amongst the Lie groups, the groups of continuous transformations; Yang and Mills indeed used it as the model for their construction.

Einstein's theory of gravity is more complicated algebraically – somewhat like the tube when allowing longitudinal translations, as well as the rotations. We have recently discussed[9] the ways and extents in which and to which it can be made to look like a local gauge theory. Some modifications and generalizations, possibly representing physical reality at high energies (where gravity becomes a quantum theory, in an as yet unverified form) correspond to larger locally implemented symmetry groups[10,11] (and *supergroups*[12-13]).

3 QUANTUM MECHANICS, STRONG AND WEAK INTERACTIONS, INTERNAL SYMMETRIES

Planck's (1900) discovery of the quantization of *action* was a beautiful and unexpected validation of the choice of this fundamental entity to represent the material and radiative content of spacetime. Euler, Maupertuis and Lagrange were vindicated in the most direct way. Action is dimensionless, it is naturally countable in terms of Planck's quanta. Relativistic Quantum Field Theory and Feynman's Path Integrals – even Schroedinger's $\psi(x)$ wave-function - all fit with this choice.

An early innovation in the domain of discrete symmetries occurred in 1927, when Dirac conceived his relativistic equation for the electron. The equation had two solutions, the one indeed representing the electron - and the other a particle with the same mass and spin, but the opposite charges (electric and leptonic - see below). Such a particle – the positron – was indeed discovered in cosmic rays within several years. It became gradually clear that aside from space (mirror) and time (reversal) reflections, nature supplies a symmetry under a similar inversion of the electric and other charges or *"internal"* quantum numbers, a symmetry between *particles* and their *antiparticles*.

The 1932 discovery of the neutron 14 ushered in two new interactions, the Strong and the Weak. The sources inducing these interactions at first appeared very ungeometrical (1932-1958). Gradually, however, a new series of conserved charges corresponding to global *internal* symmetries emerged from the study of the new interactions: *isospin* **I**[15], *hypercharge* Y[16], *baryon number* B[17] for the *hadrons* (particles participating in the *Strong Interactions*, the nuclear "glue" holding together protons and neutrons in nuclei), *μ-lepton number, e-lepton number* (and much later *γ-lepton number*) for the leptons; particles which do not "feel" the Strong Interaction. The Weak Interaction also displays a *"weak" isospin* I_w current, with a 96 percent overlap with the "strong" version in hadrons, but extending to leptons too. From 1947 to 1964, particle species – mainly the *hadrons*, the strongly interacting particles – proliferated, reaching over one hundred types.

With the discovery of the electron by J. J. Thomson in 1897 and Einstein's 1905 explanation of the *photoelectric effect* by showing that light and any other electromagnetic radiation consist of *photons*, massless particles carrying energy and momentum, the (*quantum*) particle aspect of electromagnetism was clearly realized, though much further work had to be invested before the physics (*"Quantum Electrodynamics"*, abbreviated as "QED", the *gauge theory* mentioned above) were completely understood in 1948. Photons are emitted or absorbed by any electrically charged particle, ensuring the system's invariance under the local abstract rotations symmetry of the electric phase suggested by Weyl. Yukawa had conjectured[18] in 1935 that, in analogy to the photons, the Strong Interaction, the nuclear "glue" is propagated by a *massive* particle field (the π mesons), thus explaining the interaction's short range. By the time the three necessary types of π had been found (1949) in cosmic rays, new and unexpected species – the four K mesons – had been added to the list, while the list of *baryons* had gained six new members (*hyperons*), aside from the protons and neutrons making up normal atomic nuclei. The new particles are produced in strong interactions but slowly decay, their lifetimes ranging from almost 10^3 secs in the case of the neutron's "beta decay", down to 10^{-10} seconds for some hyperons; these decays occur via the *"Weak Interactions"* the nuclear interaction identified[19] by Fermi in 1934. In addition, very short-lived hadrons, decaying through the strong interaction itself, were produced in accelerators in the fifties and sixties. These particles - known as *resonances* - were generally endowed with high values of spin angular momentum.

4 THE SEARCH FOR ORDER – STRUCTURALIST APPROACHES FAIL

The search for order was launched around 1955, after the identification 16 of *hypercharge* (or of *strangeness* S, with $Y = S + B$) by Nakano and Nishijima in Japan and by Gell-Mann in the USA. Two very different lines of thought permeated this search, a *structuralist* approach and on the opposite side *phenomenology*. The structuralists tried to guess at the inner structure of the particles, suggesting models purporting to explain the observations. The phenomenologists collected the observational information and tried to abstract this information in a mathematical formulation, assuming that structural understanding would follow. The structuralists followed several different lines. L. De Broglie, Yukawa, D. Bohm[20], Corben and others tried mechanical models, *rotators, spinning tops, vibrators*, etc. The idea was that *isospin* might correspond to true physical rotations - relative to the body frame, for instance. This would imply the possibility of adding up angular momentum and isospin, of transferring angular momentum between these categories – contradicting the observations. It turned out later on that such mechanistic models do fit something else altogether, namely the high values of the *spin angular momentum* of the above mentioned *resonances*, i.e., spin proper, rather than *iso*spin, which is mathematically similar but contextually totally different.

Another structuralist approach, unrelated to preconceived mechanistic models and thus much less dogmatic, searched for *constituent* models. Already in 1949, Fermi and Yang[21] had noted that from the point of view of the quantum numbers, π mesons could be "made of" nucleons (protons or neutrons) and their antiparticles. There was no pretence at a knowledge of the interaction involved in making the compound, just a suggestion of possible compositeness of the mesons and an identification of a set of constituents containing the observed quantum numbers. With the advent of strangeness or hypercharge, Sakata[22] added one hyperon (Λ^0) carrying $Y = -1$, to the proton and neutron (and their antiparticles) as prospective constituents of all hadrons.

It is interesting to note that both above schools of structuralism were influenced by Marxist philosophy. In the case of Sakata and his colleague Taketani, *dialectical materialism* was often referred to, and the opposite phenomenologist approach was accused of *positivism* or of *pragmatism*. Mention of positivism was generally accompanied[23] by a reminder of the defeat of Mach and Ostwald – who doubted the "true" existence of atoms – and whose discomfiture was hailed by Lenin.[24] Yet another structuralist approach was launched by G. Chew[25] under the title of *the bootstrap*, or also *nuclear democracy*. This was the antithesis of the constituent approach and carried Leibnizian features. Atomism - constituent models - was rejected and it was assumed that the hadrons were made of each other, with no possibility of distinguishing between constituent and composites. My favorite analogy is that of a jungle inhabited by tigers, lizards and mosquitoes. The tigers eat the lizards, the lizards eat the mosquitoes and the mosquitoes suck the tigers' blood, infect them and devour their carcasses. As a result, the tigers are "made of" lizards, the lizards are made of mosquitoes and the mosquitoes are made of tigers... In this approach, it was assumed that the particles with their symmetries emerge as the solutions to some highly non-linear equation.

Around 1967, a plausible system of equations was suggested by Horn and Schmid and other groups, using a phenomenological formulation. The solution, found by G. Veneziano, turned out to represent the dynamics of a string-like quantum system, again reproducing the systematics of the angular momentum excitations displayed by the resonances, but not the *internal* quantum numbers such as isospin, hypercharge, etc. Moreover, it was shown that the equations and the bootstrap idea do not contradict atomism – tigers, lizards and mosquitoes can all be made of a set of common constituents (such as the true atoms, in this example). Here in the String, hadrons made of quarks (see next section) fitted perfectly, as demonstrated by Harari and Rosner. But all of this was quickly dropped around 1971, when Quantum Field Theory took over again. Note, however, that the formalism of *String Theory*, born out of the bootstrap, has survived in an entirely different context. It is now considered as a candidate fundamental theory (a "TOE", *theory of everything*) capable of describing physics at Planck energies (around 10^{19} *GeV*) or at very short distances, at which spacetime is conjectured to be discretized (10^{-33} *cm*). All of this, however, is still very speculative.

5 THE SEARCH FOR ORDER: SYMMETRY ABSTRACTED FROM PHENOMENOLOGY – DISCOVERY OF *SU*(3)

The alternative approach consisted in the identification of a Lie group as the extended symmetry group of the Strong Interactions and of the hadrons. This could be accomplished by trying to fit the observed spectrum of hadron states into the (linear) representations of that group, by comparing observed intensity rules and measured couplings with those calculated from the Clebsch-Gordan coefficients of the group, relevant to those representations. Candidate Lie groups would be those having isospin and hypercharge ($SU(2) \otimes U(1)$) as subgroups.

The first such attempts explored ever-larger rotation groups. Even as an abstract group having nothing to do with physical rotations, the isospin $SU(2) = Spin(3)$ was regarded as $Spin(3) = \overline{SO}(3)$, the *double covering group* of $SO(3)$, i.e., of the group of *unimodular orthogonal transformations (i.e., rotations) in 3 real dimensions* – rather than as the *unitary unimodular group in 2 complex dimensions* $SU(2)$. Salam and Polkinghorne[26] first tried embedding $SO(3) \otimes U(1) \subset SO(4)$. Schwinger and Gell-Mann (as explained by Tiomno)[27] used $SO(7)$, putting the 7 mesons (3 states of the π isovector, 2+2 of the K and $\bar{K}$) as a 7-vector and the 8 baryons as a $Spin(7)$ spinor. This was further extended to $\overline{SO}(8)$ and $\overline{SO}(9)$ by Salam and Ward[28], for various reasons.

My own approach[29] was both more general and more methodical. Applying Dynkin's simplified graphical version[30] of Cartan's classification of the *semi-simple Lie algebras*[31] and realizing that every reaction allowed by the conservation of isospin and hypercharge indeed appears to occur, I could draw the conclusion that what we are seeking is a rank $r = 2$ algebra, i.e., only two diagonal (additive) charges. The existence of a third diagonal conserved additive quantum number (within the then known particles) would result in additional selection rules, forbidding reactions otherwise allowed by **I** and Y conservation. Note that this does not occur with hypercharge itself just because as we in fact already conserve both isospin and electric charge Q, we are still dealing with a $r = 2$ algebra before even mentioning hypercharge. As $Q = I_3 + Y/2$, we then automatically also conserve hypercharge a priori, so that there is no clash.

There are five $r = 2$ semi-simple Lie algebras, A_2, B_2, C_2, D_2, G_2. D_2 is the algebra of $SO(4)$, B_2 that of $SO(5)$, $C_2 = B(2)$ is the algebra of the symplectic group $Sp(4)$, i.e., orthogonal transformations in a 4-dimensional manifold with antisymmetric metric – and it turns out that $Sp(4) = Spin(5)$. Only A_2 can accommodate all 8 known baryons (N, Λ, Σ, Ξ) in one representation, assuming the spin and relative parity of the Ξ isospinor (which was not yet measured at the time) to be $J(\Xi) = 1/2^+$ and assuming an *even* relative Σ–Λ parity – which was not the favored experimental assignment in 1961. The mesons could only be assigned to a similar octet (baryon number B lay outside of A_2), the assignment thus predicting the existence of an eighth (isoscalar) meson, denoted η, soon to be discovered indeed. A variety of experimentally evaluated baryon-meson couplings fitted the group's assignments. In the exceptional group G_2, on the other hand, the baryons would have to be assigned to a 7-dimensional representation, excluding the Λ_0 hyperon, which would have to be reassigned to a different (the singlet) representation. The then known mesons would fit in a similar **7**.

Another set of prerequisites could be derived from the Weak Interactions, assuming the Weak transitions to lie within the same algebra. Only $\Delta Y/\Delta Q = 1$ transitions are observed in the hadron current in the leptonic decays of hadrons, such as $\Lambda^0 \rightarrow p^+ + e^- + \overline{\nu}_e^0$ or $K^+ \rightarrow \pi^0 + \overline{e}^+ + \nu_e^0$; G_2 contains $\Delta Y/\Delta Q = -1$ as well, whereas A_2 does not, etc.

There was one obvious conceptual "difficulty" with the choice of A_2 or (the group) $SU(3)$ in the octet version: the nucleon, considered until then as the "fundamental" hadron (especially as the proton is the only completely stable hadron) is now assigned to a "higher" representation, not to the defining one of the group. A_2 is the generating algebra of the group $SU(3)$ (unitary unimodular matrices acting over 3 complex dimensions) and thus has a 3-dimensional fundamental representation. In fact, at the advice of Yukawa, in the same year 1960-1961 a research group at Nagoya, Japan, applied $SU(3)$ to the Sakata model, with (p, n, Λ^0) in the defining representation and relegating the Σ and Ξ to other representations. The Ξ isospinor had to be assigned $J^P = 3/2$. Returning to our *octet* model, we note that in contradistinction to the Sakata version of $SU(3)$ – and also to G_2 (where the nucleons appear in a "fundamental" septet) – *the octet does not assign a fundamental role to the nucleons and they are treated very much like composite objects.*

M. Gell-Mann independently arrived – somewhat later – at conclusions similar to mine.[32] Two groups, Speiser and Tarski[33] and later Behrends et al[34] conducted in 1961-62 methodical but inconclusive searches. Glashow and Gell-Mann[35] also later conducted a methodical survey, justifying the selection of $SU(3)$.

6 UNITARY (BROKEN) SYMMETRY

Before looking in detail at the $SU(3)$ formalism, we list the key physical features relating to symmetry under a Lie group: (a) Quantum numbers, i.e., a classification in multiplets, i.e., the group representations' carrier spaces; (b) relative couplings between particles belonging to several different multiplets; (c) "intensity rules", i.e., ratios between processes involving particles belonging to the same multiplets; (d) ratios between magnetic moments, between electromagnetic mass differences, between weak transitions, etc. – all transitions involving an interaction with a perturbative description, breaking the symmetry in a regular fashion. In the specific case of $SU(3)$ this was also true of (and the same treatment was given to) the mass differences between the isospin multiplets, e.g., $N(939)$, $\Lambda(1, 115)$, $\Sigma(1, 195)$, $\Xi(1, 320)$ (the masses are in *MeV*). They seemed to correspond to an effect acting in the same direction ("λ_8") as hypercharge –

but also generated a puzzle – how could an effect due to the Strong Interactions work well perturbatively? I pointed to the solution to this puzzle in 1964[36] by suggesting that *the mass differences indeed represent an additional, as yet unexplored, perturbative interaction and do not originate in the Strong Interactions themselves,* a solution which was later incorporated in the "Standard Model" we describe in the last section (though very little can be said about the responsible interaction, now relegated to the "Higgs sector").

Aside from these predictions, *if the symmetry is local, we also have the prediction of the existence of a gauge multiplet (generally* $J^P = 1^-$ *bosons), in the adjoint representation of the group. These couple universally to the Noether-conserved currents of the symmetry, i.e., the coupling is a matrix-element of a generator of the Lie algebra.* As a matter of fact, for such a *universal* coupling, we can derive ratios for the strength of the coupling between particles in *different* representations of the gauge group – here *SU*(3) – to the *gauge bosons*, since these couplings are given by the gauge algebra generators in the appropriate representation. Hundreds – or perhaps thousands – of predictions, of all the above types, were derived and the results were compared with the experimental results, with amazing success.

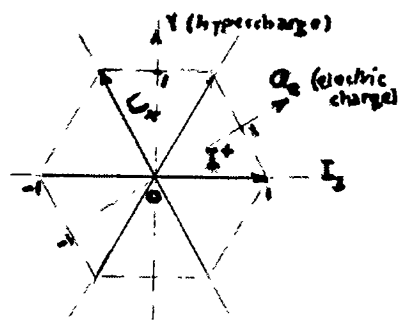
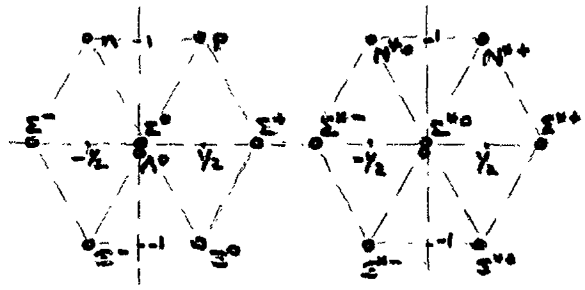
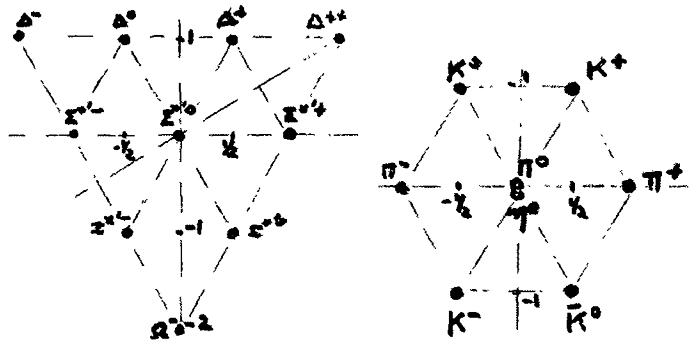


Figure 1: Root diagram of *SU*(3)

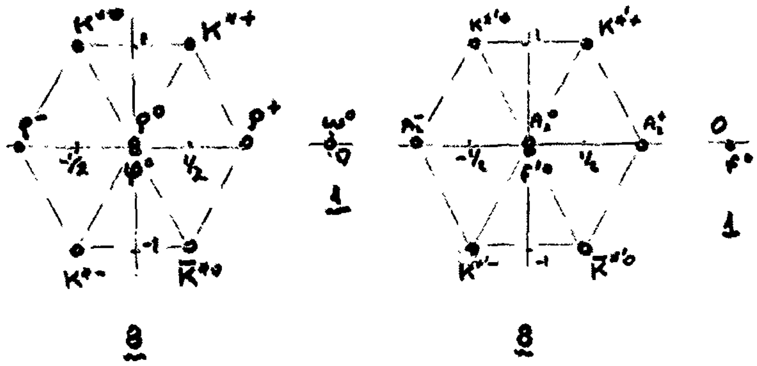
Any *simple* Lie algebra is characterized by its *root diagram.* For A_2, the generator algebra of *SU*(3), this is given by the six ends of a perfect hexagon, plus a double point in the center (see Figure 1). The horizontal axis represents *isospin* or **I**-spin; H. J. Lipkin named the two others **U**-spin and **V**-spin, following an advertisement of the times "I scream, you scream, we scream – we all scream for ice-cream." The particle *representations'* (groupings) *weight* diagrams, giving the particle content, by displaying the particles' values of the "additive" (diagonal) quantum numbers (for *SU*(3)) are all either triangles or hexagons.



(a) $J^P = (1/2)^+$ baryon $\underline{8}$ (b) $J^P = (5/2)^+$ baryon $\underline{8}$



(c) $J^P = (3/2)^+$ baryon $\underline{10}$ (d) $J^P = 0^-$ meson $\underline{8}$



(e) $J^P = 1^-$ meson $\underline{8} + \underline{1}$ (f) $J^P = 2^+$ meson $\underline{8} + \underline{1}$

Figure 2

Figure 2 displays the weight diagrams for (a, b) the $J^P = (1/2)^+$, $(5/2)^+$ (hexagonal) baryon octets (the latter is a rotational-vibrational excitation of the first, an example of a *"Regge recurrence"*), (c) the $J^P = (3/2)^+$ (triangular) baryon decimet to which we shall return in the coming paragraphs – and (d-f) the $J^P = 0^-$, 1^- and 2^+ "octet + singlet" combination bosons.

