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Symmetry in a Kaleidoscope 2

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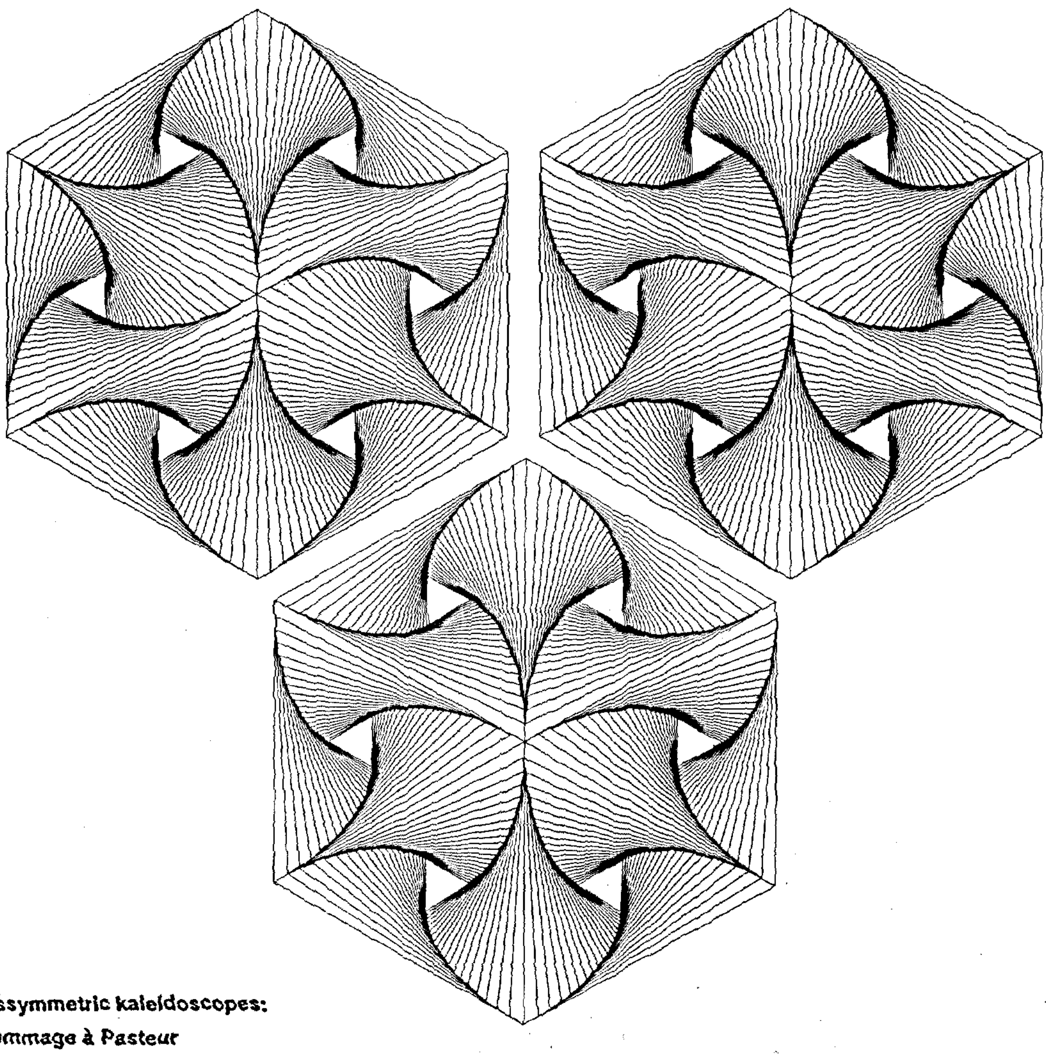

Dissymmetric kaleidoscopes:
Hommage à Pasteur

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ON ETHNOMATHEMATICAL RESEARCH AND SYMMETRY

Paulus P. J. Gerdes

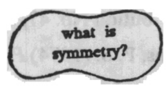
Mathematician, Anthropologist, Philosopher, (b. Hateringen, Netherlands, 1952; citizen of Mozambique).