Note that hypercharge Y is orthogonal to **I**-spin and electric charge to **U**-spin. The masses are **I**-invariant, up to electromagnetic corrections. We can immediately derive e.g., the Coleman-Glashow sum-rule for electromagnetic mass-differences. With Q orthogonal to **U**-spin, electromagnetic effects should be **U**-invariant. For the baryon $J^P = (1/2)^+$ octet, we can display the **I**-invariant masses plus the electromagnetic corrections E:

$$\begin{array}{llll} p & : m_N & + & E^+ \\ n & : m_N & + & E^0 \\ \Sigma^+ & : m_\Sigma & + & E^+ \\ \Sigma^- & : m_\Sigma & + & E^- \\ \Xi^0 & : m_\Xi & + & E^0 \\ \Xi^- & : m_\Xi & + & E^- \end{array} \tag{1}$$

yielding the sum rule,

$$m_{\Xi^-} - m_{\Xi^0} = m_{\Sigma^-} - m_{\Sigma^+} + m_p - m_n. \tag{2}$$

Similarly, magnetic moments should be **U**-spin invariant. We get

$$\begin{array}{lll} \mu(\Sigma & = & \mu(p) \\ \mu(\Xi^0 & = & \mu(n) \\ \mu(\Xi & = & \mu(\Sigma^-), \text{etc.} \end{array} \tag{3}$$

For the different masses within a multiplet, Okubo derived the formula,

$$m = m_0 + aY + b[I(I+1) - Y^2/4] \tag{4}$$

whose specific predictions for a given multiplet may be derived from **U**-spin considerations.

This is where we return to the $J^P = (3/2)^+$ triangular multiplet (Figure 2c). The four $\Delta(I=3/2)$ states were discovered by Fermi in the fifties; the three $\Sigma(I=1)$ and the two $\Xi(I=1/2)$ states were reported at a conference at CERN (Geneva) in July 1962. At the same time, G. and S. Goldhaber reported the inexistence of similar resonances in K-N scattering (i.e., $Y = 2$). Up to this point, all the reported states could have been assigned to either one of two multiplets, the decimet of Figure 2c or a 27-state hexagonal multiplet with $I = 2$, $Y = 2$ states. The Goldhabers' negative search enabled me[37] to select the decimet assignment and to predict the precise properties of its missing member, the Ω^- hyperon, with $J^P = (3/2)^+$, $I = 0$, $Y = -2$, $m = 1673$ MeV (note that the Okubo formula simply becomes an *equal-spacing* rule, for the masses in the *triangular* decimet). M. Gell-Mann reached the same conclusions independently.[38] The experimental validation[39] of the existence and properties of the Ω^- in February of 1964 brought about the adoption of $SU(3)$ (in our baryon octet version – the Sakata model has no room for such an assignment).

7 THE QUARK MODEL

Chemistry came of age around 1868, when Mendeleev correctly identified the *order* in the patterns listing chemical elements, with their characteristic properties as abstracted from the phenomenology – *valencies* being one important example. It took, however, more than half a century and the work of Roentgen, Becquerel, J. J. Thomson, Rutherford's uncovering of the structure of the atom in 1911, Bohr's 1916 quantum version and finally Chadwick's 1932 discovery of the neutron – before the *structural* explanation of this order was understood. Biology provides us with yet another example of a classification (Carl von Linne's, in the 17th century) providing the first step in a prolonged effort, finally leading to a structural understanding (the uncovering of the genetic code in the 1950's) – with the work of Gregory Mendel representing the key intermediate stage. Progress in physics between Kepler's identification of three regularities in the motion of the planets in the solar system – and Newton's discovery of the laws of mechanics half a century later - can be considered as another such example.

The analogous sequence, from our 1960/61 classification of the hadrons, to a structural understanding of this order, appears to have gone faster. Early in 1962, with H. Goldberg, we conceived[40] a "mathematical model" which would produce the combinatorics leading to just such a pattern. *(1) The basic constituents had to be fermions with baryon charge* B = *1/3*, and *(2) they would have to come in three types.* All observed baryons and their $SU(3)$ representations would thus correspond to specific symmetry patterns of sets of three "bricks", selected out of the "available" three types. $SU(3)$ symmetry itself is the expression of the equality between the three types with respect to the Strong Interaction, the invariance of the latter under transitions mixing or exchanging the three types. The known mesons, on the other hand, whatever their spins, are composed of a "brick-antibric" pair. The "basic bricks" are now named *quarks*, following a suggestion by M. Gell-Mann, who about two years later provided the mathematical idea with a more physical formulation[41]; the name "quarks" was borrowed from a phrase in Joyce's "Finnegan's Wake", "three quarks for Master Mark". The "types" are now known as *flavors*, a term also selected by Gell-Mann, the inspiration this time coming from Baskin-Robbins, i.e., palatal rather than literary. Another presentation of the same quark-brick idea was made by G. Zweig in 1964.[42] To display their flavors, the three quarks were denoted *u, d, s* (for "up", "down", "strange" or "sideways", or also "singlet"). Between 1974 and 1996, three more quark flavors were discovered (the fourth was predicted) and named *c, t, b* (for "charm", "top" and "bottom" or also "truth" and "beauty"). Both cosmological-astrophysical calculations and accelerator experiments appear to indicate that there are no more than six flavors altogether. *The electrical charges* of all quarks are fractional, 2/3 for *u, c, t* and -1/3 for *d, s, b*, in units of *e*, the absolute value of the charge on the electron.

Between 1964 and 1968, the *"naive" quark model* (i.e., without a dynamical explanation) provided hundreds of experimentally verifiable predictions and passed all tests beautifully. Two simple examples relate to the ratio 3:2. F. Gürsey and L. Radicati[43] had suggested a methodology for non-relativistic situations ("$SU(6)$"). Applied to the magnetic moments, it required the ratio between the proton's and the neutrons' (experimentally, both displaying major unexplained "anomalous" contributions) to have the value -3:2. The two numbers were known since the early fifties but had not been compared. The values in Bohr-magnetons are $\mu_p = 2.79$, $\mu_n = -1.96$, i.e., a ratio of -1.42. The difference fits well with the corrections due to the breaking of $SU(3)$ and the non-relativistic treatment. Another typical prediction is due to E. Levin and L. Frankfurt[44], who evaluated the ratio between the nucleon-nucleon and pion-nucleon total cross-sections, in the *high-energy "asymptotic"* limit. In this limit, the scattering of a particle and of its antiparticle on the same target should yield the same cross-sections, according to a theorem in Quantum Field Theory. In the comparison due to Levin and Frankfurt, the target is a nucleon in both cases; one scattered beam is made of three quarks, the other of quark-antiquark pairs, i.e., the equivalent of two quarks. The ratio should thus be approaching 3:2 in the asymptotic limit. Many other predictions of that nature were published by Lipkin and Scheck 45 and others[46], all experimentally validated.

An evaluation of the *"bare"* masses of the quarks indicate masses $m_u = 5\ MeV$, $m_d = 9\ MeV$, $m_s = 160\ MeV$, $m_c = 1.4\ GeV$, $m_b = 5.0\ GeV$, $m_t = 170.0\ GeV$. Note that all masses are dwarfed when compared with that of the t – a mystery as yet unexplained by theory, as is the entire issue of the origin of these masses. The strong interaction appears to add about 305 *MeV* per quark, whatever the flavor. As a result, e.g., the proton, which is a *uud* compound, acquires a mass of 5 + 5 + 9 + 915 = 934 *MeV* (the correct figure is 938 *MeV*), the neutron (a *udd* compound) should acquire a mass of 5 + 9 + 9 + 915 = 938 *MeV* (939.6 *MeV* experimentally) and the Λ_0 (*uds*) 5 + 9 + 160 + 915 = 1089 *MeV* for the physical 1,115 *MeV*.

The "direct" experimental proof of the existence of quarks was achieved in 1967-69 at the Stanford Linear Accelerator ("SLAC") in deep-inelastic electron-nucleon scattering[47], somewhat in the spirit of Rutherford's 1911 probing of the structure of the atom. At the time, Rutherford had probed a very thin metal sheet with a beam of *alpha particles* (later identified as He nuclei, with a positive electric charge +2e). Most projectiles went through with just a slight deflection, but in a small number of cases, the projectile was strongly reflected, almost as if it had hit very heavy and hard "rock" and was bouncing back elastically. From these results, there had then emerged a picture of an "empty" atom, with orbiting electrons – and a nucleus taking up only 1:100,000 of the radius (or 10-10 of the area) and containing all but 1:2000 of the atomic mass.

In 1957-59, Hofstadter and colleagues had probed the meson cloud surrounding nucleons, scattering on nucleon targets elastic photons emitted from recoiling electrons. Instead of the expected 0^- pion cloud, the probing (as shown by Y. Nambu) had revealed clouds of $J^P = 1^-$ mesons, surrounding the nucleon; they turned out to be coupled, in an approximation to the Yang-Mills 8 dynamical mode for local symmetries, to **I**-spin, to hypercharge and to baryon number. These mesons were later produced directly and constitute the 1^- octet + singlet multiplet of Figure 2e, coupled a la Yang-Mills to the entire set of $U(3)$ currents, i.e., $SU(3)$ plus baryon charge. They decay into $J^P = 0^-$ mesons, which populate the outer reaches of the nucleon. In the 1967-69 version, R. Taylor and his colleagues[47] probed the "insides" of protons and neutrons by hitting them with high-momentum-transfer photons, again emitted from recoiling highly-accelerated electrons. The experiments involved large values of the momentum-transfer and *highly inelastic* scattering. The photons emitted by the recoiling electrons were found to interact with *point-like electric charges "floating" within the nucleon* and accounting for about 50 percent of the nucleons' momentum – and with fractional values of the charges, fitting the quark picture. The other 50 percent of the proton momentum is carried by matter which is electrically neutral.

Four puzzles still remained unresolved. The first related to the nature of the Strong Interactions *at very short distances or very high energies*, displaying what looks very much like *quasi-free* quarks, floating pointwise within the nucleon "sea". The second related to the same subject – the nature of the Strong Interactions – but at *the low-energy or long-distance end: why are there no free quarks? Are they confined inside hadrons?* Neither do we observe two-quark or four-quark hadronic systems. This is puzzle #3: are all but $3n$ (n an integer) systems forbidden? Puzzle #4 related to *the apparent breakdown of the spin-statistics correlation in quarks*. The Ω^-; $J^P = (3/2)^+$ hyperon (1,672 *MeV*), for instance, is an *sss* combination, the Δ^{++}, $J^P = (3/2)^+$ Fermi resonance (1,235 *MeV*) is *uuu*, etc; all also have their three quark spins aligned symmetrically (to add up to $(3/2)^+$). With their $J = 1/2$), quarks have to be fermions; how then can they form totally symmetric combinations in these hadrons?

8 QCD AND THE STANDARD MODEL

A plausible solution to all four puzzles emerged with the developments in Quantum Field Theory in 1971-73. We described in section 2 the *Yang-Mills model* of a *local symmetry*, indirectly inspired by Einstein's General Theory though much simpler, a dynamical theory, geometrical in its nature. When it was published, the Yang-Mills "theory"[8] was just a model, a mechanism searching for an application. In 1958-62 it was successfully applied to **I**-spin, etc., and then to the hadron's $SU(3)$ and in this role it is an important component of the *effective* low-energy strong interaction, involving the 1^- mesons first encountered in the Hofstadter experiments we described in the previous section – and coupled *universally* as explained in section 6. These gauge mesons are, however, massive, though such masses break the gauge symmetry. Meanwhile, however, between 1957 and 1971, a number of theorists had worked on developing a quantized version of the Yang-Mills model, similar to QED. When this was completed in 1971, the model was restudied as to its physical features. It then turned out (1973) that these features of the exact quantized theory were just what was needed in order to understand the puzzles we listed. *These theories indeed have a coupling which weakens at shorter ranges*, where the relevant charged matter currents become effectively uncoupled. In the opposite direction, with increasing distance, the force increases like in a spring. *Quarks are thus confined*, though there is as yet no exact proof that the confinement is absolute. Another feature deserving to be mentioned: QCD is mediated by an octet of 1^- massless (but confined too) *gluons* – explaining for instance the nature of the electrically neutral stuff found to carry 50 percent of the nucleon momentum (see section 7).

What is the charge generating the Strong Interaction, at the quark level? To explain all four puzzles, this had to be yet another $SU(3)$, which was named *color* by Gell-Mann. *Each of the six flavors of quarks appears in three colors.* This explains the *spin-statistics correlation* - the quarks enter in configurations which are *antisymmetric in the color labels*, and are thus normal fermions. It also explains *saturation at three* – the fact that configurations involving $3n$ quarks are favored energetically. *Quantum Chromodynamics (QCD)* is the name of this basic theory of the strong nuclear force. 48-49 Yukawa's original suggestion of the 0^- pion mediating the nuclear interaction – or the approximate Yang-Mills interaction mediated by the 1^- mesons – these remain true but only as "effective" higher order contributions of QCD, itself acting at the quark level. Note that $SU(3)$ flavor and $SU(3)$ color "commute", i.e., *QCD is invariant under flavor transformations. The breaking of* SU(3) *flavor is due to the quarks' bare masses, part of the input in QCD.*

We mentioned in sections 3 and 5 the characteristics of the Weak Interactions and their currents, such as the hadron current appearing in the neutron's *beta-decay* $n \rightarrow p + e^- + \overline{\nu}$; or the related $p + \overline{\nu} \rightarrow n + \overline{e}^+$ through which Reines and Cowan proved in 1955 that neutrinos exist. This is roughly the **I**-spin transitions current ($n \leftrightarrow p$), relating to a symmetry of the Strong Interactions, which explains why this Weak coupling is unmodified by the Strong Interactions and stays equal to the one measured in muon decay $\mu^- \rightarrow \nu_\mu + e^- + \overline{\nu}_e$. It became clear in 1957 that the Weak Interactions involve currents very similar to the electromagnetic. S. Weinberg and A. Salam, using an algebraic $U(2)$ construction due to S. Glashow, raised in 1967-68 the possibility that the Weak and Electromagnetic Interactions might represent at higher energies (above 100 *GeV*) pieces of one single interaction.[50-51] The theory became precise with the progress achieved in the construction of the quantum version of the Yang-Mills model. It requires four 1^- universally coupled mesons, three of which acquire large masses when the symmetry is broken (they were indeed discovered in 1982) and the fourth – remaining massless – is the electromagnetic potential. The *electroweak theory* is thus a Yang-Mills theory with "spontaneously" broken symmetry. It turns out that such a mechanism can also be derived in a purely geometrical formulation – applying a recently developed new branch of geometry – *noncommutative geometry*.[52-53] I had hit on this formulation, using "naïve" arguments[54-56], *before* the emergence of the new mathematics; it was later shown that my formulation is that of the new geometry.[57] The *electroweak* theory, proved experimentally, is thus also a dynamical and geometrical theory and forms, together with QCD, the *Standard Model.* This is a well-tested dynamical theory of the fundamental forces other than gravity, covering phenomena as explored below 500 *GeV.*

The entire Standard Model is thus geometrical – in line with Plato's guess. Gravity, the remaining interaction outside it, is also geometrical – as established by Einstein – but it is as yet a *classical* theory. There are several candidate models for a quantum adaptation, but at the writing of this article, we do not yet know which is the correct version. We can only guess, at this stage, that both Quantum Gravity and beyond it, the unifying prospective theory which will encompass both that Quantum Gravity and the Standard Model (present candidates: *supergravity, the superstring, "M-theory" or the supermembrane,*) will also end up fulfilling Plato's adage.

REFERENCES (NOTES AND BIBLIOGRAPHY)

[1] Noether, E. (1918) Nach. d. Kgl. Ges. d Wiss , *Math.-phys, Klasse* 2, 17

[2] Minkowski, H (1908) *Lecture at the 80th Assembly of the German Researchers and Physicians*, Cologne.

[3] London, F (1927) *Z.f Phys.* 42, 375.

[4] Kaluza, Th. (1921) *Sitz. Preuss Akad. Wiss.*, 966, Klein, O. (1926) *Nature* 118, 516.

[5] Whittaker, E. (1949) *From Euclid to Eddington*, Cambridge University Press.

[6] Weyl, H (1929) *Z. f. Phys.* 56, 330

[7] Wu, T. T. and Yang, C N (1975) *Phys. Rev.,* D12, 3845.

[8] Yang, C. N. and Mills, R L (1954) *Phys. Rev.* 96, 191

[9] Ne'eman, Y. (1997) Gauge Theories of Gravity, Opening address at Jadwisin (Poland) symposium to be published in *Acta Phys. Polon,* B.

[10] Hehl, F. W., Heyde, P. v d., Kerlick, G. D. and Nester, J. M (1976) *Rev. Mod. Phys* 48, 393.

[11] Hehl, F. W., McCrea, J D., Mielke, E. W. and Ne'eman, Y. (1995) *Phys. Rep* 258, 1.

[12] Freedman, D. Z., Ferrara, S. and Nieuwenhuizen, P. v. (1976) *Phys Rev.,* D13, 3214, Deser, S and Zumino, B. (1976) *Phys. Lett.*, B62, 335

[13] Cremmer, E. and Julia, B. (1978) *Phys Lett.*, B80, 48 and (1979) *Nucl Phys.*, B159, 141.

[14] Chadwick, J. (1932) *Proc. Roy. Soc.*, A136, 692.

[15] Heisenberg, W. (1932) *Z f Phys.*, 77, 1.

[16] Nishijima, K (1955) *Prog. Theoret. Phys* 13, 285; Gell-Mann, M. (1956) *Nuovo Cim. Suppl.* 2, 848.

[17] This is generally attributed to E. P Wigner.

[18] Yukawa, H (1935) *Proc. Phys. Math Soc. (Japan) (3)* 17, 48.

[19] Fermi, E (1934) *Z. f. Phys.* 88, 161.

[20] Broglie, L et al. (1963) *Phys. Rev.* 129, 438, 451.

[21] Fermi, E. and Yang, C.N. (1949) *Phys. Rev.* 76, 1730.

[22] S. Sakata, *Prog. Theoret. Phys. (Kyoto)* 16 (1956) 686

[23] Fujimoto, Y (1970) Foundations of S. Sakata's Theory etc., *Pric Lenin Symp.;* see also appendix by Fujimoto, Y. to M. Taketani's letter to Nobel Committee, published in *Soryushiron Kenkyu (Study of Elementary Particles),* Nov. 1970, p. 262.

[24] Lenin, V. I. (1927) *Materialism and Empirio-Criticism* (vol XIII of Coll Works), International Pub., New York, pp. 220-230.

[25] Chew, G. (1961) *The S-Matrix Theory of the Strong Interactions*, W.A. Benjamin pub., New York.

[26] Salam, A. and Polkinghorne, J.C. (1955) *Nuovo Cim* 2, 685.