Address: Instituto Superior Pedagógico, Caixa Postal 3276, Maputo, Moçambique;

Fields of interest: Ethnomathematics, geometry, history of mathematics, philosophical problems of mathematics and mathematics education.

Publications: *Zum erwachenden geometrischen Denken*, (1985), Dresden; *Marx demystifies mathematics*, (1985), Minneapolis; *Ethnogeometrie, Kulturanthropologische Beiträge zur Genese und Didaktik der Geometrie*, (1988), Bad Salzdetfurth; *Mathematics, Education and Society*, co-editor, (1989), Paris.

QUESTION 1



Symmetry is such an overwhelming phenomenon both in nature and in culture, that it is easy to forget to question: *why*? Why do fans all over the world have an axial symmetry (Fig. 1a)? Why are fire drills always twirled at a right angle to the drill-stick (Fig. 1b)? Why do combs normally display a bilateral symmetry (Fig. 1c)? Why do most cooking pots have a rotational symmetry? Why do many baskets, when seen from above, show a double bilateral symmetry pattern (Fig. 1d)? Why do most string figures have a line symmetry (Fig. 1e)? Why are bellows symmetrical (Fig. 1f)? Why do boats, shields, musical instruments, boomerangs take on a symmetrical shape?

At first sight one might think that symmetry arose in human culture as a blind copy of symmetry in nature. In reality however, e.g. hand axes were initially *not* symmetrical, but they became so as the result of the production traditions of thousands of generations (Frolov, 1977-78, p. 151). Rotational symmetry of order 2 is (almost) absent in nature (cf. Brew, 1946, p. 269), but frequent in human culture (see the example in Figure 2).

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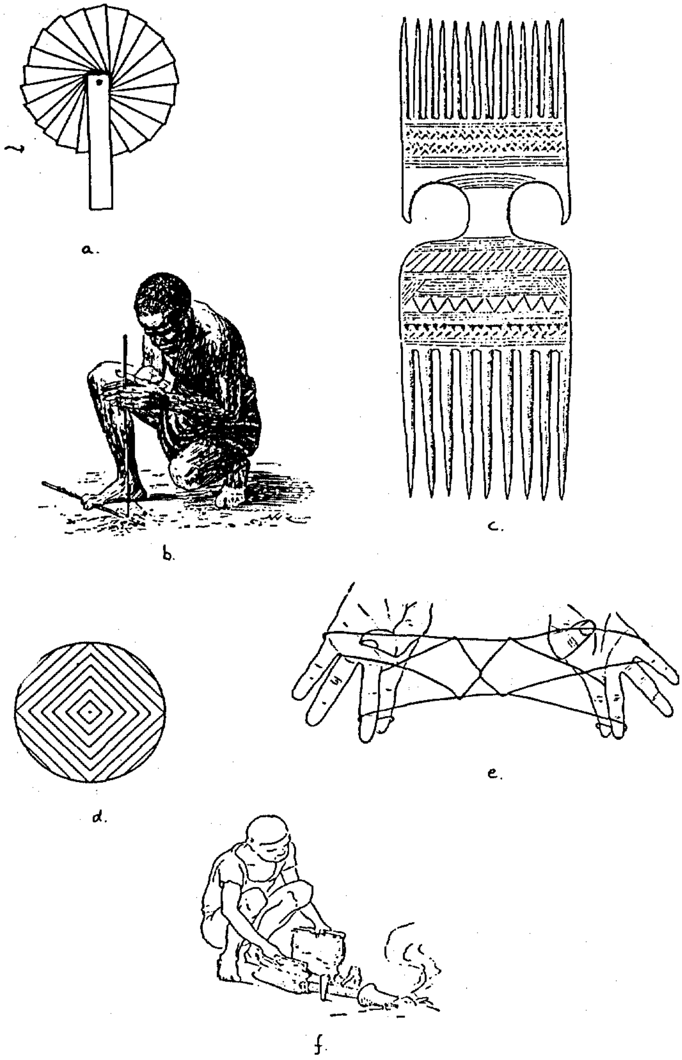

Figure 1: Examples from Mozambique.

So far, we may conclude that the question "*What is symmetry?*" cannot be answered completely in an abstract way, detached from history or from any cultural context.

The answer is changing over time. "*Thinking in terms of symmetry*" is human and it is a cultural-historical product: human beings *learn*t to think in terms of symmetry, learnt to see symmetry in their artifacts and in nature, and learnt to esteem symmetry as an esthetical value.

As there are so many different forms in nature, it has to be explained *why* man gradually became capable of observing certain forms in nature. There are no forms

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in nature that are *a priori* destined for human observation. The capacity of man to recognize geometrical forms in nature and in his own products has been developed through his *labour activity* ["Tätigkeit", in German] (cf. Gerdes 1985; 1988a).

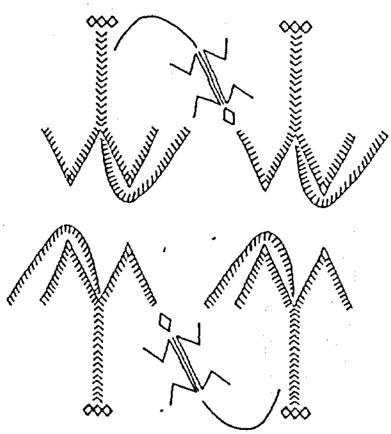


Figure 2: Traditional Makonde tattooing (northern Mozambique).

Regularity and symmetry of man-made objects are the result of creative human labour. The real, practical advantages of an invented regular and symmetrical form for an artifact lead to a growing consciousness of this regularity and symmetry. The same advantages stimulate man to compare this artifact with other labour products and with natural phenomena. The regularity and symmetry of a product generally simplifies its reproduction and in this way both the consciousness of its form and the interest in it are reinforced. The growing consciousness and interest develops at the same time a positive valuing of the invented form, and this is also used where it is *not necessary* for material, objective reasons; this form becomes experienced as *beautiful.*

The cylinder, cone, and other symmetrical forms of recipients, the regular hexagonal hole pattern of baskets, fishtraps, hats, snowshoes, etc. may *appear* at first sight as the result of instincts or of an innate feeling for these forms, or, mechanically, as the imitation of natural phenomena. In reality, however, these forms have been *created* by man in order to satisfy his daily needs. Working with the materials at his disposal, man learned to understand which were the necessary forms in order to produce something useful. Some examples will clarify my ideas.

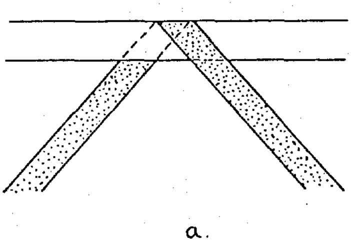
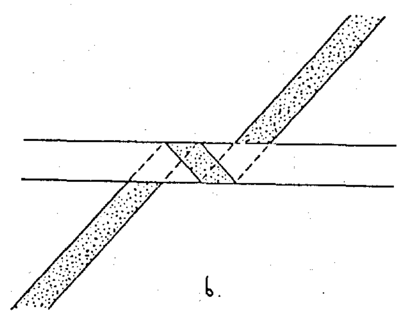


Figure 3.



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Folding one strand once over a second strand, automatically an axial symmetry is obtained (Fig. 3a). Folding it twice over the second strand, one arrives immediately, and independent from human will, at a rotational symmetry of order 2 (Fig. 3b).

In the coastal zones of Mozambique fish is dried to be sold in the interior. How should the fish be dried? What happens if you place the fish around the fire like in Figure 4a. Some fish will be grilled, while others remain moist. Through experience, the fishermen discovered that it is necessary, when the wind does not blow, to place the fish *equidistant* from the fire, i.e. in a *circle* (Fig. 4b).

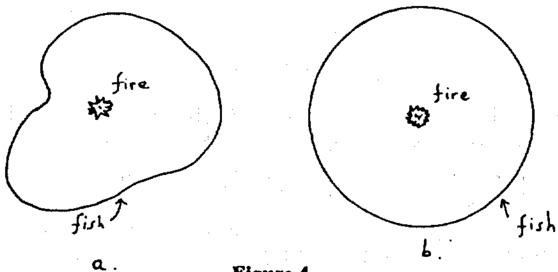

Figure 4.