[27] Gell-Mann, M. (1957) *Phys Rev.* 106, 1296; Schwinger, J. (1957) *Ann. Phys. (NY)* 2 (1957) 407. J. Tiomno, *Nuovo Cim.* 6 (1957) 69

[28] Salam, A. and Ward, J.C (1961) *Nuovo Cim.* 20, 419.
[29] Ne'eman, Y (1961) *Nucl Phys* 26, 222.
[30] Dynkin, E B (1947) *Uspekhi Mat. Nauk.* 2, n. 4
[31] Cartan, E. (1894) *These*, Paris.
[32] Gell-Mann, M (1961) *Caltech rep.* CTSL-20, unp.
[33] Speiser, D and Tarski, J (1961) IAS Princeton rep., unp
[34] Behrends, R.E., Dreitlein, J , Fronsdal, C and Lee, B.W. (1962) *Rev Mod. Phys.* 34, 1.
[35] Glashow, S. and Gell-Mann, M. (1961) *Ann. Phys (NY)* 15, 437
[36] Ne'eman, Y. (1964) *Phys. Rev.*, B134, 1355.
[37] Goldhaber, G. (1985) The encounter in the bus, in *From* SU*(3) to Gravity*, Gotsman, E. and Tauber, G., eds., Cambridge University Press, p. 103-106; see also Cahn, R.N. and Goldhaber, G. (1988) *The Experimental Foundations of Particle Physics*, Cambridge University Press, p.116.
[38] Gell-Mann, M. (1962) *Proc 1962 (Geneva) ICHEP*, p. 805.
[39] Barnes, V.E. et al. (1964) *Phys. Rev. Lett.* 12, 204.
[40] Goldberg, H. and Ne'eman, Y. (1963) *Nuovo Cim.* 27, 1.
[41] Gell-Mann, M. (1964) *Phys. Lett.* 8, 214.
[42] Zweig, G (1964) CERN rep TH401, unp.
[43] Gürsey, F. and Radicati, L (1964) *Phys. Rev. Lett.* 13, 173
[44] Levin, E M. and Frankfurt, L L (1965) *JETP Lett.* 2, 105
[45] Lipkin, H.J. and Scheck, F. (1966) *Phys Rev Lett.* 16, 71; see also Lipkin, H J. (1973) Quarks for Pedestrians, *Phys Rep* 8, N3
[46] See for example Cabibbo, N , Horwitz L.P and Ne'eman, Y (1966) *Phys. Lett* 22, 336.
[47] Bloom, E.D. et al. (1969) *Phys. Rev. Lett.* 23, 930; Breidenbach, M. et al. (1969) *Phys. Rev. Lett.* 23, 933.
[48] Fritzsch, H. and Gell-Mann, M (1972) in *Proc. XVIth ICHEP (Chicago)* 2, 135.
[49] Weinberg, S (1973) *Phys. Rev. Lett.* 31, 494.
[50] Weinberg, S. (1967) *Phys. Rev. Lett.* 19, 1264.
[51] Salam, A. (1968) in *Elementary Particle Theory*, Svartholm, N., ed., Almquist Vorlag AB, Stockholm.
[52] Quillen, D (1985) *Topology* 24, 89
[53] Connes, A. (1994) *Noncommutative Geometry*, New York: Academic Press, p. 661; Connes, A. and Lott, J. (1990) *Nucl. Phys.*, B18 (Proc. Suppl.), 29.
[54] Ne'eman, Y. (1979) *Phys Lett*, B81, 196; Fairlie, D.B (1979) *Phys Lett*, B82, 97
[55] Thierry-Mieg, J. and Ne'eman, Y. (1982) *Proc. Nat. Acad. Sci.*, USA, B79, 7068.
[56] Ne'eman, Y and Sternberg, S. (1990) *Proc. Nat. Acad. Sci.*, USA, 87, 89.
[57] Coquereaux, R., Haussling, R., Papadopoulos, N.A. and Scheck, F. (1992) *Int. J. Mod Phys.*, A7, 2809.

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SYMMETRY AS KEY-NOTION IN THE INTERRELATIONSHIP BETWEEN SCIENCE AND CONSTRUCTIVIST ART

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"12. *Symmetry and repetition.* The new architecture has suppressed symmetry's monotonous repetition as well as rigid equality which results from division into two halves or the use of the mirror image. It employs neither repetition in time, street walls, nor standardized parts. A block of houses is as much a whole as a single house. The laws governing single houses apply also to both blocks of houses and the city as a whole. In place of symmetry, the new architecture proposes *a balanced relationship of unequal parts*; that is to say parts, because of functional characteristics, differ in position, size, proportion and situation. The equivalence of these parts is gained through an equilibrium of unequality rather than of equality. In addition, the new architecture has granted equal value to the 'front', 'back', 'right', and possibly also to the 'above' and 'below'." (Theo van Doesburg, *Towards Plastic Architecture*, Paris 1924)[1]

As architecture, painting and sculpture always include a compositioning within a restricted area (like a building-site, the format of the canvas), which is nothing else than a balancing of elements each with its own particular visual "force", this balancing of forces made symmetry the key-notion both in abstract art and in architecture. This paper will focus on art and discuss that the ideas as exemplified by van Doesburg are not merely coincidental or a result of merely studying composition-possibilities. This can be stated, because many abstract working artists have always been interested in cross-references to scientific knowledge. Obviously under influence of the culture of which they are a part and the *Zeitgeist* they lived in or still live in. Contrasts, rhythms, complementarity of form and colour, their interrelationships and (visual) counterbalancing are the tools. Compositioning as such is nothing else than a sensitive balancing around the situation of absolute symmetry. This feature of art becomes clear in this century in the emergence and further development of the art that one in general knows as "constructivist art". It is interesting to re-study the analogies and interrelationships between this art and the development of scientific knowledge in direct relation to historic events in society. As the complete material on this subject concerns an immense amount of information, I do hope it can be understood that in this paper only some examples can be presented. It still will become clear, however, how the tension between symmetry and *a*symmetrical symmetry are key-notions in the search for order and structure in the phenomena of reality. For further reading a small selection of literature is suggested at the end of this paper. As this seems to be appropriate this paper starts with a more or less extensive survey of some aspects of the pioneer-period of constructivism, than unfortunately only a very few aspects of later developments can be discussed in relation to the limitations of length and density of this paper. Finally, some contemporary artists will be discussed briefly.

THE PIONEER-PERIOD

This period concerns the birth of constructivist art. It emerged between approximately 1910 and 1922 as result of simultaneous interests and efforts by several artists in different groups and different countries in Europe. Constructivism in art is slightly different from constructivism in science or constructivism in psychology. Constructivism in science means constructivism in mathematics, which is considered the equivalent of intuitionism founded by the Dutch mathematician L. E. J. Brouwer. Brouwer considered mathematics to be inseparably linked to the thinking of the individual. The mathematical activity is considered to consist of constructions and activities in the mind of the mathematician and as such is opposed to the formalistic opinion that mathematics is merely a formal manipulation of symbols. Brouwer and followers considered mathematics to be absolutely autonomous. Other constructivist mathematicians were Bishop, Beeson and Stolzenberg. The constructivist aspect of Euclidian geometry often refers to "straightedge and compass constructions" (see Greenberg 1994, p. 49). This obviously is in its *Zeitgeist* quite well interrelated to constructivism in early abstract art. Especially Russian constructivism has made its mark on history, although one should also recall the Dutch *De Stijl*, Polish groups *Blok* and *AR*, Hungarian *MA*, Balkan *Zenith*, the *Hannover Abstracts*, and many others in Belgium, Italy, Czechoslovakia, Yugoslavia and elsewhere. It is worthwhile to have a look at some of the books in the proposed list of literature.

When one hears of constructivism for the first time it may seem that there would be a clear understanding about its definition. Like with so many *Zeitgeist*-related directions, schools or groups in art this is not the case. But, then again, the same occurs in science, philosophy or in politics. One is only reminded of the very different interpretations on humanism in philosophy or the differences in definitions of socialism in the several decades of the 20th century. When one studies the available information one finds quite easily that constructivism is a notion more or less related to a specific concept of art, philosophy and science, but still without a clear definition. Art historian and scholar on constructivism Patricia Railing discusses the difficulties around the concept of constructivism even during its years of emergence in Russia[2]. The Russian artist/architect/designer/theorist El Lissitzky[3] lists two groups considering themselves constructivists: the *Obmohku*[4] and the *Unovis*[5]. But, there were also the brothers Naum Gabo and Antoine Pevsner, who published *The Realistic Manifesto* in 1915, which can be considered to also have set the line of thinking on constructivism. Or take Alexei Gan, Rodchenko and Tatlin, and not to forget the Hungarian artist/designer/theorist and important educator Moholy-Nagy. He first made himself quite well-known in Hungary, before becoming one of the major professors at the *Bauhaus* in Germany. The *Bauhaus*, which makes up theories for a multi-disciplinary approach towards an improvement of our everyday environment and all the products in it, still sets the tone for all contemporary industrial design. Escaping fascism Moholy-Nagy moved to the USA where he first held influential courses on art and photography in New York before becoming one of the founding professors of *The New Bauhaus* at Chicago, which later was renamed the ITT School of Design that still exists today. Besides, as mentioned above, there were many more simultaneous developments all through Europe in the 1920's and early 1930's[6]. An important general characteristic was to try to visualize the cosmic powers of nature. The powers which scientific knowledge and theories showed the public to exist "under" or "behind" the visible reality of our existence. To be able to show this, the artists believed in trying to show clear "constructions" as a kind of reconstructions of the motions of nature. As such, many opted for clear and "honest" visualizations with the aim to approach the problem in a kind of scientific sense. The result was that sculptors like Tatlin or Gabo wished to discard colour for its illusionary properties and preferred to use the properties of the material – like unpainted wood, steel or glass. Symmetry was not studied as most important feature by these artists, although a kind of symmetry can be found in some paintings like by Malevitch, Kliun or Gontcharova. The purest symmetry was used by Rodchenko in a very nihilistic sense when he painted his highly idealistic pure monochrome paintings in 1921 (one in red, one in yellow, one in blue), declaring this the end of painting. This bold semi-political statement is quite typical for his *Zeitgeist* as painting was understood by many Bolshevik artists as academic, elitist and the metaphor for the bourgeoisie. It is somehow ironic and typical for human history that first in the 1960's and also currently one can find quite so many artists painting only monochromes, believing that more then the monochrome surface would be too much, too. It is needless to say that Rodchenko's statement was very wrong. Painting was and is not finished. Just like today the existence of the computer has not "finished" painting, sculpture or architecture.

The emergence of simultaneous interests of artists, architects and theorists in constructivism was not an opportunist response to incentives from the international art market as one could expect in today's world. In the 1920's or 1930's there still was virtually no art market! The inspiration to take up constructivism came first from their commitment to contribute as artists to the improvement of our knowledge on reality – with the desire to be as important as scientists in this matter. The big impulses came however from politics and economics. These led artists to take up their responsibility to try to help improve society as it is by improving all design[7]. This was on one hand the result of the political changes in Europe: the communist revolution in Russia, the need for an international non-elitarist society for all "working-class" people after the slaughter of the World War I (of which the elite of the nobility and bourgeoisie were considered guilty, sacrificing working-class people by the thousands for their political games). On the other hand, lots of excitement and inspiration to accomplish this new mission for artists came from modern science, especially from theoretical physics and astronomy. Although most artists will not have actually studied Poincaré, Minkowski, Bohr or Einstein, they picked up general ideas on space/time and translated their understanding into artistic concepts of a fourth dimension. Exceptions were some of the pioneers of constructivist art, like I would like to mention Michael Matjushin and Kazimir Malevitch who actually did study scientific knowledge. Especially 19th century knowledge as published by the American Hinton, translated into Russian around 1910. To Matjushin and Malevitch also the ideas of projective geometry became well-known through the theories of Lobachevsky. El Lissitzky has referred in one of his lectures to the space/time curvature of Einstein. Around 1919 El Lissitzky did change his suprematist linear motion as simulated in his paintings into compositions that clearly simulate a motion that is curved – often curling back into itself. Let us not forget that the highly influential and gifted painted/theorist/teacher Wassily Kandinsky first followed courses in physics and engineering before starting to paint. He has tried all his life to make clear the importance of connecting art and science into a new synthesis. This, in my opinion, explains why he worked out an artistic translation and application of his ideas on the importance of the point, the line and the plane exemplifying motion and dynamics as similar to theories in the modern physics of his days. Around the turn of the 19^{th} and the 20^{th} centuries, in philosophy and in science one of the hot debates was our fascination for *geometry*[8]. Other *Zeitgeist*-related subjects were keen interests in the *idea,* in the *sensation* and in *construction.* In relation to the sensation for instance the influence on our physical well-being of certain colours in our environment were explored. Obviously, these subjects did not come out of the blue at the start of the 20^{th} century, but are logical results from discussions in the 19^{th} century. Although it would take too long a detour to explain the relationships, or to prove what I say here, it should be sufficient for now to know that the debate on *a free will* from the beginning of the 19^{th} century and later discussions on the object versus the subject led to ideas on a basic structure of nature as a whole[9], to ideas on continuity, progress and the notions of space and time, finally leading to think about the construction of reality. Also, in the 19^{th} century new theories on colour[10] and light, but also discoveries like the principles of the conservation of energy, the discoveries of electromagnetism and of X-rays clearly influenced the interests of scientists, philosophers but also of artists around the turn of the 19^{th} and the 20^{th} centuries. As evidence of how subjects can be *Zeitgeist*-related, I would only like to mention that Maxwell's electromagnetism and new ideas on the properties of light, space and time obviously inspired scientists like Minkowski or Einstein, or a mystic like Ouspensky, but coincided with the emerged interest in the occult. In occultism mystical theories are discussed between our selves and the universe, on the connection between everything through superstructures or supervised by higher beings. Serious interests in knowledge from ancient and oriental cultures became natural (a kind of late 19^{th} century "New Age", as we would call it today). Anthroposophy as initiated by Madame Blavatsky and later developed by Rudolf Steiner into a cultivated philosophy is its offspring. In a different way in this occult interest there were analogous interests in electromagnetism, light and space/time as in science. Like the reflection of cosmic energies inside our bodies. Artists tried to combine both approaches. To find the structure of nature, to resemble the construction of reality, they tried to search for the universal. To find the universal, they had serious interest in certain aspects of folk art[11] and in other cultures. Proof of this can be found with artists like Malevitch or Kandinsky who clearly knew about for instance Hindu or Islamic art[12]. Besides, literature on Hindu or Hebrew philosophies, but also for instance the scientific ideas on space and time by the earlier mentioned American Hinton were translated into Russian before or around 1910. Obviously, the publications on new discoveries and theories in astronomy or on X-rays were sources of information to the artists[13]. An artist like Malevitch was fascinated by the evidence that we as humans on our little planet Earth are part of the universal motions and developed his famous paintings of geometric shapes "flying" past us as planets. This was the start of his *Suprematism.* The discovery of X-rays was exhibited to a large public, showing how it enables us to look through our bodies, showing the bones in one's hand or leg. To the kind of artists discussed here, this proved that matter does not exist indeed! It inspired artists from Cubism or Futurism to discard of painting solid shapes and instead opted for intersecting planes and shapes that give the impression to be virtually transparent. Cubism and Futurism both also tried to translate their understanding of space/time[14]. These ideas and the scientific interests in light stimulated artists like Moholy-Nagy and later the American pioneer of photography Quigley to explore photography into a new medium for abstract art by using only light directly to shape forms. Moholy-Nagy also experimented with light itself in his sculptural installations. In the translation of all these sources of inspiration or analogous, *Zeitgeist*-related subjects, these artists used no exact or pure symmetry (even Malevitch's famous "Black Square" is not perfectly symmetrical), but the kind of dynamic equilibrium of unequal parts of which van Doesburg speaks in the quote at the start of this paper.

The notion of constructivism, therefore, is from the start no mere painter's idea of a certain kind of composition, but a clear example of the connection between artistic explorations and scientific knowledge in a certain cultural and political climate that by its characteristics has sparked the desire to make this connection work. Anyway from the artists' point of view – not so many scientists have ever really been interested in art, especially in thinking and writing artists, although this seems to slowly change now. The different aspects of the *Zeitgeist*, therefore, must always be taken into account when one tries to understand any development in history. Constructivism, as it also has been used by later generations of artists is based on the assumption that the world around us is constructed on some kind of logic, consisting of elementary particles or basic units or cells and governed by laws by which their construction and interaction is defined. To the early generations of constructivists our image of the world, like exemplified in artworks, should therefore also be constructed out of basic units and be based upon a certain logic. A related idea was to present works of art to the observer in which the system or structure or construction method can easily be retraced. Purity and honesty of form, colour, and material were considered important to find the universal in combination with the individual – exemplified in the personal interpretation of every different artist – and thus to present the essence of reality (with obvious relations to for instance Hússerl's phenomenology). Visual illusions and complexity were also considered decorative, elitarist, bourgeois and simply *wrong*. As was said before in this paper, the artists of constructivism obviously hoped to become as fundamentally important to the ideas and knowledge on reality as the scientist. The direct link as seen by these first generations of constructive artists between art – applied design or industrial design – architecture – urban design and everything in between[15] basically has formed the accepted fundamental notion all through the 20th century concerning all design of any object or product or environment that have shaped our societies[16].

In modern psychology constructivism is a general theoretical position which proposes that perception and perceptual experience are, according to Gregory, constructed out of *"volatile fragmentary scraps of information, for a part caught by the senses, for another part from memories-banks - that are itself constructions from bits and pieces out of the past."*[17]. In this psychology the perceptual experience is not understood as merely a response to a stimulus, but as a construction of hypothesized cognitive and affective operations. In sociology social constructivism is the family name for all theoretical approaches that emphasize the social character of the human construction of meaning. Social construc*tion*ism is introduced by K. J. Gergen for a theoretical-scientific approach of psychology and its field of enquiry stating that the phenomena studied by psychology are in fact historical and social constructions. More recently, constructivism has been taken up by psychotherapy and is considered to be part of the postmodernist approach. This approach in modern constructivist or constructionist psychotherapy is based on the assumption that nobody makes his or her image of reality based on objective observation, but as a construction of subjective "realities"[18].

AFTER WORLD WAR II

The idealistic search of the early constructive artists for a complete renovation of society by desiring a new world, a new man and (opted by some) a new order was taken over by the national-socialists in their particular understanding. Even worse, the fascist propaganda-machine used the knowledge on media-communication which was developed by constructivist artists: strong symbols in clear designed graphics and dynamic photography. It is somewhat ironic, that the Nazis not only choose to copy from the constructivists the effective use of black and red for their propaganda and emblems, but even used a typical constructivist layout and typography for posters of the exhibition of *Entartete Kunst*. In the late 1940's and in the 1950's one can find in general that the artists retreated in their studios, focussing on painting and sculpture in a more classical sense. This was not only the result of the public mistrust for any art that in appearance or statement was associated by them to the new order of dictatorial fascism, but of course was also stimulated by a slowly emerging art market. As in that market as a logical result in the post-war traumata no geometric art, but more personal and emotional tachistic or expressionistic art became successful and ruling, many postwar constructivist artists had a hard life in merely surviving. This was the same on both sides of what later became known as the Iron Curtain. Similar to science, psychology and sociology (or linguistics), the constructivist artists from the 1950's and into the 1960's researched the notion of structure and structuralism. This development can be explained as a further result from the interests in these fields earlier in the 20th century as was explained. It is interesting to further explain the growing interest in a strict ordering of elements by referring to Quantum mechanics, especially to Quantum Field Theory and a general interest in the notions of "fields" or "nets" as metaphors to explain all interactions and interconnections of events. But also to structural chemistry[19], which led for instance to the development of the DNA-structural model by Crick and Watson in 1956. Interests that became emphasized by new ideas, like by the theory in linguistics by the notorious Noam Chomsky. Chomsky stated that the structure of a language is more essential than its phonetics, revolting against Roman Jakobson's emphasis on phonetics[20]. In the 1960's Claude Lévy-Strauss followed in cultural anthropology with his structural anthropology, emphasizing the overall or underlying structure of a culture. In that period we can find in constructivist art many links to the *Zeitgeist*-related interest in structure, like in structuralism in art. To this generation of artists the American artist Charles Biederman was a prime source, especially with his book *Art as the Evolution of Visual Knowledge* from 1948. Biederman's own main sources or influences on his ideas were the artists Courbet, Monet, Cézanne, the philosopher Whitehead and the semiotician Korzybski. He focussed on the development through human history of the visualization, the interest to translate natural light and the structure of nature at a deeper level into artworks with the aim to understand reality better. As the notion "evolution" already indicates, he believes strongly in the general progress of knowledge and of mankind itself. He emphasizes the coloured relief as the new medium, being between painting and sculpture. The interest in structure is exemplified in art for example in the artists' magazine "Structure"[21] with regular contributions by the artists Joost Baljeu, Kenneth and Mary Martin, Anthony Hill, Richard Paul Lohse, Charles Biederman, Peter Lowe and Jean Gorin. Another example is the academic publication "The Structurist"[22]. The interest of these artists in the notion of space/time was especially discussed in relation to the notion of "growth" and illustrated by following van Doesburg's idea of a repetition of certain units or cells. This interest in simulating motion quite naturally inspired artists to use actual motion in their sculptures. As such, kinetic art became known[23]. Examples are the French artist Nicolas Schöffer, the Russian late 1950's group *Dvezhenye* (movement) initiated by the artist Lev Nussberg and including Francisco Infante, the Spanish group *Equipo57* of Angel Duarte or the American and probably best known-kinetic artist George Rickey. A different, but still familiar direction in constructive art was born just before the war in Switzerland: *konkrete Kunst* (concrete art). It can be seen as the furthered version of van Doesburg's *Concrete Manifesto* from 1931. "Concrete" would be a work of art that does not refer to any natural object then to itself. The main artists were Max Bill and Richard Paul Lohse. Others were Camille Graeser, Hans Hinterreiter and Verena Loewensberg. Their concept is still being followed by many artists in Europe, although most have not studied the ideas and works of Lohse enough. The modular and serial structures in Lohse's highly coloured paintings are much more complex as in general concrete artists seem to have noticed. His works are so fascinating as he includes both rational and intuitive properties.

These magazines, groups and ways of working went well into the 1960's. In general they all led to an interest in searching an even more objective, a more logical or rational and more "scientific" approach to art. As we will see, it led to another and more strict use of symmetry. The moment constructive art almost came to a dead end.

ULTIMATE OBJECTIVITY

The 1960's and early 1970's focussed even more on the search for the optimum of objectivity. There should be no illusion whatsoever and no mimetic effects. The eye should see what the eye sees. This semi-scientific study by artists of perception led to simultaneous interests in Optical Art, Minimalism and Systems Art. Symmetry became essential once more. This should be understood as to be quite natural as the search for the ultimate objectivity in art is the equivalent to the search for the absolute balance between equal pictorial parts in an artwork. As such, one can draw a line between these artistic efforts and democratic ideas in the politics of the 1960's (the ideal that everybody is the same and has the same rights). These interests are exemplified by a group like the French/German group *GRAV*[24], including artists like François Morellet, Vera Molnár, Julio Leparc, Klaus Staudt. In France, England and the USA important incentives for "Optical Art" came from Michael Kidner, Bridget Riley, Richard Anuskiewicz, Victor Vasarely. By its sheer popularity and the highly commercial successful applications in fashion also known as "Op-Art". The art that became the metaphor for the hippie-cult and, without the intention of the artists, became very fashionable. In Germany, Italy, Belgium and the Netherlands similar initiatives for the pure objective came from groups like *Zero* and *Nul*, including artists like Macke, Piene, Ücker, Fontana, Manzoni, Peeters, Schoonhoven, Armando, Dekkers, Leblanc and many others. Especially in this art pure symmetry (absolute mirror symmetry) became the prime tool to represent absolute zero-subjective art. In sculpture one should not forget the Buckminster Fuller-pupil Kenneth Snelson with his magnificent constructions that exemplify a clear connection between art and engineering. Also, in the 1960's the first artistic experiments to find structure with aid of the computer were explored, like by Frieder Nake, Herbert W. Franke, Georg Nees, Gottfried Jäger, Vera Molnár, Peter Struycken, Ernest Edmonds, Manfred Mohr.

AMERICA'S EMPHASIS

For a long time, almost quite parallel to the existence of the Cold War, American absolute objectivism in Minimal Art ruled the international geometrical art. This art advocated absolute objectivism in great sizes of paintings and sculpture. Pure symmetry, repetition of form and size, exemplifying the material in its "own" right, using colour only in a "functional" way (as emphasis of the shape, not as something with its own meaning) should get rid of any emotional content. It was a clear artistic analogy to the architectural concept of pre-fabricated building and to Functionalism in architecture. Within American art history it was the natural response of a younger generation of artists to the large and emotional paintings of the American expressionists (Pollock, Kline, Motherwell, Rothko, De Kooning, Frankenthaler), a logical follow-up after Colour-Field painting (Leon Polk Smith and later Elsworth Kelly) and a rational protest to the success of their contemporaries in Pop-Art (Warhol, Rauschenberg, Rosenquist, etc.). Representatives were Donald Judd, Richard Serra, Frank Stella, Sol Lewitt, Carl André and Agnes Martin. Seen in a larger context and not only in the American context, the link to early Russian constructivism is evident in this art (just compare some of André's wooden sculptures to Rodchenko's), although the Americans miss a lot of the substantial contents the Russians did have. The success and the impact on the European artists by these Americans can only be explained as part of the general glorification of the USA as major world power of democracy and because of the fast growing art market in the USA at that time. The pure symmetry of the compositions can be understood as the desire for ultimate democracy of the political movements in the 1960's. Also analogies can be found in scientific ideas around that time. Ideas like David Bohm's Implicate Order and interests by many in oriental thought reflect the search for an order connecting everything at a deeper level, making everything of similar importance, making everybody the same. Only after the Cold War ended in the late 1980's, several of the Minimal Art artists (like Judd) admitted they had been inspired by early European (Russian) artists. Symmetry was to become the metaphor of this pure American emphasis.

FROM CHAOS TO COMPLEXITY INTO THE FUTURE

Analogous again to the developments in science and to the general interest in society, at the end of the 1970's and into the 1980's sincere constructivist artists became doubtful about the "reductionist" approach and opened up for instance to the implications of the concepts of Chaos Theory and even to the less general popular (less romantic sounding) Theory of Complexity in the late 1980's. Artists like Frank Stella, not necessarily pure constructivists anyway, understood these great changes in understanding reality through his own artistic development. Stella's highly rational black paintings of the 1960's, which brought him fame, have changed dramatically into reliefs that became proceedingly more complex during the 1980's. To some they may seem only "expressionistic chaotic" or even "baroque", but when one studies his progress through the years it becomes evident how his art has grown towards complexity in a most logical way from pure geometric to almost only organic shapes.

Michael Kidner (Figure 1) in England had been reading scientific books for a long time – out of a personal interest in knowledge on the process of reality and to search for a better understanding of what he was looking for as an artist. He is one of those exceptional artists who rather focusses on a neverending artistic research for structure and complex order. Kidner is especially interested in units that are interrelated like life-cells, each with their own individual behaviour and still related in shape and purpose. He combines a highly rational approach with intuitive experiments with unusual combinations of materials. For instance, he makes a circle out of a thin fiberglass rod and attaches broad elastic bands differing in widths at certain rationally decided intervals on spots along this circle, crossing the circle in a diagonal and symmetrical fashion. The result is often quite surprising, even to himself as well. The different tensions in the material, caused by the battle between the different widths of the elastic with the resistance of the fiberglass circle, make the perfect and symmetrical shaped flat circle jump into a three-dimensional and more complex shape – a kind of *Möbius* strip. The change from flat and perfect circle into its three-dimensional shape can be understood as a quick process of actual breaking of symmetry. Currently Kidner is exploring the artistic implications of Boolean nets using a kind of Penrose-tiling structure.

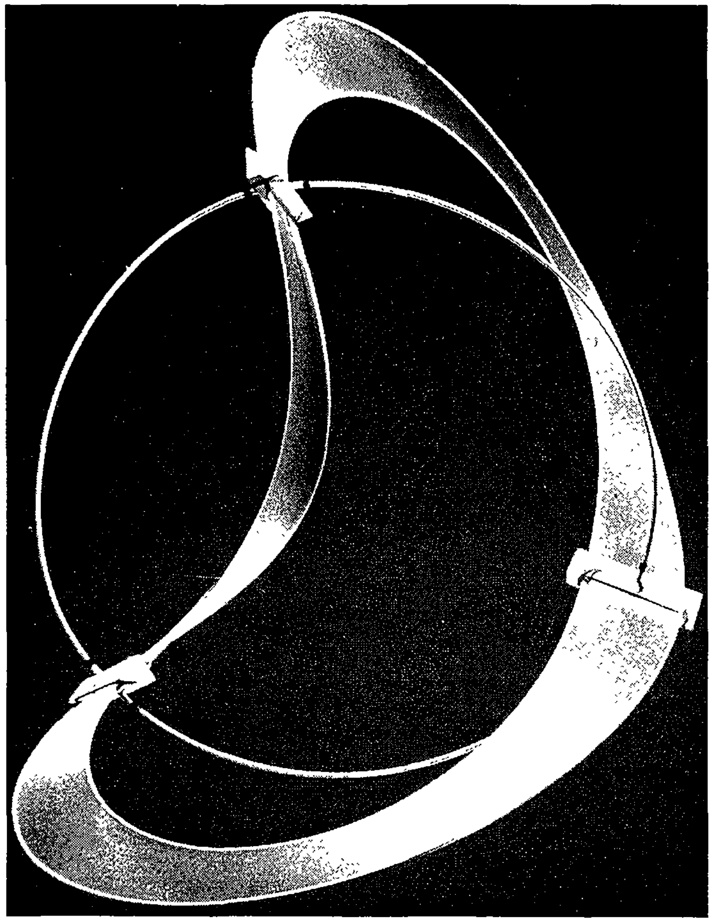
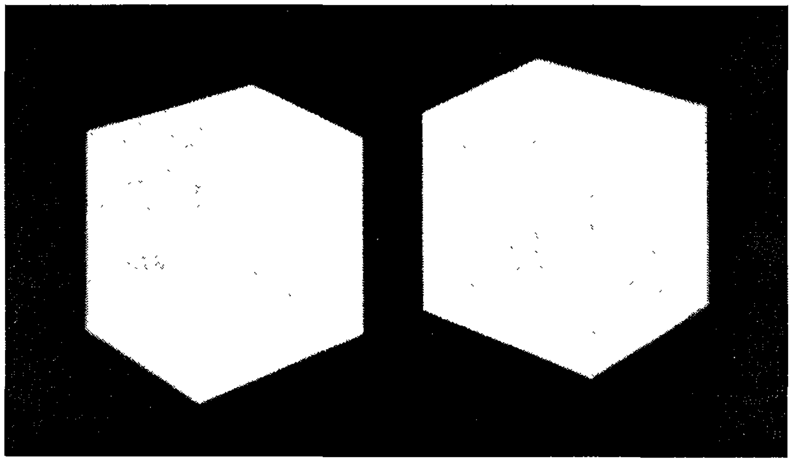


Figure 1: Michael Kıdner (1989) Cırcle and Oval in Countertension, 114x70x50 cm, fibre glass and nylon stockıng

American painter James Juszczyk (Figure 2), who can be considered to further the artistic colorist research of Josef Albers, works on the breaking of symmetrical compositions by a delicate range of colours and perspective projection methods. At a closer look one starts to see how intelligent the breaking of symmetry is accomplished in very little differences between two cubes. The cubes look to be in exact mirror-symmetry to each other, but evidently are not. The sensitive choice of colour-hues makes the pleasure and sensibility of looking at these pictures and give them a meditative functioning. This is not without reason. Juszczyk is also a sincere student of Zen-philosophy and aims to translate his meditation experiences into painting. Symmetry and breaking of symmetry also play important roles in modern Zen (as can be found in the Japanese culture) in relation to our position and direct interrelationship to nature. This connects this approach as such clearly to the modern science's interest in a process "on the edge of chaos", emphasizing continuous change.

Figure 2: James Juszczyk



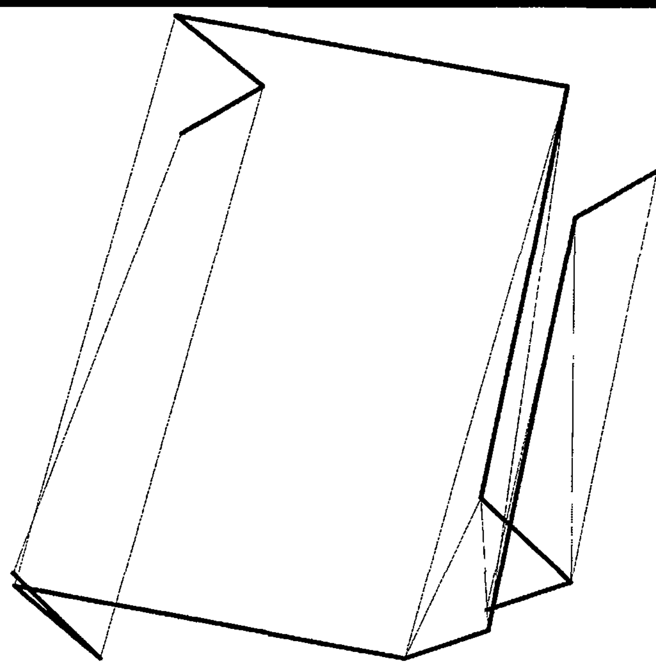


Figure 3: Manfred Mohr

Manfred Mohr (Figure 3), born in Germany and living in the USA, has been mentioned before. He is one of the very few artists using a computer while knowing how to programme the software he needs to develop his paintings. He is especially fascinated by exploring the hypercube and distorting it through a certain sequence of actions in time. Earlier works showed parts of the hypercube growing and interlocking like a complex crystal. Mohr's further interest is the abstract observation of studying a multi-dimensional motion projected – or flattened – onto a two-dimensional plane. His recent works have become so complex that one cannot "read" the picture anymore in the sense of still recognizing the turning and swirling motions of the hypercube. His fascinating new compositions show very complex simulations of a dynamic equilibrium in complex space/time. Although highly abstract artworks in only black/whites, it is not so strange to consider that they are translations of abstract explorations of space/time as contemporary scientific diagrams explaining highly complex theories on space/time are similar visualizations using black lines connecting and disconnecting at certain points (compare the Feynman-diagrams).

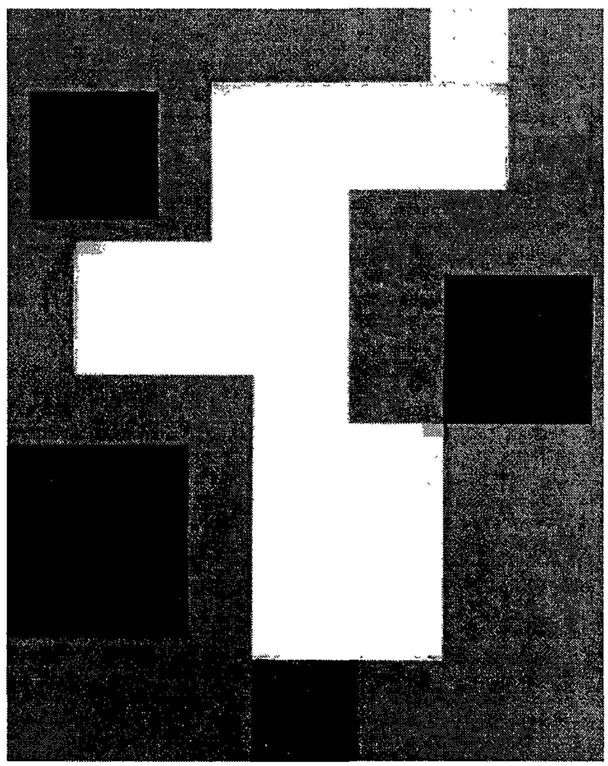


Figure 4: Rodney Carswell (1995) Cut (figured in green), RC95 46, oil, wax, canvas, wood, 72x58x4 in.

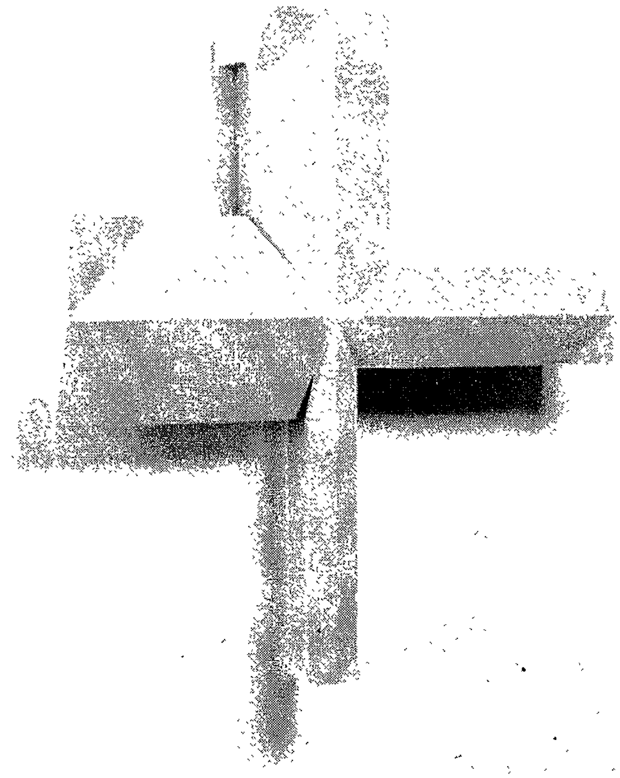


Figure 5: Richard Rezac (1996) Untitled, RR96 14, painted wood, 39½x36¼x8½ in.

Two other contemporary American artists, Rodney Carswell (Figure 4) and Richard Rezac (Figure 5) exemplify in their art a dynamic equilibrium in different ways. Carswell makes shaped canvasses, or paintings with some geometrical shapes cut-out. Although painting is virtually a two-dimensional technique, Casrwell manages a multi-layered dimensionality by the simple act of cutting out a shape. The vision of the observer as a result jumps from a square plane of the canvas to the shape of the canvas itself and further to the cut-out square. Rezac is mainly a sculptor. This can definitely be observed as only when one sees all sides of his wall-works one will notice the very three-dimensional properties of these pure shapes. Like in Juszczyk's paintings also in Rezac's works a breaking of symmetry is accomplished by sometimes minor differences on the left, right, top or bottom of a composition. Carswell and Rezac fit with their choice of form and colours very well into the tradition that was started by American Minimalsim (especially Robert Mangold's use of shaped canvasses and exploring the cross-composition), but they belong more to our times as they have included a higher level of complexity and dynamics.

The Belgian painter Jean-Pierre Maury (Figure 6) since the 1960's has been developing combinations of the same elements into abstract and geometrical paintings following a gradual process towards higher levels of complexity. His structured paintings are often in only one or two colours and present a complex interaction of optical effects using form, distortion of form, direction and simulation of motion. The overall impression of his works show a position "on the edge of chaos", but then at the "order"-side of this edge. His paintings clearly reflect a dynamic equilibrium under pressure of change. They often balance between a more symmetrical state and an a-symmetrical state.