The Makonde artisans in North-Mozambique weave their *chelo*-basket in the following way. In order to make its border, they bend a rectangular strip of wood and bind its ends firmly together. Automatically (independent of human will), the border becomes *circular* (=symmetrical), as the homogeneous material of the strip forces it to become so. The artisan weaves a rectangular mat and binds its sides to the border of the basket (Fig. 5a). He wets the mat and then presses it uniformly downwards with one of his feet (Fig. 5b). Finally he trims the end pieces and binds the rest of the bottom to the border. Experience has shown the artisans that the mat has to be a *square* (rotational symmetry of order 4). If it were not a square, then, for lack of equilibrium, the basket would easily fall over to one side. In order to press the mat downwards, it is necessary to bind first all its four sides to the border, and not only two or three of them. This has to be done exactly at their *midpoints*, otherwise the right angles between the strands of the mat would be unevenly distorted under the pressure of the foot which would result once more in a lack of equilibrium.

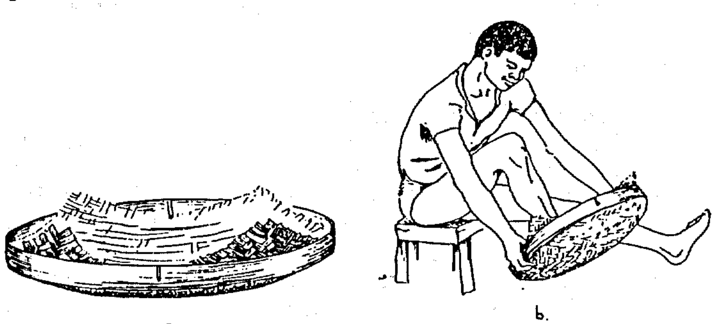

Figure 5.

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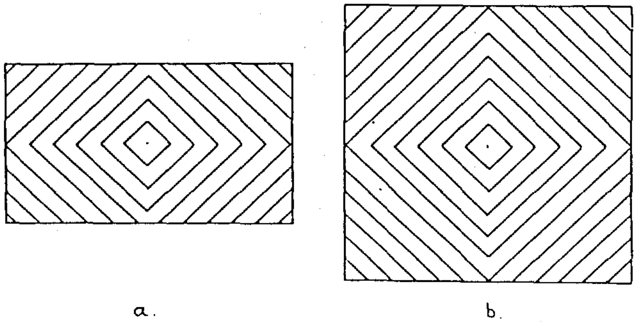

Figure 6.

How may one know if the mat is square and where are the midpoints of its sides? The Makonde artisans use a solution invented by many people. The solution avoids measuring. One weaves the mat in such a way, starting from its future center, that the weaving pattern *immediately* shows if the mat is square or not and where are the midpoints of its sides. Figure 6 gives examples. The weaving patterns have to display almost automatically a double bilateral symmetry or a rotational symmetry of order 2 or 4 (see Figure 7, cf. Figure 1d).

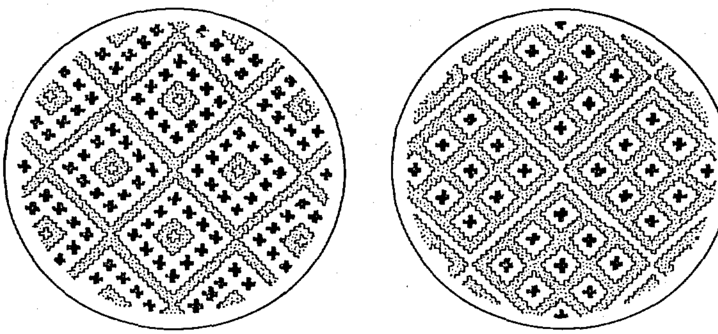

Figure 7: Examples from Guyana.

In my research I use symmetry/dissymmetry of patterns in the reconstruction of possible original patterns that have been lost with time. Some examples will clarify this heuristic role of symmetry.