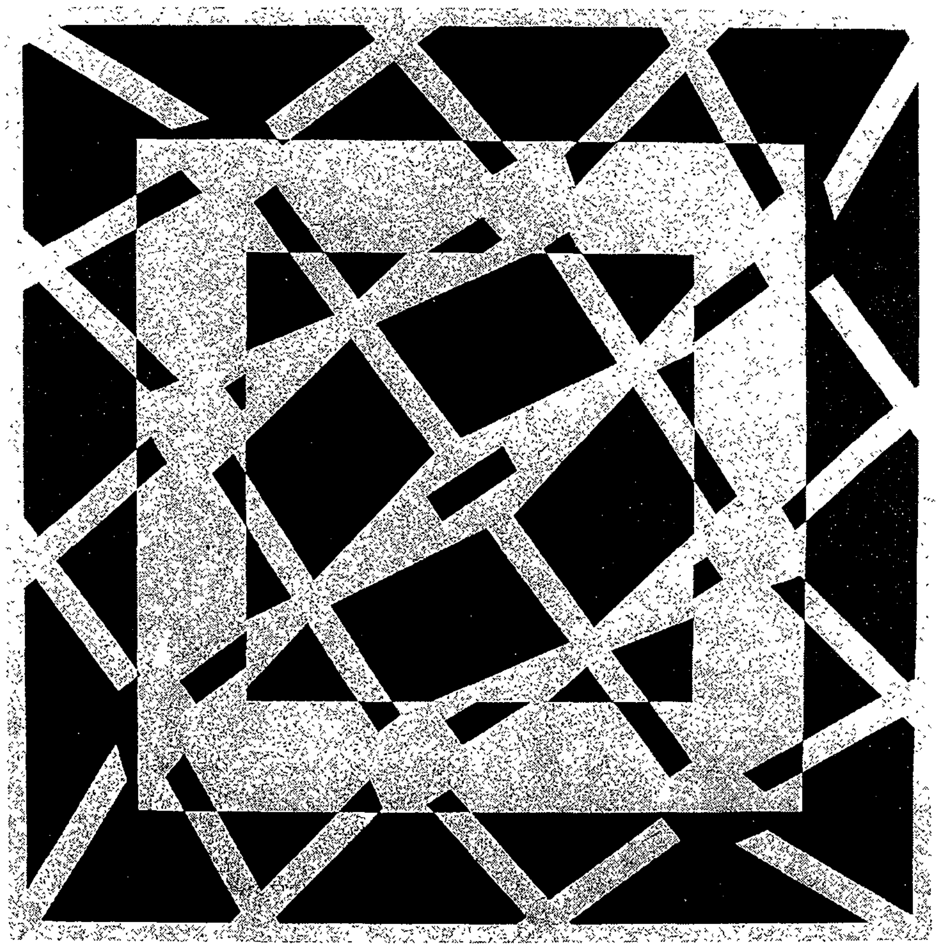


Figure 6: Jean-Pierre Maury

An interesting analogy to Maury's art can be found in the mobiles of the American artist Tim Prentice (Figure 7). Prentice develops a kind of sculptural curtains containing many square or rectangular forms that are suspended individually to a flexible grid-construction. The result is that with the slightest draft or change of air the in principle pure symmetry of the composition dances away into every variety of symmetry-breaking. Photographic reproductions of Prentice's art show remarkable resemblances to Maury's form-field paintings. Although both have a different concept in their art they also both seem to reflect the theories of self-organization by contemporary scientist Ilya Prigogene. From experience I know that at least Maury is informed on Prigogene's ideas[25].

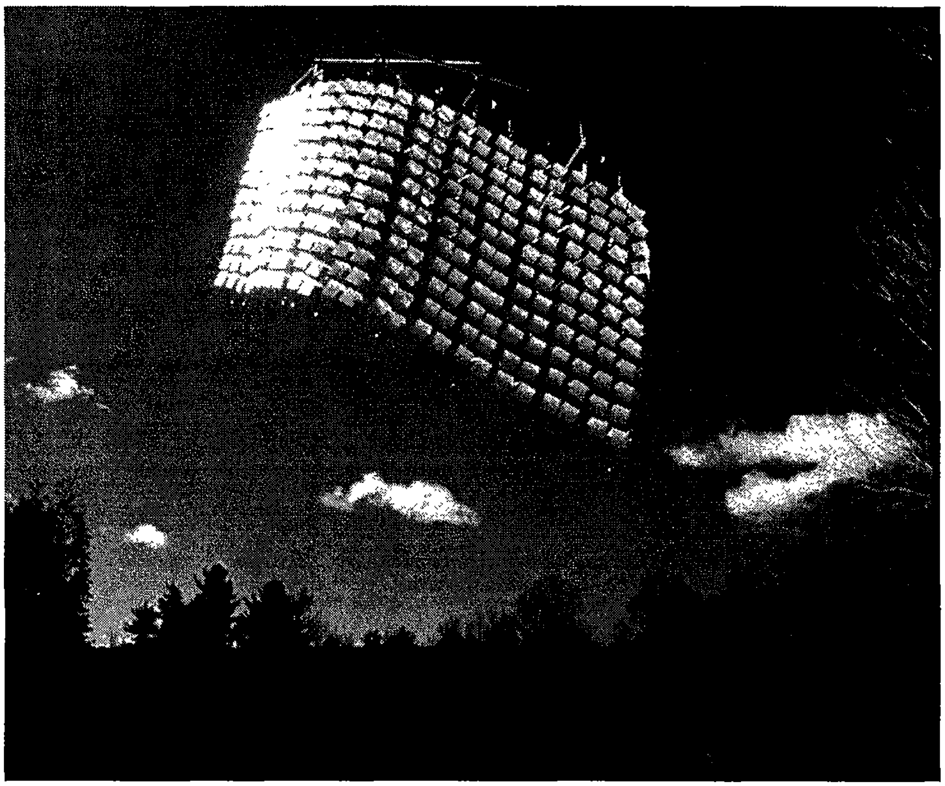


Figure 7: Tim Prentice

The German painter Michael Bette (Figure 8) paints a different complexity. He offers us an intertwining and highly pulsating complex structure on basis of geometrical, sometimes almost amorphous (certainly organic) shapes and unusual colour-combinations, making the "reading" of a system impossible. His paintings are still based on a certain personal language of shapes and rules for the choice of colours. His art-language has been developed through the years towards more complexity. Maybe a comparison can be made to how Stella also developed his art to a state of almost complete chaotic complexity. Bette's art presents us with a possible interpretation of (coloured) Quantum Fields. His compositions are virtually chaotic on the edge of becoming ordered. One might say, that Bette approaches the moment of pure order (or of symmetry) from the opposite side as Maury or Prentice do.

Figure 8: Michael Bette (August 1995), acryl/moleskin, 170x140 cm

Figure 9: John Okulick (1996) Outer Walls, printed wood, 45x43½x8 in

Again different, but still maybe comparable, one can draw an analogy between the paintings by Bette to the reliefs of American artist John Okulick (Figure 9). Okulick's reliefs maybe show resemblances to certain early 1980's wall-works by Stella, but they are not similar which one can find after serious comparison. They are a kind of assemblages of rectangular and circular shapes in compositions that simulate strong motions returning into themselves. Said otherwise: a combination of centripetal and centrifugal motions. An extra visual quality is reached as Okulick distorts the shapes by a clever use of perspective-projection creating the impression that one perceives them from a different angle. His wall-works seem to virtually go through the wall by optically tilting backwards. The use of perspective obviously means that also the notion of direction as such is involved in his compositions, exemplified by the juxtaposing of rectangular shapes. This effect is emphasized by his use of linear structures on some of these shapes. His use of large shapes in combination with smaller shapes, like large parts of circles and small ellipsoid shapes enhance the complexity of motions simulated. Small circular shapes simulate small or concentrated rotations, large parts of circles show parts of larger rotations. On top of these properties he applies colour, too. Not purely functional colour to contrast the different shapes, but colour as colour, meaning that it forms an extra element in the whole composition.

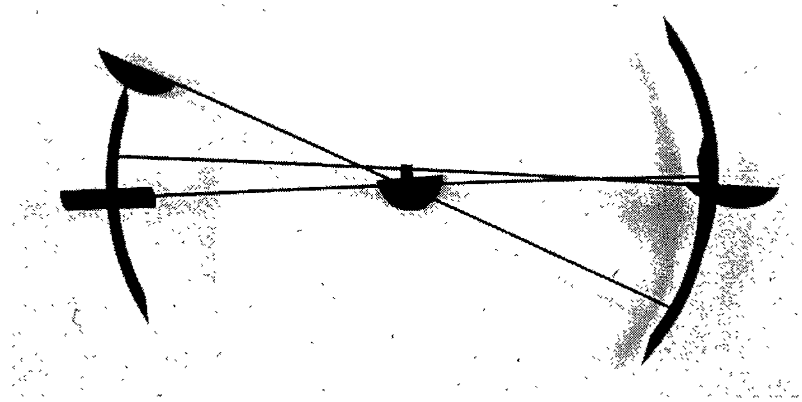


Figure 10: Pedro de Movellan (1995) Rotating Arcs, blued steel/pain/A1, 41x5½ in.

Pedro de Movellan is a young American artist who explores motion in a very different – in an actual way. His art (Figure 10) concerns wall-constructions, suspended sculptures or free standing sculptures that will not be moved by the wind as happens with Rickey's or Prentice's art, but by a kind of interaction of the observer with the sculpture. The observer touches the sculpture gently and some kind of sequence of motion starts. De Movellan uses combinations of polished wood, brass and stainless steel. This choice of material and only using the colour of the material itself relates him to the first generations of constructivist artists. The contrasts between the materials and the reflections of light on the material play important roles in our perception of the process from a state of perfect equilibrium into a state of a disturbed equilibrium into finally a state of perfect equilibrium again. The natural process from pure symmetry through a continuous changing state of breaking of symmetry towards a returning into a state of perfect symmetry can be followed quite well in his handsomely crafted sculptures.

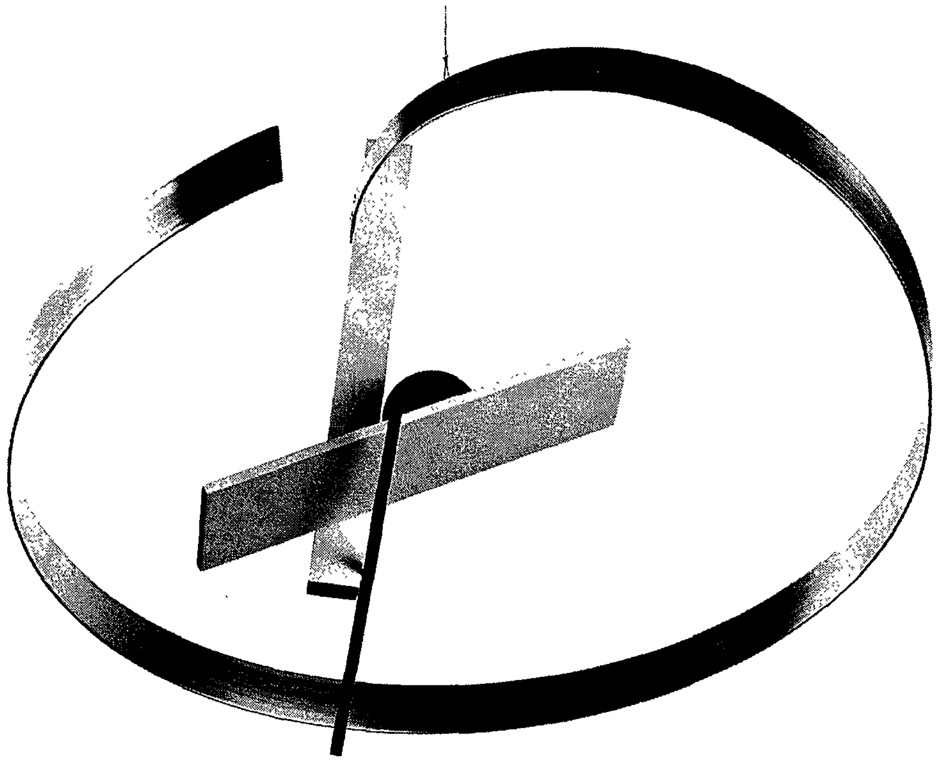


Figure 11: Fré Ilgen (1997) *Dance on my Own*, suspended object, wood/acrylic paint/stainless steel, 52x74x75 cm

In a different way also interactive and visually maybe more related to Okulick and Bette, I would like to introduce you to my own art (Figure 11). As an artist I am exploring a better understanding of reality by the simulation of complex movements of which our reality seems to consist. This I do by making highly three-dimensional sculptures consisting of straightedge geometrical shapes and bent (organic?) stainless steel strips. The simulation of motion is enhanced by a careful use of contrasts: contrasts in length, width, in height, in colour and in direction. The colour-choice is based on a colour ordering inspired by the colourwheel of Ostwald. This means that the eyes will follow more or less unconsciously the ordering from bright yellow to orange, to red, violet, blue, blue-green, green, green-yellow back to yellow and thus create an extra movement in our vision. As colourwheels are ideal models, I am tempted to prefer colour combinations as we can see in nature itself. This implies the combination of pure colours (like primaries) with mixed colours (like a dark green or very light violet).

Furthermore I use the notion of visual memory. As the coloured parts are coloured differently on both outer ends one can never see all the colours at once. When a suspended sculpture or the observer changes position another colour becomes visible – and the earlier perceived colour disappears out of vision. Nevertheless, our visual memory fills in this gap of information and will thus also influence our perception of the sculpture. A phenomenon that seems essential to our everyday experience. Another aspect clearly visible in my art is the notion of "direction". Along the length of the coloured parts directions are suggested. Because of the different coloured outer ends these directions seem to be virtually cut off and led back into the composition. That is why in combination to the bent stainless steel strips I refer to my compositions as "an oscillation with a direction". As I try to understand what I am doing, why I am doing it and for what reason other people seem to respond to my art, I am involved in art-theory. This has led me to research very different fields of knowledge in history and of today, not only in art history or art theory, but also in science and philosophy. Especially in theoretical physics, astronomy, the cognitive sciences and computer-science. A research that is continuously proceeding. I am convinced that just like science and philosophy also through fine art the human being tries to understand his or her reality better. This fact makes the finding of analogies between these obviously different disciplines a logical result and no fiction.

AFTERWORD

Symmetry did not change so much in appearance in constructive art from the pioneer days of abstract art onwards till the end of the 20th century. It is always a bit disturbing to have to conclude that there is not so much real progress of human knowledge as we are grown up with to believe. In society one can see that ideas of new generations are not always "new" or a next step in this supposed progress, but quite often repetitions of earlier accomplishments. This is the same in art. It is somehow interesting to see how also in constructive art or art that by the use of geometry is often automatically associated to constructive art not so much has actually been developed along the length of the 20th century. This becomes especially evident in the purist tradition which one can find everywhere and through all generations. The preference by many artists to still keep painting monochromes or monochrome-dualities (two similar planes of contrasting colours) with the usual easy aesthetic effect is only one example. Another example is that one cannot state that either geometrical art or figurative (mimetic) art would be "the only real art". Both are accepted as obvious means to understand our reality. The phenomenon of non-progression can quite probably be understood to be similar to the academism in physics to still believe in the Newtonian "reductionist" and pure empirical approach. But, then again, maybe "progression" is just a fiction of our culture? Of course, progression does exist, but only more slowly like in a spiral looping slowly upwards. Anyway, the artists I discussed here seem to my opinion to better reflect the spirit and ideas on reality of our *Zeitgeist* as one can also find in science and philosophy. Van Doesburg's remark on "a balanced relationship of unequal parts" still is feasible for contemporary art, especially for the art discussed here. The discussed artists apply this concept in their paintings and sculptures like Yin and Yang forces. To my opinion this can well be understood as so many artistic translations of the universal law of the conservation of energy. Within art history the difference to the early constructivist artists lies in the purpose, the sensibility and complexity. The artworks have no particular political statement anymore. Complexity in form and colour are not considered to be "wrong" anymore. The contemporary artists learned from the gradual developments of abstract art through the 20th century to handle the artistic means like shape, colour and composition more sensitively using much more complex schemes. Thus, also their ways of breaking symmetry happens more consciously. A breaking of symmetry, which also in the most recent ideas in science has become a major concern. In biology, for instance, a state of pure symmetry is understood as the equivalent to death. In the multi-disciplinary approach to physics, artificial intelligence and economics these new ideas are combined in a "Theory of Complexity" and its central notion is "a state on the edge of chaos". This is the equivalent to the moment a symmetrical state is on the verge to be broken. Continuous change has become the new paradigm. This seems quite well illustrated by the art discussed in this paper. Also artists today have a better understanding of what they are doing, just like science also advanced bit by bit and knows something more about the complexity of reality and the role of symmetry/breaking of symmetry. As also science found out, the previously discussed artists all are different but still all accept the importance of continuous change in the dynamics of the compositions in their art. This art still has a further future when the artists will continue their quest with integrity and with a serious interest in the knowledge in science in their own *Zeitgeist*. As final remark to this paper I would like to state that constructivist art should not be labeled "constructivist" anymore, as our current knowledge on our perception of reality makes the concept of a construct in one's mind not very feasible anymore. Interactions between our mind, our body and the phenomena of reality are more important then only the processes in our head. The high complexity of the art of most artists discussed here show that their concepts can not be visually analyzed so easily anymore which make them quite different from earlier constructive art (the "construct" is cannot be re-traced easily). When a new notion should be opted for this tradition of artistic research, maybe "extensionalism" would be a better notion. The word "extension" reflects better our current understanding of our perception (the interaction).

Already our breathing process, the interactions and synchronizations between the rhythmic energies that make our bodies what they are with the rhythmic but irregular streams of energies from outside our bodies (like our dependence on the sun) make this very evident and not merely a "New Age" romanticism.

ACKNOWLEDGEMENT

For the permission to reproduce the artworks discussed in this paper I am grateful to the artists and to Maxwell Davidson Gallery for Prentice and De Movellan, to Nancy Hoffman Gallery for Okulick, to Feigen Inc Gallery for Carwell and Rezac.

NOTES

[1] this quote has been published in De Stijl, Series XII 6-7, (1924), 78-83, this translation is quoted from Baljeu (1974, pp.144) (emphasis mine); see list of literature;

[2] This is discussed in Railing, *PROUN: The Interchange Station of Suprematism and Constructivism*;

[3] In his famous lecture "New Russian Art" from 1922 in Hannover, Germany, presented in relation to an exhibition of his own art;

[4] including the Stenberg brothers, Meydunyesky, Yoganson;

[5] Syenkin, Chasnik, Klutsis, Ermolayeva, Khidekel, Kogan, Noshov, and others led by Malevitch and El Lissitzky;

[6] see especially Bann, Turowski and Stanislawski/Brockhaus, list of literature;

[7] every object in our environment has some effect on our perception of reality, therefore, on our well-being; the artists researched how more consciously developed design could have a more positive effect; this coincided with the emergence of serious interests in hygiene as a social topic, like in working class households, but also this coincided with the emergence of commercial graphics (advertisement) in the West and the need for new propaganda-tools in Russia after the revolution;

[8] the discussion on our fascination for geometry, both Euclidian and Non-Euclidian, is exemplified by the publications of Poincaré, Hússerl and Russell around the turn of the 19th-20th centuries;

[9] exemplified in Schopenhauer's "worldwill" in philosophy and reflected later in the 19th century in Cézanne's search for a structure parallel to the structure of nature through painting in his typical style of "modulations" or repetitions of certain interrelated quantities of paint;

[10] like Goethe or Helmholtz;

[11] especially in the application of a certain choice or in the repetition of similar shapes (often geometric) and colours (often primaries) on every object and clothing; Kandinsky is known to have stated that his visit to a farm in the north of Russia, where everything of the house was decorated in a non-figurative folk-art style, was instantly perceived by him as walking into an abstract painting; he has written himself that this inspired him to find pure abstract painting;

[12] in 1910 there was a large exhibition on Islamic art in Munich where for instance Kandinsky lived at the time; from Popova, the contemporary of Malevitch, the Tretjakov Gallery in Moscow owns a study of an Islamic geometrical composition; in old 7th century Tantric illustrations one can find marvelous similarities to Malevitch's paintings of the famous black square or black circle,

[13] a very good example I found in a book-antiquariat: a 1922 publication of the encyclopedia by the famous and popular French publisher Larousse on *"Le Ciel"*, including many photographs of stars, nebulae, comets, explanations of star- and galaxy-structures and with an essay on Einstein's theories (!),

[14] in Cubism. the looking at an object or a portrait from many different angles at the same time as was already experimented on by Cézanne in his still-lifes, in Futurism the simulation of dynamic motion by a repetition of the object or by repetition of its contours, Futurism as such inspired the simulation of motion as we today know from popular comics;

[15] examples: graphic design or theatre set-design; this development was stimulated by a fast growth in commissions for artists to make designs for advertisements, posters, murals, industrially produced products for the new mass consumer-markets, stage design and architecture for promotional political purposes, including promotional activities to enhance the public awareness on national health and hygiene - as was mentioned before;

[16] important schools institutionalized these ideas, making their impact so large: the German Bauhaus, the Russian vChutemas, both from 1919 till about 1933; later the American New Bauhaus at Chicago and in the 1960's the German *Hochschule fur Gestaltung Ulm* (initiated by the Swiss artist/architect Max Bill);

[17] see Reber, pp 119, see list of literature;

[18] Barbara Held discusses in a recent paper the confusion and different interpretations in psychotherapy around the meaning of "consctructivism"; it is interesting to note that "constructivism" started in the early 20[th] century in art and mathematics with a lot of confusion about its meaning to be taken up by a very different field at the end of the same century in a similar state of confusion about its meaning;

[19] this field was initiated by Linus C Pauling's idea to structure the spatial arrangements of atoms in molecules and crystals and the interactions that bond substances, published in his "The Nature of the Chemical Bond" in 1939,

[20] Jakobson, by the way, originally belonged to a group of artists, poets and early linguists called the "Moscow Circle" in the time of early constructivist pioneering of graphic-design; later he initiated the Prague School of Phonetics before going to the USA, where he acquired his fame;

[21] published in 1958-1964 by the Dutch artist Joost Baljeu in English only,

[22] published and edited since 1960 by Canadian artist Eli Bornstein, University of Saskatchewan, Saskatoon, Canada;

[23] it must be stated here, that the early constructivist pioneers already experimented with actual motion, especially Naum Gabo and Moholy-Nagy (and Rodchenko, Klucsis and Kobro made suspended sculptures), but also one should recall Tatlin with his flying machine "Letatlin" or El Lissitzky with his moving stage-designs and exhibition-design projects; maybe one could also mention Austrian Friedrich Kiesler with his fabulous stage-design, which later led him to his utopian project for an "endless house";

[24] GRAV = *Groupe de Recherche de l'Art Visuel;*

[25] Maury and Prentice do not know each other as yet, but are of the same generation, their mutual interests could also be the natural result of living in the same *"Zeitgeist"*, could not they?

REFERENCES (LIST OF LITERATURE, SUGGESTIONS FOR FURTHER READING)

Baljeu, J (1974) *Theo van Doesburg*, Studio Vista.

Bann, S 1990 (original: 1974) *The Tradition of Constructivism*, Da Capo Press Inc.

Dabrowski, (1985) M. *Contrasts of Form - Geometric Abstract Art 1910-1980*, New York· The Museum of Modern Art.

Greenberg, M J. (1994) *Euclidian and Non-Euclidian Geometries - Development and History*, Third Edition, New York. W. H. Freeman and Co.

Hemken, Dr K-U and Stommer, R. *K. I. - Konstruktivistische Internationale Schöpferische Arbeitsgemeinschaft 1922-1927 - Utopien für eine Europäische Kultur*, Verlag Gerd Hatje.

Jaffé, H. L C., *De Stijl· 1917-1931 Visions of Utopia*, Walker Art Center.

Kandinsky, W. (1977) *Concerning the Spiritual in Art*, Dover Publications Inc (original *Über das Geistige in der Kunst*, or original in English *The Art of Spiritual Harmony*, 1914)

Kandinsky, W (1979) *Point and Line to Plane*, Dover Publications Inc. (original *Punkt und Linie zur Flache*, 1926).

Railing, P (1989) *From Science to Systems of Art - On Russian Abstract Art and Language 1910-1920 and other essays*, Artists Bookworks.

Railing, P. (1990) *On Suprematism - 34 Drawings - K.Malevitch*, Artists Bookworks

Reber, A. S. (1994) *Woordenboek van de Psychologie*, Uitgeverij Bert Bakker (original: *The Penguin Dictionary of Psychology*, 1985)

Rickey, G. (1967) *Constructivism - Origins and Evolution*, New York: George Brazilier (a reprint has recently become available).

Rotzler, W. (1977) *Konstruktive Konzepte - Eine Geschichte der konstruktiven Kunst vom Kubismus bis heute*, Zürich· ABC Verlag.

Schirn Kunsthalle, Frankfurt; *Die Grosse Utopie - Die russische Avantgarde 1915-1932* (also published as *The Great Utopia* by New York: Guggenheim Museum)

Stanislawski, R. and Brockhaus, C. (1994) *Europa - Europa - Das Jahrhundert der Avantgarde in Mittel- und Osteuropa*, Kunst und Ausstellungshalle der Bundesrepublik Deutschland.

Tuchman, M. (1986) *The Spiritual in Art· Abstract Painting 1890-1985*, New York Alsbeville Publishers.

Turowski, A. (1986) *Existe-t-il un Art de l'Europe de l'Est? Utopie & Ideologie*, Les Editions de la Villette.

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GEOMETRIES IN THE EAST AND THE WEST IN THE 19TH CENTURY

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1 INTRODUCTION

In the 19th century several geometries were discovered, and F. Klein classified them in terms of transformation groups. Since then many publications about several such kinds of geometries have been made. But those publications only discuss geometries only in the West. Were there several kinds of geometries in the East in the same age? In this paper we will give an answer to the question by giving some examples taken from the old Japanese mathematics. Different approaches to the geometry between the East and the West and related topics will be given.

The mathematics we will present here is one developed in Japan during the Edo era (1615-1868), and is called *wasan* (*wa* and *san* means Japan and mathematics, respectively). Its root is in Chinese mathematics from the late 13th century to the late 16th century which was introduced to Japan through the Korean peninsula in the late 16th century, and developed rapidly during the Edo era. At the beginning of the Meiji era (1868-1912), the new government adopted Western mathematics in the new school system, and *wasan* had to end its short life, but *wasan* tradition was maintained for a while.

They used symbolic calculation as a main tool. It was also used in their geometry. Almost all the *wasan* books have the style of problem books, and the contents need not be systematic. Hence it is rather exceptional to find a geometric *wasan* book which is written in a systematic style or on a certain geometric purpose. But in some books we can see that the problems are solved by a peculiar technique. Also there are a few which are written on a geometric purpose. We will show that we can see a certain geometric worlds in those books.

2 BŌSHA JUTSU

Bōsha jutsu is a theory of tangents of various touching circles. It was founded by Ajima [1] and developed by Umemura [16]. The basic formulas are

$$l^2 = \frac{c^2 t_{AB}^2}{(c+b)(c+a)} \quad \text{and} \quad l^2 = \frac{c^2 t_{AB}^2}{(c-b)(c-a)}$$

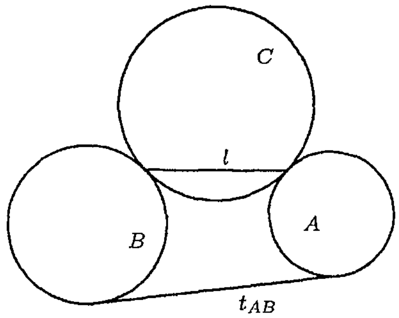


Figure 1a.

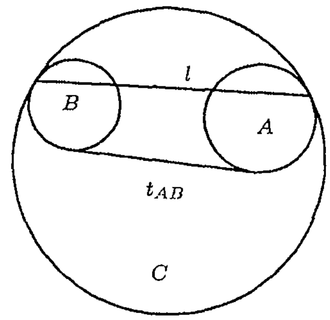


Figure 1b.

for Figures 1a and 1b, where a, b, ... are the radii of the circles A, B, ... , t_{AB} is one of the external tangents of A and B, and l is the distance between the two points of tangency of A, C and B, C. As a custom of *wasan* they used diameters of circles instead of radii in the original context to express these relationships, but we translate them using the term "radii" of circles as just shown above.

The next formulas for four tangent circles are

$$(b+d)^2 t_{CA}^4 - 8bd(b+d)(c+a)t_{CA}^2 - 16abcdt_{CA}^2 + 16b^2d^2(c-a)^2 = 0,$$

$$(b-d)^2 t_{CA}^4 + 8bd(b-d)(c+a)t_{CA}^2 + 16abcdt_{CA}^2 + 16b^2d^2(c-a)^2 = 0,$$

$$(b+d)^2 t_{CA}^4 + 8bd(b+d)(c+a)t_{CA}^2 - 16abcdt_{CA}^2 + 16b^2d^2(c-a)^2 = 0$$

for Figures 2a, 2b and 2c respectively. Umemura showed that these formulas could be applied for problems involving tangent circles by solving many problems. One of his simple results is $t^2_{BD}t^2_{CA} = 64abcd$ for Figure 3.

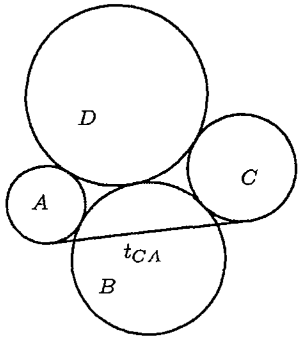


Figure 2a.

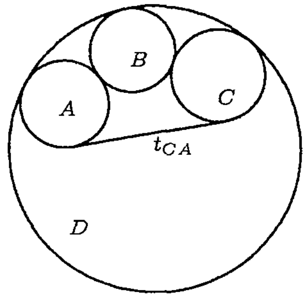


Figure 2b.

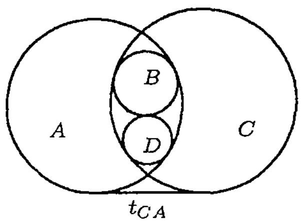


Figure 2c.

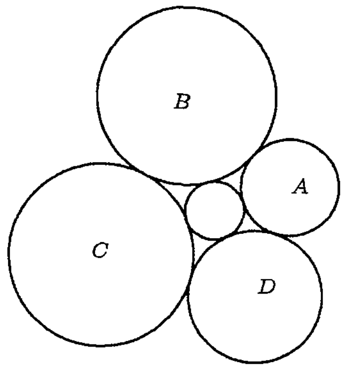


Figure 3.

As stated above, those formulas change the signs according to various tangency. It is pointed out that we must calculate to determine the signs and this is a defect of the method [7]. But those formulas can be unified if we consider oriented circles and oriented lines and if we regard tangent circles as anti touching oriented circles [13]. In this sense these results belong to the geometry of oriented circles. Since there is no idea of orientations or directions in *wasan* geometry, *bōsha jutsu* is not a counterpart of geometry of oriented circles. But it is interesting to see that such a geometry was made in Japan in the same age as Laguerre was considering oriented lines and oriented circles with his inversion.

3 *KYOKUKEI JUTSU*

Kyokukei jutsu (or *kyokugyō jutsu*) is a technique which transforms figures into symmetric ones and uses symmetric polynomials [3]. A typical problem is as follows:

Given the three exradii a, b, c of a triangle, find the inradius r (Figure 4a). The solution is consisting of several steps as follows:

1. Suppose a limiting (or an ideal) figure where the relation $a = b = c = x$ holds (in this event the triangle is equilateral) (Figure 4b).

2. Find the relation between x and r in the limiting figure (we can easily get $3r - x = 0$).

3. Regard the relation as an equation of a suitable degree ($3rx^2 - x^3 = 0$ in our case, but there in no explanation why we choose the cubic equation).

4. Substitute $(ab + bc + ca)/3$ and $abc/3$ for x^2 and x^3 respectively.

5. And we get $r = abc/(ab + bc + ca)$.

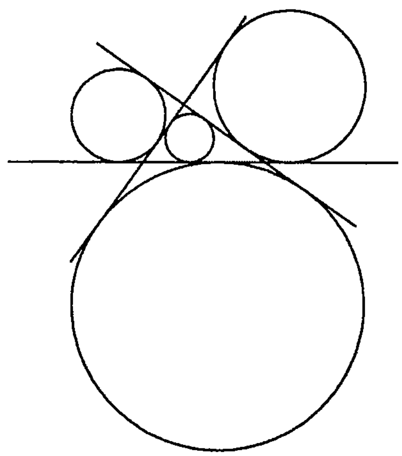


Figure 4a.

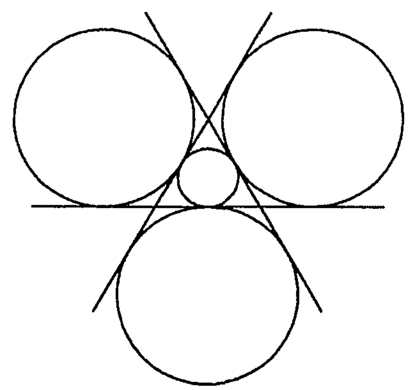


Figure 4b.

Another example is as follows: Given three inradii of the four incircles of the four inscribed triangles as in Figure 5a, find the remaining inradius. Let the radii of the circles A, B, C, D be a, b, c, d. The solution is:

1. Suppose a limiting figure where the relations $a = c = x$, $b = d = y$ hold (see Figure 5b).

2. Then we have $x = y$.

3. Replacing x and y by $(a + c)/2$ and $(b + d)/2$, we have $(a + c)/2 = (b + d)/2$.

4. Hence $a = b + d - c$.

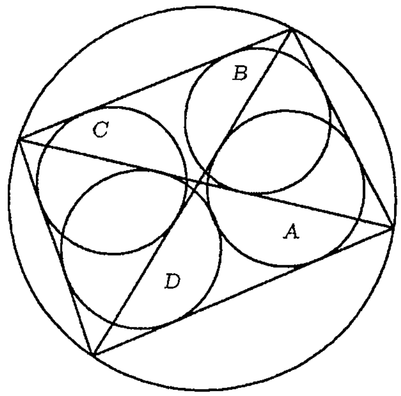


Figure 5a.

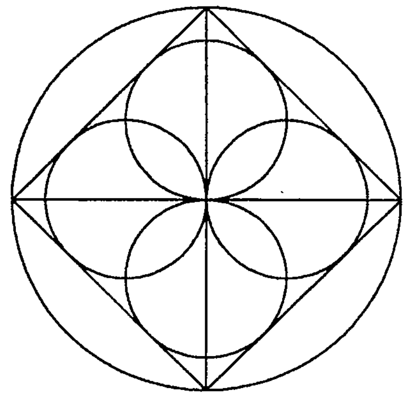


Figure 5b.

It is easy to point out several defects of this method: The metric relations in the limiting figures are not preserved in the original figures in general. There are many ways to replace variable x and y in the examples by symmetric polynomials. Indeed the method had drawn serious criticism from other *wasan* mathematicians in the same age. We can even find an incorrect answer to a problem in the book. However there are serious defects in this method, the method requires certain symmetry to the figures. Therefore the book is a collection of certain symmetric figures consequentially. It seems that the author of the book had already known the answers to most of the problems, or had got answers by another way, then he adopted these method to obtain the answer. It is very interesting that the author of the book considered symmetry of the figures connecting with such symmetric polynomials. We cannot find Western counterpart of this technique.

4 *SAMPENHŌ* AND OTHER SIMILAR TECHNIQUES

Sampenhō (or *Sanhenhō*) is a technique due to Hōdoji, which transforms tangent circles into parallel lines to obtain a certain metric relationship of the figures [8]. Similar techniques were also used by a couple of other *wasan* mathematicians in the same age. Let us see an easy example: Given two internally tangent circles, let us inscribe successively touching n small circles as in Figure 6a, where the figure is symmetric in the vertical line through the two centers of the given circles. Find the radii of the smallest circles A and B at the both ends. To obtain the common external tangent t_{AB} of A and B of radii r, Hōdoji considers a figure such that the radii of the two large circles become larger and larger and gets Figure 6b. Since the tangent in the transformed figure is $2(n - 1)r$, Hōdoji concludes that $t_{AB} = 2(n - 1)r$ also holds in the original figure (Figure 6a) with no explanation. Then by the formula of *bōsha jutsu* for Figure 2b, he obtains the relationship between the radii of the two given radii, t_{AB} and r. Eliminating t_{AB} by replacing it by $2(n - 1)r$, he gets the final answer.

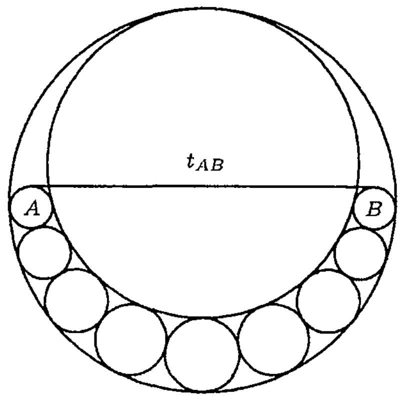


Figure 6a.

Figure 6b.

As in the example, the feature of the method is the way to get tangents of two circles. He gets common tangents of several kinds of tangent circles in the transformed figures, then he uses them as the relation in the original figures with no explanation. Those transformed figures exactly coincide with the ones obtained by suitable inversions. Let us see other examples. In the following four examples, we will cite only the transformed figures, and state what relations are derived from them. Those relations are also used as ones in the original figures.

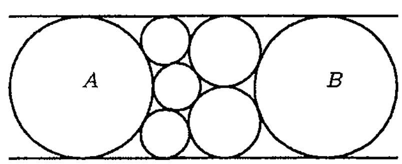


Figure 7: $t^2_{AB} = 2(\sqrt{2} + 1)^2ab$

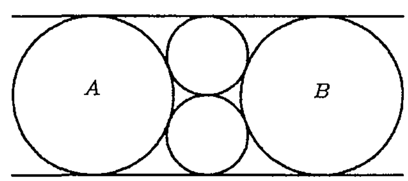


Figure 8: $t^2_{AB} = 8ab$

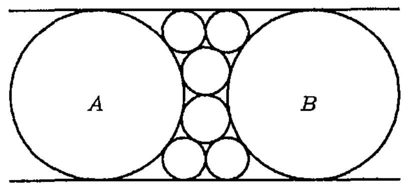


Figure 9: $t^2_{AB} = (25/4)\cdot ab$

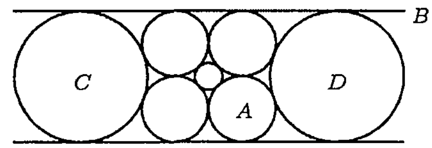


Figure 10: $t^2_{AB} = 8ab$, $t^2_{CD} = (4\cdot\sqrt{2} + 9)cd$

However there is also a gap or a jump of logic, it seems that the *wasan* mathematicians did not criticize the method. What is the Western counterpart of this? It is very similar to inversive geometric technique to consider parallel lines instead of tangent circles. Is Hodoji's method inversion? Does he use inversion? This point was controversial between Hayashi [4, 5, 6] and Mikami [11, 12]. Hayashi states that the method was correspond to inversion [4] at first. Countering this, Mikami mentions that there is no reason that the method is inversion [11], and he concludes that it is not inversion [12]. Hayashi also writes that the method is using not transformation as inversion, but deformation [5]. Also he says that the method is not inversion but it corresponds to inversion [6]. However there were several arguments between them, it seems that both grant that the method is not inversion, as a result.

In [10, p. 77] it is said that there is no evidence that the method is using inversion, and it is conjectured that they had already known the relationships between the tangents and the radii in the original figures and Hodoji tried to derive the relationships from the transformed figures. But in [7, p. 135], it is said that the method corresponds to inversion and since it is discussing metric relationships, there are a lot of difficulties, therefore some unknown parts are still left even by today's mathematics.

It seems that he points out the difficulty of how to prove results after the gaps of the method. However proofs that the results after the gaps being true can be found in [12], [10] (some can also be found in [4]). In [14], it is stated that Hodoji's acumen founded inversive method between circles and lines. Quoting the above difficulty in [7], it is shown that Hodoji's method can be explained by suitable inversions [15], and the authors conjecture that the method may be done by such inversions.