During the harvest month, Tamil women in South India draw designs in front of the thresholds of their houses. In order to prepare their drawings, they set out a rectangular reference frame of equidistant points. Then curves are drawn in such a way that they surround the dots without touching them. The (culturally) ideal design is composed of a single closed line. However some of the reported threshold designs do not conform to the standard as they are composed of two, three or more superimposed closed paths. Figure 8a shows an example, made out of three separate closed lines (Figs. 8b, c, and d). The outer part of this design displays a rotational symmetry of 90° (Fig. 9a), but, on the other hand, the inner part displays only a rotational symmetry of 180° (Fig. 9b). Is this asymmetry (90°-180°) related to

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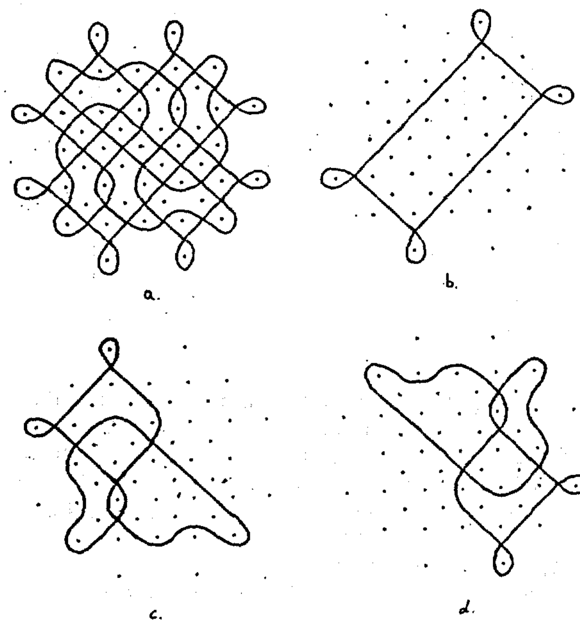

Figure 8.

the fact that the design does not conform to the cultural standard? If we adapt the inner part in such a way that it displays also a rotational symmetry of 90° just as the outer part (Fig. 10a), then we arrive at a design (Fig. 10b) rather similar to the reported design, and it satisfies at the same time the norm, as it turns out, to be composed of only one closed, smooth path. The reported plural closed line pattern (Fig. 8a) is probably a "degradation" of the reconstructed original design (Fig. 10b), a consequence of deficient transmission from one generation to another (for more details and related examples, see Gerdes, 1989).

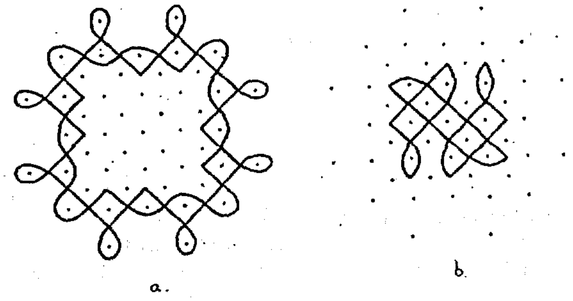

Figure 9.

Many cylindrical baskets with a square bottom display a wall decoration with exactly or almost a rotational symmetry. When such a basket has only *almost* a rotational symmetry, I pose the question "What went wrong?" For instance, during

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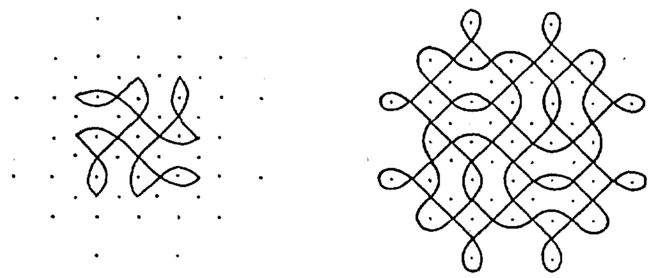
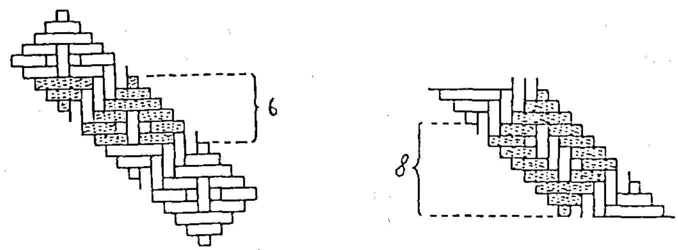


a. b.