As mentioned above, recent publications following to [7] are affirmative on the question while earlier ones are not. After seeing a simple history of the issues on the method, let us give our evaluation. We agree with [10]. In the terms of inversive geometry, the ratio of the square of the tangent of two circles and the product of their radii are preserved by inversion [9, p. 122], and we can show that the relations obtained by transformed figures are also true in the original figures in fact. But we can find no reason that the method is inversion. In the preface of the manuscript, Hodoji says that his method is useless for beginners, so it is not allowed to tell the method to such persons. There are several versions of the manuscript, and among them we can find a sentence in which he says that if one had not solved a thousand problems, one could not be allowed to see the manuscript. It seems that he thought that after enough experiments, the method can be applied correctly. In other words one must know many relationships between the radii and tangents of circles to use the method. If some rule, how to apply his technique, was given in his manuscript, one could use it even without enough experiments. But we cannot find such rules, and can guess that he could not establish such rules. It seems that the sentences in the preface suggest this.

5 CONCLUSION

Since the three techniques are collecting problems which are able to be solved by each of the methods, they are resulting to correct figures which are suitable for the techniques. Therefore each of [1], [3], [8], [16] consists of figures with certain common properties respectively. Hence they have their own geometric worlds in this sense. But there was no notion of transformation. As mentioned in § 4, Hayashi says that sampenho is using not transformation as inversion, but deformation. It seems that he notes that there is no notion of mapping figures in the method. Indeed as we have shown, those techniques to deform figures only just occurred at the close of *wasan* age.

Since *wasan* geometry was not influenced by Euclid, axiomatic treatment was not made. Results were always expressed as problems and answers, which was far from theorems and proofs. Also *wasan* geometry was concerning about metric relationships, they did not care for projective properties of figures.

Rhombuses frequently appear in *wasan* problems while parallelograms do not. In most cases rhombuses are drawn so that one of the diagonals is horizontal. Ellipses can also be seen though neither parabolas nor hyperbola. It has been said that one of the reason is that they thought that ellipses were sections of cylinders. In addition to this, it seems that the Japanese prefer to consider an inner world of a closed figure. Indeed most *wasan* figures are of such kind.

Wasan left many results and some of them were made earlier than those of Western's. For example, a solution to Malfatti's problem; to draw three circles in a given triangle each of which is tangent to the other two and to two sides of the triangle, can be found in [2]. Malfatti solved this in 1803. But *wasan* geometries were very primitive if we regard them as systematic ones. It might be natural if we take its too short history, two centuries, and a dearth of Euclid into consideration. There were neither mathematics journals nor universities in those times. Since the Japanese attached little importance to mathematics, one could not live by mathematics. The situation forms a striking contrast to the fact that a European mathematician could devote himself to study mathematics as a professor at a university or at an academy.

REFERENCES

[1] and [2] Ajima, N. (1739-1798) Shi-enbosha no Kai (undated) in Naonobu Ajima's complete works (1966), Fuji Junior College, Tokyo. Fuji Junior College (compiled by Hirayama, A and Matsuoka, M.)

[3] Akita (1835) *Sampō Kyokukei Shinan*

[4] Hayashi, T (1937) On Hodoji's sampenhō I, *Wasan kenkyū shūroku (Collected papers of wasan) Volume 1*, Tokyo Tokyo Kaiseikan, 878-889.

[5] Hayashi, T. (1937) On Hodoji's sampenhō II, *Wasan kenkyū shūroku (Collected papers of wasan) Volume 1*, Tokyo: Tokyo Kaiseikan, 890-906.

[6] Hayashi, T. (1937) Problems in *wasan* taste I, *Wasan kenkyū shūroku (Collected papers of wasan) Volume* 2, Tokyo: Tokyo Kaiseikan, 825-832

[7] Hirayama, A. (1965) *Peoples in wasan history*, Tokyo: Fuji Junior College

[8] Hodoji (1860) *Kanshinko Sampen* (manuscript).

[9] Johnson, R. A. (1960) *Advanced Euclidean Geometry*, Dover, New York.

[10] Katoh, H. (1956) *A study of wasan miscellanea III*, Tokyo. Maruzen.

[11] Mikami, Y (1932) On deformation methods by Ushijima, *Hōdōji and Tsuchiya*, Tokyo Butsuri Gakkō Zasshi, Oct, 419-428; Dec., 534-540.

[12] Mikami, Y (1933) On deformation methods by Ushijima, *Hōdōji and Tsuchiya, part 2*, Tokyo Butsuri Gakkō Zasshi, Jan, 13-19, Feb, 51-60, Mar, 106-110, Apr., 146-154; May, 188-190

[13] Okumura, H. (1990) Generalization of the results in the old Japanese incircle geometry by the terms of cycles and rays, *Sūgakushi Kenkyū*, No. 124, 24-53 (with English summary).

[14] Shimodaira, K. (1970) *A history of wasan, volume* 2, Tokyo. Fuji Junior College

[15] Suzuki, F and Michiwaki, Y. (1988) On Hōdoji's "Sanhenhō" and Casey's inversion theorem, *Sūgakushi Kenkyū*, No. 116, 62-77 (with English summary).

[16] Umemura (1804-1884) Bōsha Shōkai (undated manuscript).

All the references except [9] are in Japanese.

Symmetry: Culture and Science
Vol. 10, Nos. 1-2, 199-200, 1999

BOOK REVIEW

Douglas R. Hofstadter: *Gödel, Escher, Bach: an Eternal Golden Braid, 20th-anniversary Edition*: with a new preface by the author, New York: Basic Books, 1999, P23 + xxi + 777 p.

One should not need to introduce the contents of this book to the readers, for it became a classic in the twenty years what have passed since its first publication. Hundred-thousands red it in different language editions throughout the world. After the great success of this epoch-making work one should not need to explain to the readers of *Symmetry: Culture and Science*, what did these three geniuses Gödel, Escher and Bach bring together, as well as what is the relation of the message of the theme to symmetry.

Nevertheless, there remained certain misunderstandings around the topics and aims of the work, what inspired the author to emphasise in the new *Preface* what he wrote to the Twentieth-anniversary Edition.

Interdisciplinary approach caused always some confusion, both in readers and booksellers, librarians. Where to classify this book: to math, general science, philosophy or cognitive science? Really, one of the goals of this work was just to evaporate the solid barriers among disciplines, fields of knowledge. Such barriers exist only in artificially sliced curricula, not in the round pie of human knowledge. We can state: Douglas Hofstadter successfully contributed to making these barriers broken down. In part, GEB delved "fugues and canons, logic and truth, geometry, recursion, syntactic structures, the nature of meaning, Zen Buddhism, paradoxes, brain and mind, reductionism and holism, and colonies, concepts and mental representations, translation, computers and their languages, DNA, proteins, the genetic code, artificial intelligence, creativity, consciousness and free will - sometimes even art and music, of all things! - that many people find impossible to locate the core focus."

The author makes a new attempt to tell the goals and the essence of GEB in his new *Preface*. Frankly, he does it as excellently, as excellent the original book itself was. This relatively short, 23 pages *Prolegomena* is as delightful reading, as the whole book itself was. An individual intellectual work in itself. However - this is the reviewer's opinion - there is needless to explain the book for the sophisticated readers, and hopeless for those who did not understood its goals. Fortunately, most understood, this was the key of the success. Yet, we were poorer, if this *Prolegomena* would have not been born.

For example, he presents us new contributions to Kurt Gödel's relation to Bertrand Russell before he concludes: "For GEB, the most crucial aspect of Gödel's work is its demonstration that a statement's meaning can have deep consequences, even in a supposed meaningless universe." Then he explains again the "level-crossing feedback loop", and the role of the person, the "I". In this respect he expresses his surprise, that the elapsed time did not bring out that debate what he had intended to provoke. Really, why did not?

One can learn a lot from the history how the idea and the work were born. This makes us better understand the attitudes of the author to his intellectual product. Highlighted by these personal backgrounds I can propose to re-read the book, at least partially, even for those, who have red it twenty (or less) years ago. Note, not only the author looks at his spiritual son with more mature eyes, our mind has evolved too during the decades. Anecdotes like, how the French translation helped to reconsider a prejudice, colour the story.

Finally, the reader may be curious why did not Hofstadter bring the GEB up to date after twenty years? I will not share his response with you, let's leave some experience for the reading his new *Preface* and for the mind-inspiring, revealing adventure of re-reading the book.

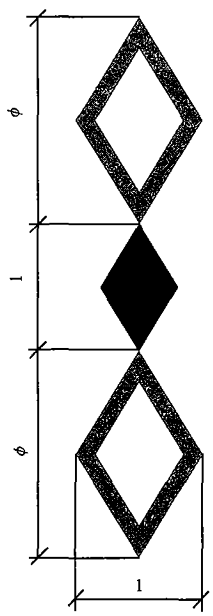
György Darvas

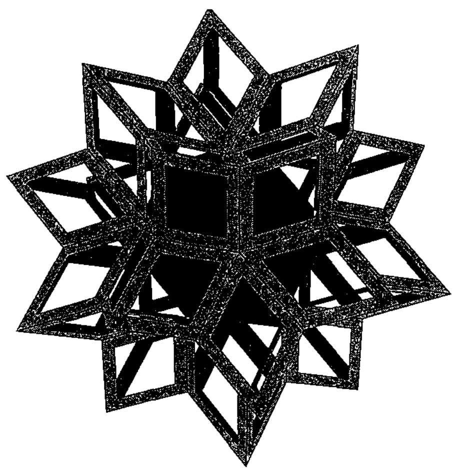
Lesson 91

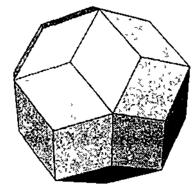
RT AND RH RELATIONSHIP

RT = rhombic triacontahedron
RH = rhombic hexacontahedron
Take a golden rhomb and combine it with two rhombs having size ϕ times larger in a way that the vertices meet, and the longer diagonals of all rhombi are aligned
Place the unit consisting of these three rhombs on each face of an RT The larger rhombs form the faces of RH.

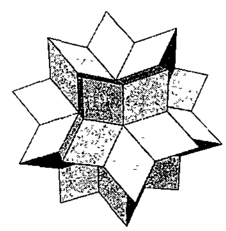
$$\phi = \frac{1}{2}\,(1 + \sqrt{5}\,) = 1.61803$$



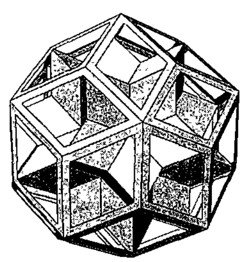


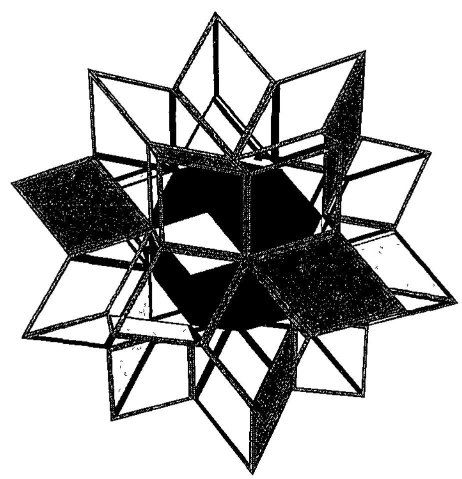


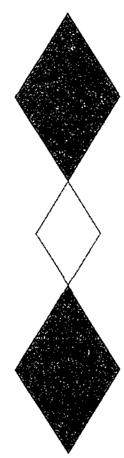
RT



RH

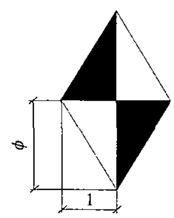


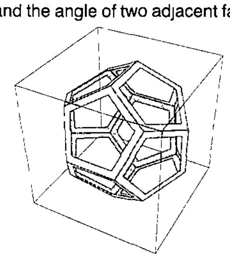


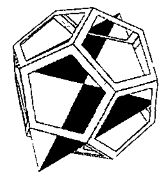


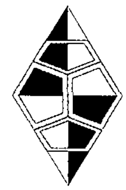
Lesson 94	GOLDEN RHOMB AND DODECAHEDRON RELATIONSHIP	

The angles of golden rhomb are 2 ArcTan[$1/\phi$] and 2 ArcTan[ϕ], corresponding to the angle between two non adjacent faces and the angle of two adjacent faces of the dodecahedron, respectively



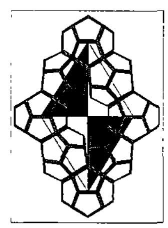


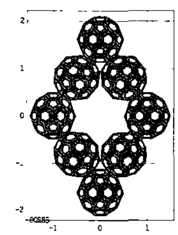


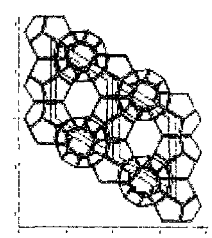


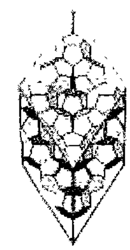
Place 4 dodecahedra on vertices and 4 dodecahedra on the edges of the golden rhomb. Alternatively, place 8 truncated isosahedra (bucky balls) the same way.

It is possible to place dodecahedra or bucky balls on the vertices and edges of any rhombic polyhedra, e.g. golden rhomhedron shown below.



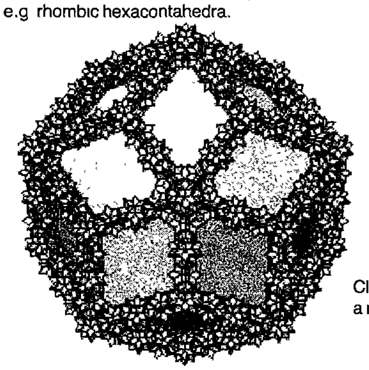




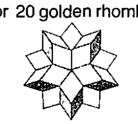


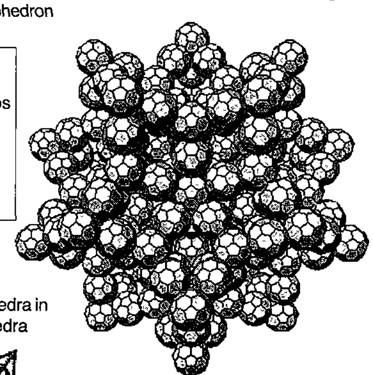
Notice that the quantity of dodecahedra placed along the edges of the rhombic polyheron can be any odd number, and the dodedcahedra can be replaced with fitting objects, e.g rhombic hexacontahedra.

This is a cluster of 195 buckly balls located on the vertices and edges of a 20 pointed star made of 20 golden rhombohedron

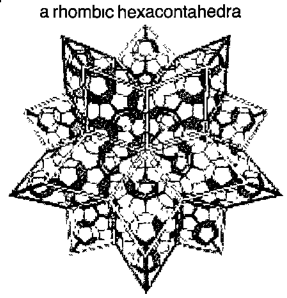


A rhombic hexacontahedron or 20 golden rhombs

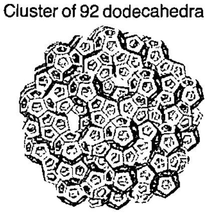




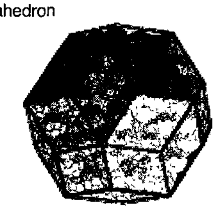
Cluster of 195 dodecahedra in a rhombic hexacontahedra



Cluster of 92 dodecahedra



Cluster of 92 bucky balls in a rhombic triacontahedron

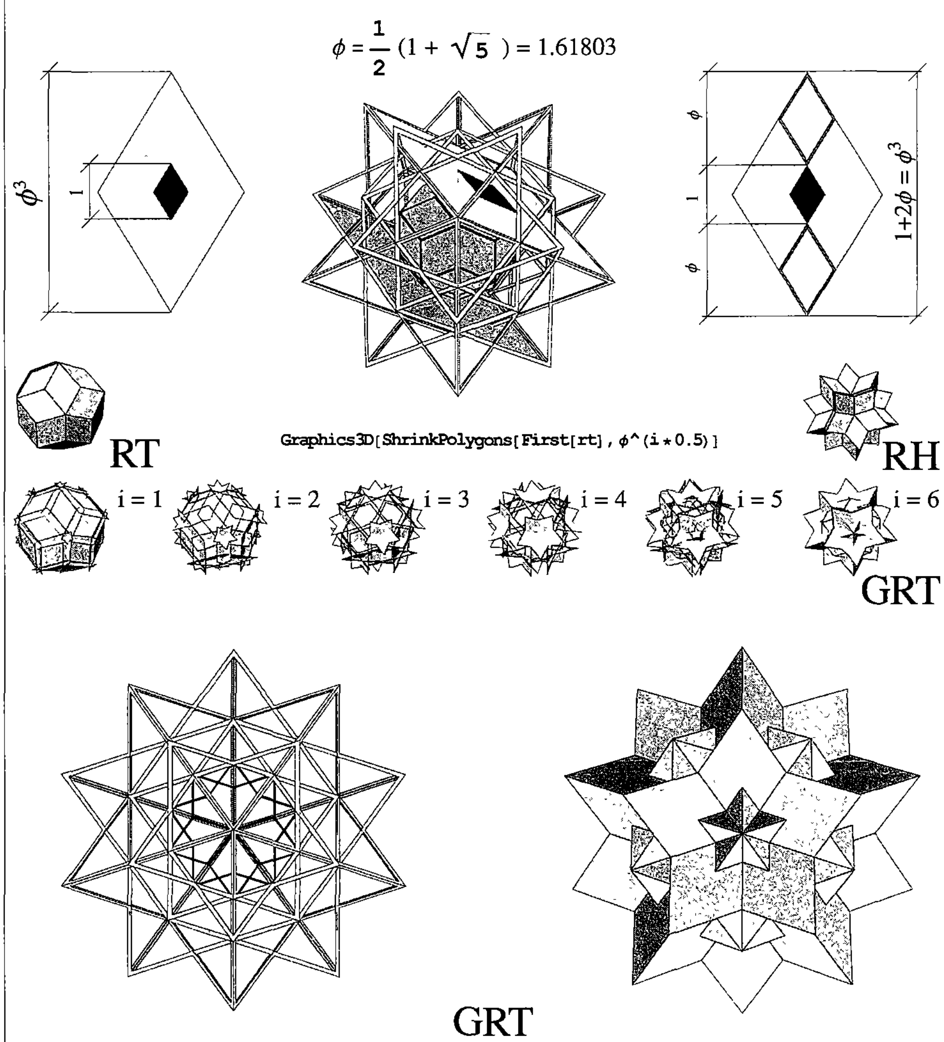


Lesson 92

GRT - GREAT RHOMBIC TRIACONTAHEDRON

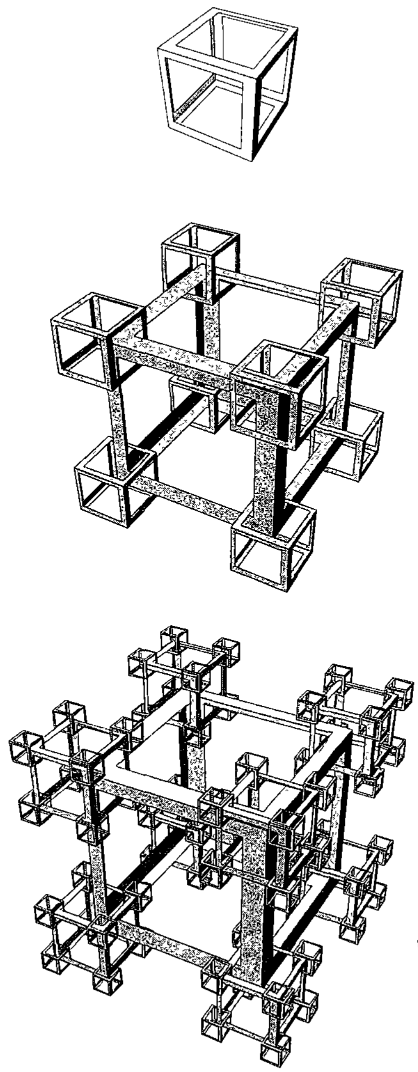
Combine a golden rhomb and a ϕ^3 times larger golden rhomb with their centres coinciding Place this unit on each face of an RT in a way that the small golden rhomb coincides with the faces of RT. The large rhombs are the faces of the GRT

Alternatively, expand the faces of an RT in their own planes to a size ϕ^3 times larger.
Observe that the rhombs used in Lesson 91 for creating the RH closely fit into the larger rhomb. It means that RH and GRT are partly identical

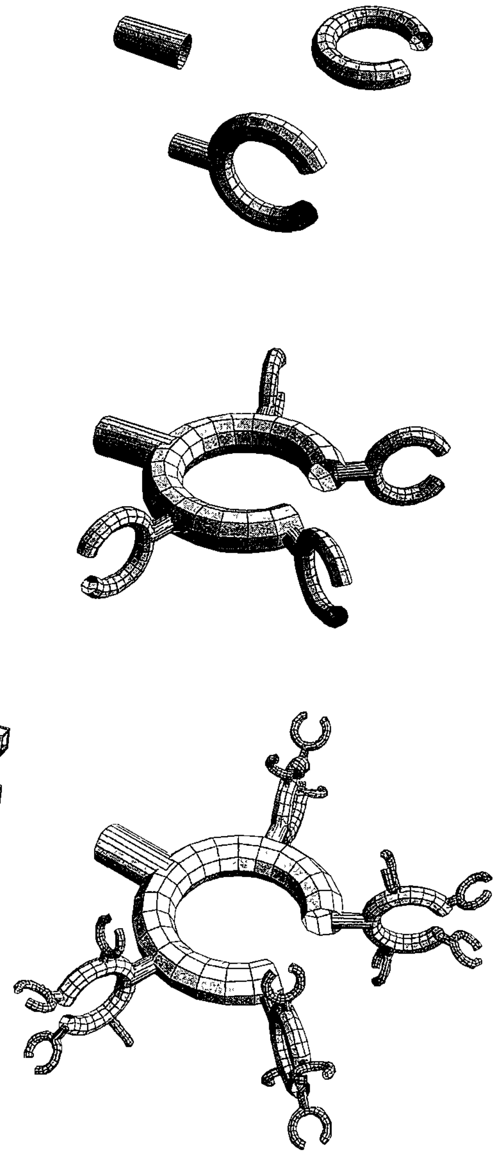


Lesson 61	CUBE AND TORUS FRACTALS	61. lecke

Take a cube, and place a smaller cube on all its vertices Reduce the size of the unit thus produced, and place the unit on the vertices of the original cube. Repeat this procedure to increase the resolution.