Figure 10.

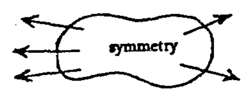
my visit to Brazil, I encountered an Indian basket, that displayed on each horizontal layer of its wall 16 woven dented squares (Fig. 11a) and one dented rectangle (Fig. 11b). The dented rectangle that breaks the rotational symmetry of the basket had probably been caused by an "error of calculation", as each quarter of the bottom had two strands too much to generate the exact rotational symmetry of order 16 (for more details and analysis, see Gerdes, 1988f). In the concrete case of the basket under consideration, in order to achieve the rotational symmetry of order 16, the inventor of the ornamentation had to know that $4 \times 6 = 6 \times 4$ (*arithmetical symmetry* or commutativity).



a. b.

Figure 11.

QUESTION 2



In order to give an idea of the many possible and multiple links between the ethnomathematical research of symmetry and other scientific and/or cultural spheres, I will present a concrete example.

The Tchokwe people of northeast Angola are well known for their beautiful decorative art. When they meet, they illustrate their conversations by drawings on the ground. Most of these drawings belong to a long tradition. They refer to proverbs, fables, games, riddles etc. and play an important role in the transmission of knowledge from one generation to another. Just like the Tamils of South India,

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the Tchokwe people invented a similar mnemonic device to facilitate the memorization of their standardized drawings. After cleaning and smoothing the ground, they first set out with their fingertips an orthogonal net of equidistant points. The number of rows and columns depends on the motif to be represented. By applying their method, the Tchokwe drawing experts reduce the memorization of a whole design to that of mostly two numbers and a geometric algorithm. Most of their drawings display bilateral and/or rotational (90° or 180°) symmetries (see Fig. 12). The symmetry of their pictograms facilitates the execution of a drawing. This is important as the drawings have to be executed smoothly and continuously. Any hesitation or stopping on the part of the drawer is interpreted by the audience as an imperfection and lack of knowledge, and assented with an ironic smile. A beautiful collection of 287 different Tchokwe sand drawings has been published by Fontinha (1983).

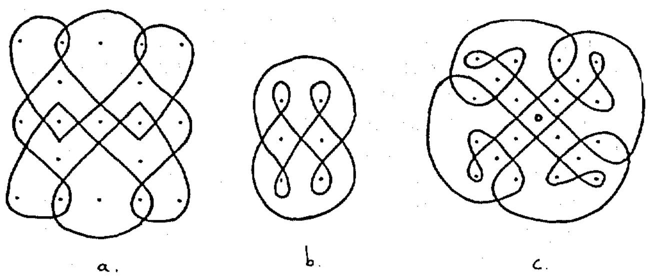

Figure 12.

The ethnographer and musicologist Kubik was very impressed by the order, regularity, and symmetry of the Tchokwe graphic tradition. He analyzed the "deep structure" of the drawings and saw a parallel with the shaping and composition of music in some regions of Africa (e.g. the Kiganda musical system in Uganda): the same type of symmetry and regularity that enable one to deduce nearly automatically the whole composition, as soon as a basic part is composed (Kubik, 1987).

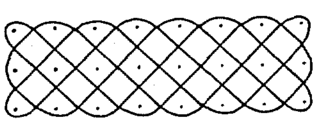

Figure 13.

In my own research on the Tchokwe sand drawings, I was initially mostly interested in the reconstruction of the mathematical knowledge that had been present at the invention of some (types) of these designs. Among other results, it came out that in order to draw one type of design (see the example in Fig. 13), made out of only one closed curve, both numbers (length and width) have to be relatively prime. This led me to the formulation of a didactical, geometrical model for the determination of the greatest common divisor of two natural numbers (see Gerdes, 1988b) and of a physical model for the determination of prime numbers (see Gerdes, 1987; 1988a). Symmetry considerations speeded up the velocity of the second model.

In the same way as in the example from the Tamils given before, symmetry considerations helped me also to find possible original designs of reported sand

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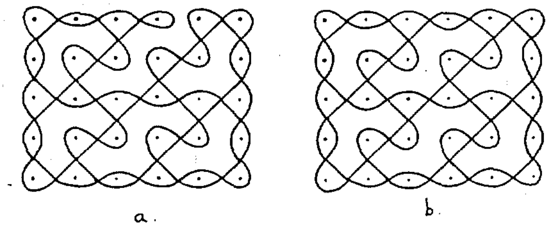

Figure 14.

drawings (Gerdes, 1988i). Figure 14a gives an example. The reported drawing lacks rotational symmetry and is furthermore composed of two closed curves instead of only one, as generally preferred in Tchokwe culture. The reconstructed design (Fig. 14b), which is made out of only one closed curve, displays a rotational symmetry of order 2 and is relatively easy to draw as the underlying geometrical algorithm facilitates the execution of the drawing. This reconstructed design is similar in structure to Figure 15, reported by Vergani (1986, p. 286), that represents the marks on the ground left by a chicken when it is chased (cf. also Fig. 15 in Ascher, 1988). Both conform to the same geometrical algorithm, only the dimensions of the reference frame have changed. In this sense, one might say that Figure 15 constitutes an *extension* of Figure 14b. In the same way, Figure 16 is an extension of the reconstructed Tamil threshold design (without border ornamentation), shown in Figure 10b. The extensions of such Tchokwe or Tamil designs display the same symmetries as the 'mother" figures and, for appropriate dimensions of the reference frame, they are composed of only one closed curve.

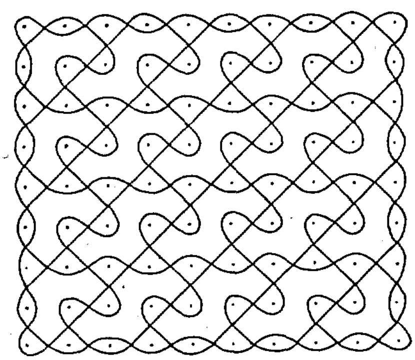

Figure 15.

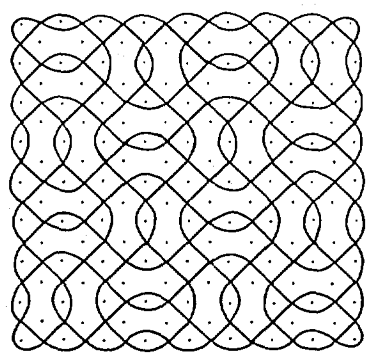

Figure 16.

The first use of these extended Tchokwe and Tamil patterns I arrived at, was a didactical one. Well known as a pedagogical tool, there are arithmetical problems of the type "*Find the missing numbers*". For instance: 2, 2, 6, 10, 22, .., 86, .., ... Which are the missing numbers? As a variant on this theme, a series of geometric problems has been elaborated: "*Find the missing figures*" (Gerdes, 1988g). Figure 17 gives an example. These problems have the objective to develop a sense for geometric algorithms and symmetry.