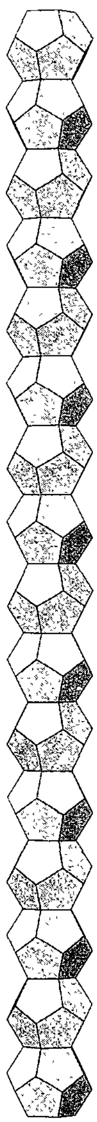
The same procedure can be used to produce other fractals, e.g. from a tube and a torus.

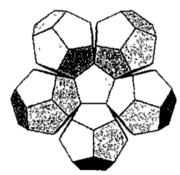


Lesson 33	DODECAHEDRON TOWER	

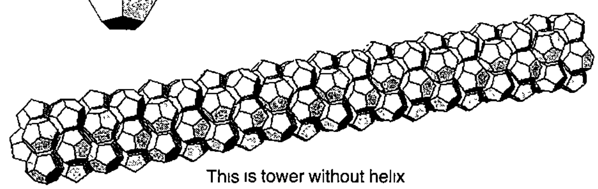
Place a series of dodecahedra on the top of each other with their faces aligned Attached five identical dodecahedra to each of the tower level thus created

Observe that the dodecahedra of the tower are aligned along five helices of right rotation and five helices of left rotation.

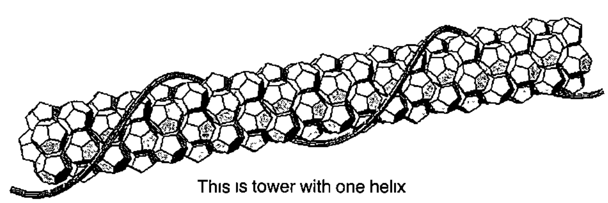




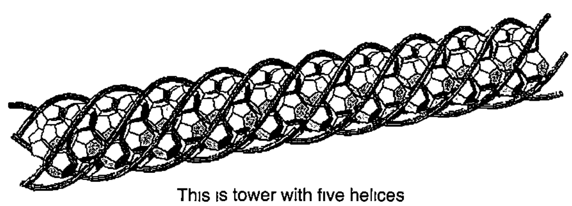
This is the top view of one level of the tower consisting of 6 dodecahedra



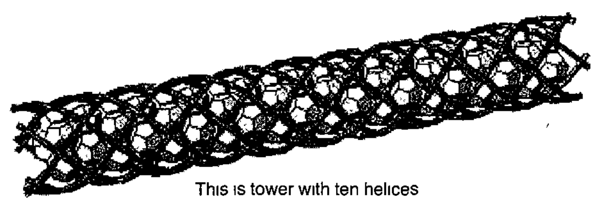
This is tower without helix



This is tower with one helix

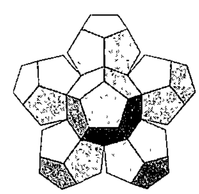


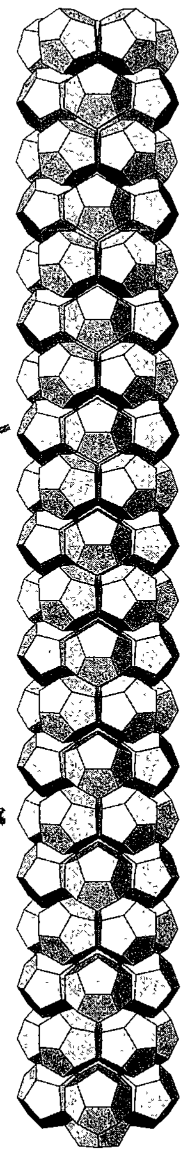
This is tower with five helices



This is tower with ten helices

This is the bottom view of one level of the tower consisting of 6 dodecahedra





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- All papers and notes are published in English and they should be submitted in that language. The quarterly reviews and annotates, however, non-English publications as well

- In the case of complicated scientific concepts or theories, the intuitive approach is recommended, thereby minimizing the technical details. New associations and speculative remarks can be included, but their tentative nature should be emphasized. The use of well-known quotations and illustrations should be limited, while rarely mentioned sources, new connections, and hidden dimensions are welcomed

- The papers should be submitted either by e-mail attachment, or on (PC formatted) computer diskettes (rtf format preferred) Typewritten texts will not be rejected in the evaluation process, but the preparation of these items takes longer; reviewed and accepted papers, however, will be requested in electronic form. For any method of submission, three hard-copies of the text are also required, where all the necessary editing is marked in red. No manuscripts, diskettes, or figures will be returned, unless by special arrangement

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- Only black-and-white, camera-ready illustrations (photos or drawings) can be used Scanned illustrations inserted in the electronic version are preferred. The required (approximate) location of the figures and tables should be indicated in the main text by typing their numbers and captions (Figure 1. [text], Figure 2: [text], Table 1: [text], etc.), as new paragraphs. The figures, which may be slightly reduced in printing, should be enclosed on separate sheets. The tables may be given inside the text or enclosed separately.

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- Subtitles (numbered as 1, 2, 3, etc.) and subsidiary subtitles (1 1, 1 1 1, 1 1 2, 1 2, etc.) can be used, without over-organizing the text

- The use of references is recommended The citations in the text should give the name, year, and, if necessary, page, chapter, or other number(s) in one of the following forms . Weyl (1952, pp. 10-12) has shown..., or ... as shown by some authors (Coxeter et al., 1986, p. 9; Shubnikov and Koptsik 1974, Chap. 2; Smith, 1981a, Chaps. 3-4, Smith, 1981b, Sec. 2 12; Smith, forthcoming). The full bibliographic description of the references should be collected at the end of the paper in alphabetical order by authors' names; see the sample This section should be entitled *References*.

Sample of heading (Apologies for the strange names and addresses)

SYMMETRY IN AFRICAN ORNAMENTAL ART
BLACK-AND-WHITE PATTERNS IN CENTRAL AFRICA
Running head Symmetry in African Art
Section: Symmetry Art & Science

Susanne Z Dissymmetrist	and	Warren M Symmetrist
8 Phyllotaxis Street		Department of Dissymmetry, University of Symmetry
Sunflower City, CA 11235, U S A		69 Harmony Street, San Symmetrino, CA 69869, U.S.A.
		E-mail. symmetrist@symmetry.edu

Abstract: *The ornamental art of Africa is famous ...*

Sample of references

In the following, note punctuation, capitalization, the use of square brackets (and the remarks in parentheses). There is always a period at the very end of a bibliographic entry (but never at other places, except in abbreviations). Brackets are used to enclose supplementary data Those parts which should be italicized - titles of books, names of journals, etc. - should be underlined in red on the hard-copies In the case of non-English publications both the original and the translated titles should be given (cf., Dissymmetrist, 1990).

Asymmetrist, A. Z. (or corporate author) (1981) *Book Title· Subtitle*, Series Title, No. 27, 2nd ed , City (only the first one)· Publisher, vii + 619 pp ; (further data can be added, e g ,· 3rd ed , 2 Vols , *ibid*, 1985, viii + 444 + 484 pp with 2 CD-s; Reprint, ibid , 1988; German trans., *German Title*, 2 Vols , City. Publisher, 1990, 986 pp.; Hungarian trans)

Asymmetrist, A. Z., Dissymmetrist, S. Z., and Symmetrist, W M (1980-81) Article or e-mail article title. Subtitle, Parts 1-2, *Journal Name Without Abbreviation*, [E-Journal address: http·//sym journal edu (if applicable)], B22 (volume number), No. 6 (issue number if each one restarts pagination), 110-119 (page numbers); B23, No 1, 117-132 and 148 (for e-journals any appropriate data).

Dissymmetrist, S. Z. (1989a) Chapter, article, symposium paper, or abstract title, [Abstract (if applicable)], In: Editorologist, A B and Editorologist, C D , eds , *Book, Special Issue, Proceedings, or Abstract Volume Title*, [Special Issue (or) Symposium organized by the Dissymmetry Society, University of Symmetry, San Symmetrino, Calif., December 11-22, 1971 (those data which are not available from the title, if applicable)], Vol. 2, City. Publisher, 19-20 (for special issues the data of the journal).

Dissymmetrist, S Z , ed. (1990) *Dissimmetriya v nauke* (title in original, or transliterated, form), [Dissymmetry in science, in Russian with German summary], Trans. from English by Antisymmetrist, B. W , etc

[Symmetrist, W. M] (1989) Review of *Title of the Reviewed Work*, by S. Z. Dissymmetrist, etc (if the review has an additional title, then it should appear first; if the authorship of a work is not revealed in the publication, but known from other sources, the name should be enclosed in brackets)

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Sample of biographic entry

Name: Warren M. Symmetrist, Mathematician, (b Boston, Mass., U.S.A., 1938).

Address: Department of Dissymmetry, University of Symmetry, 69 Harmony Street, San Symmetrino, Calif. 69869, U.S A *E-mail*· symmetrist@symmetry edu .

Fields of interest: Geometry, mathematical crystallography (also ornamental arts, anthropology - non-professional interests in parentheses)

Awards: Symmetry Award, 1987, Dissymmetry Medal, 1989

Publications and/or *Exhibitions*: List all the symmetry-related publications/exhibitions in chronological order, following the conventions of the references and annotations. Please mark the most important publications, not more than five items, by asterisks. This shorter list will be published together with the article, while the full list will be saved in the data bank of ISIS-Symmetry.

There are many disciplinary periodicals and symposia in various fields of art, science, and technology, but broad interdisciplinary forums for the connections between distant fields are very rare. Consequently, the interdisciplinary papers are dispersed in very different journals and proceedings. This fact makes the cooperation of the authors difficult, and even affects the ability to locate their papers

In our 'split culture', there is an obvious need for interdisciplinary journals that have the basic goal of building bridges ('symmetries') between various fields of the arts and sciences Because of the variety of topics available, the concrete, but general, concept of symmetry was selected as the focus of the journal, since it has roots in both science and art.

SYMMETRY: CULTURE AND SCIENCE is the quarterly of the INTERNATIONAL SOCIETY FOR THE INTERDISCIPLINARY STUDY OF SYMMETRY (abbreviation: ISIS-Symmetry, shorter name: Symmetry Society). ISIS-Symmetry was founded during the symposium *Symmetry of Structure* (*First Interdisciplinary Symmetry Symposium and Exhibition*), Budapest, August 13-19, 1989. The focus of ISIS-Symmetry is not only on the concept of symmetry, but also its associates (asymmetry, dissymmetry, antisymmetry, etc.) and related concepts (proportion, rhythm, invariance, etc.) in an interdisciplinary and intercultural context. We may refer to this broad approach to the concept as *symmetrology*. The suffix *-logy* can be associated not only with knowledge of concrete fields (cf., biology, geology, philology, psychology, sociology, etc.) and discourse or treatise (cf., methodology, chronology, etc), but also with the Greek terminology of proportion (cf., *logos*, *analogia*, and their Latin translations *ratio*, *proportio*)

The basic goals of the *Society* are
(1) to bring together artists and scientists, educators and students devoted to, or interested in, the research and understanding of the concept and application of symmetry (asymmetry, dissymmetry);
(2) to provide regular information to the general public about events in symmetrology;
(3) to ensure a regular forum (including the organization of symposia, congresses, and the publication of a periodical) for all those interested in symmetrology

The Society organizes the triennial Interdisciplinary Symmetry Congress and Exhibition (starting with the symposium of 1989) and other workshops, meetings, and exhibitions. The forums of the Society are *informal* ones, which do not substitute for the disciplinary conferences, only supplement them with a broader perspective.
The Quarterly - a non-commercial scholarly journal, as well as the forum of ISIS-Symmetry - publishes original papers on symmetry and related questions which present new results or new connections between known results. The papers are addressed to a broad non-specialist public, without becoming too general, and have an interdisciplinary character in one of the following senses
(1) they describe concrete interdisciplinary 'bridges' between different fields of art, science, and technology using the concept of symmetry;
(2) they survey the importance of symmetry in a concrete field with an emphasis on possible 'bridges' to other fields.

The Quarterly also has a special interest in historic and educational questions, as well as in symmetry-related recreations, games, and computer programs.

The regular sections of the Quarterly:
- **Symmetry: Art & Science** (papers classified as humanities, but also connected with scientific questions)
- **Symmetry: Science & Art** (papers classified as science, but also connected with the humanities)
- **Symmetry in Education** (articles on the theory and practice of education, reports on interdisciplinary projects)

There are also *additional, non-regular* sections.

Both the lack of seasonal references and the centrosymmetric spine design emphasise the international character of the Society, to accept one or another convention would be a 'symmetry violation'. In the first part of the abbreviation *ISIS-Symmetry* all the letters are capitalized, while the centrosymmetric image ¡SIS! on the spine is flanked by 'Symmetry' from both directions This convention emphasises that ISIS-Symmetry and its quarterly have no direct connection with other organizations or journals which also use the word *Isis* or *ISIS*. There are more than twenty identical acronyms and more than ten such periodicals, many of which have already ceased to exist, representing various fields, including the history of science, mythology, natural philosophy, and oriental studies ISIS-Symmetry has, however, some interest in the symmetry-related questions of many of these fields.

ISIS-Symmetry
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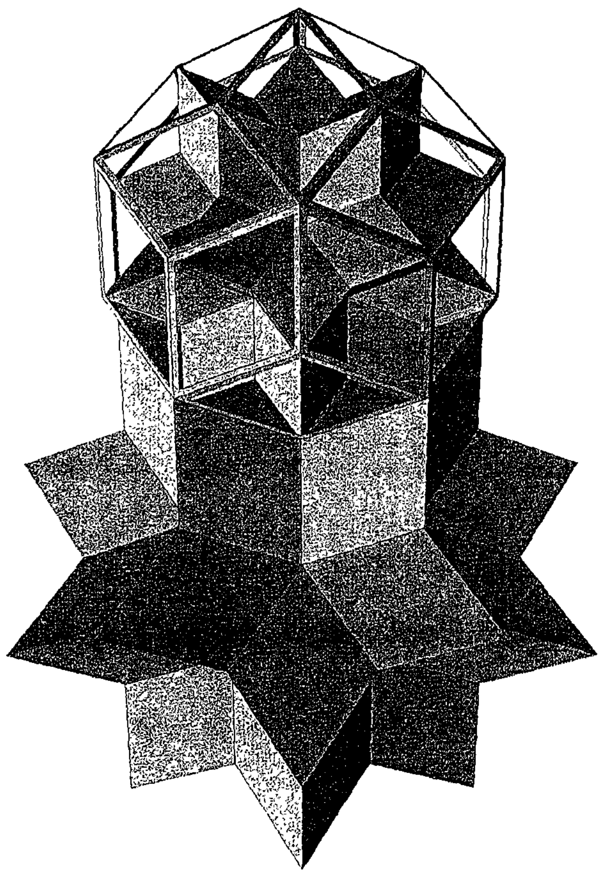
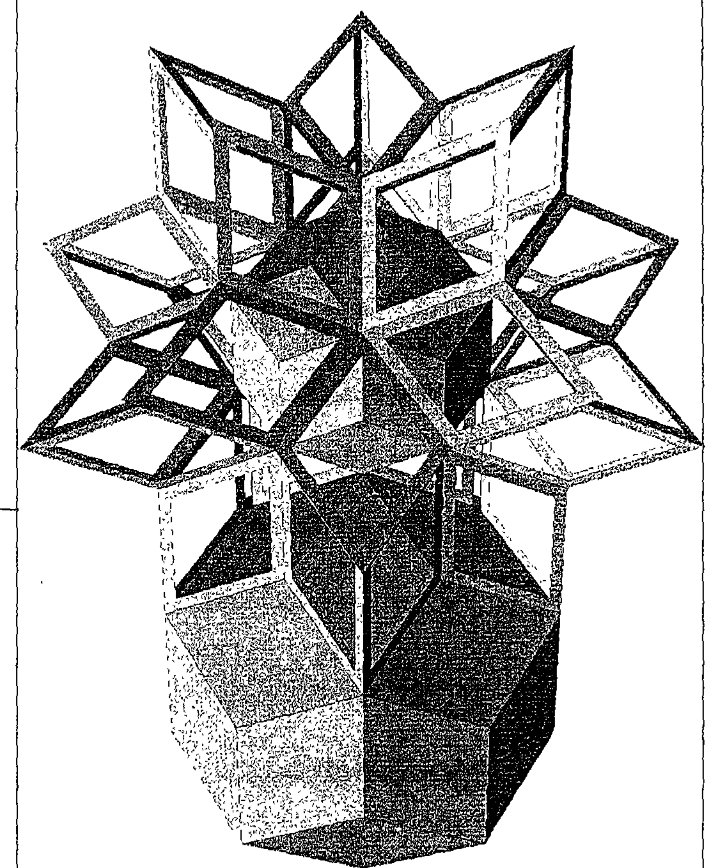


Symmetry



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