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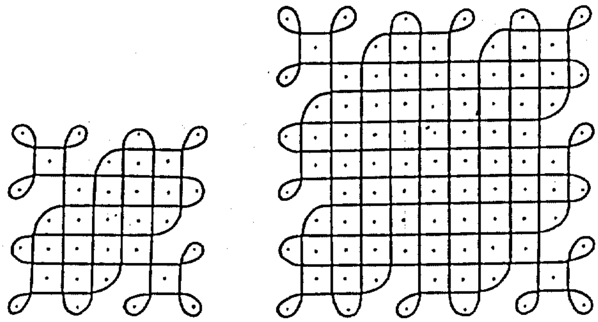
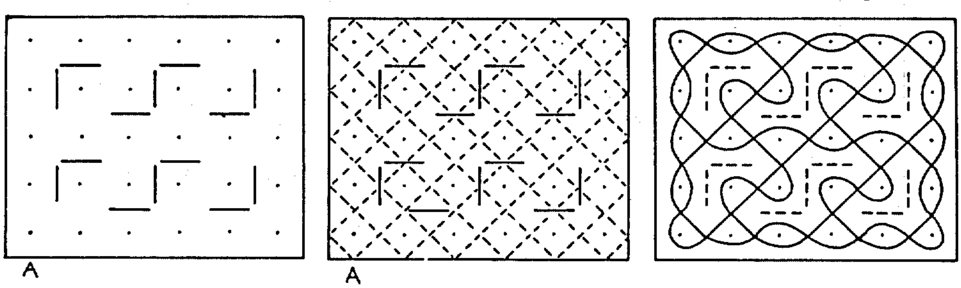


A B

Figure 17.

As I found this type of original (reconstructed) Tchokwe and Tamil pictograms and their extended versions *beautiful*, I looked for a common *construction principle*. The curves of the type considered may be generated in the following way. Each of them is the smooth version of the poligonal path described by a lightray emitted from point A (see Fig. 18a). The ray is reflected in the sides of the "circumscribed" rectangle of a (basic) point reference frame. It encounters on its way through the point reference frame double-sided mirrors which are placed, at regular intervals, horizontally in the middle between two vertical-neighbour frame points and vertically in the middle between two horizontal-neighbour frame points (see Fig. 18 for an example). The underlying "mirror-patterns" of the designs display the



a b c

Figure 18.

same axial and rotational symmetries as the designs themselves. Once was this common construction principle formulated, it became possible to find a whole class of single closed curves that satisfy the same principle. Figure 19 gives examples. The class of curves I found in this way is attractive and interesting for many reasons. The curves are esthetically appealing. They may be used for instance in textile design. By filming them, starting the curve at one point, one sees a geometrical algorithm at work. Possibly they may be applied in the codification of information, in the development of laser memory circuits for optical computers, in the study of the topology of large scale integration chips etc., as suggested by my colleague Petrossiuk (1988).

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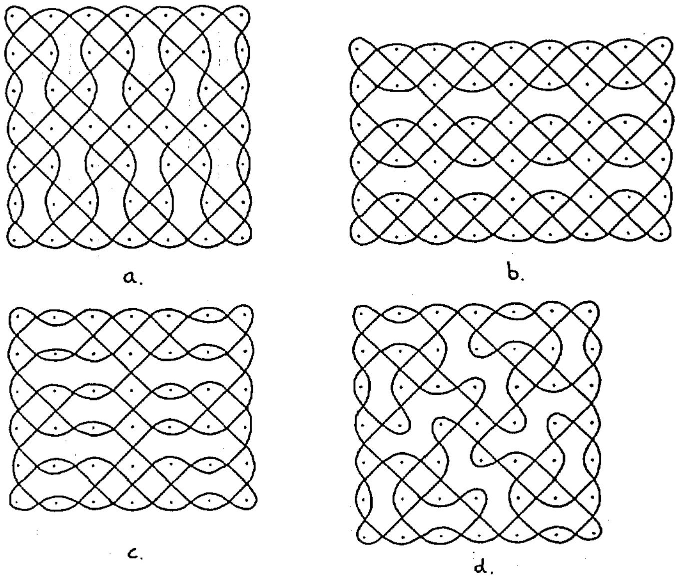

Figure 19.

Such applications become more probable if one takes into account the underlying *arithmetical symmetries* of the designs that belong to the considered class of curves. If one draws such a design on squared paper (Fig. 20a gives an example) and enumerates the squares through which the curve successively passes, modulo 4, i.e. 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, ... (see Fig. 20b) one always obtains a scheme like the one in Figure 21. These underlying numerical schemes display interesting symmetries. In the example there is a vertical line symmetry and a horizontal "semi"-symmetry: a "0" on the one side of an axis always corresponds to a "3" on the other side and vice versa; to a "1" on the one side of an axis corresponds always a "2" on the other and vice versa.

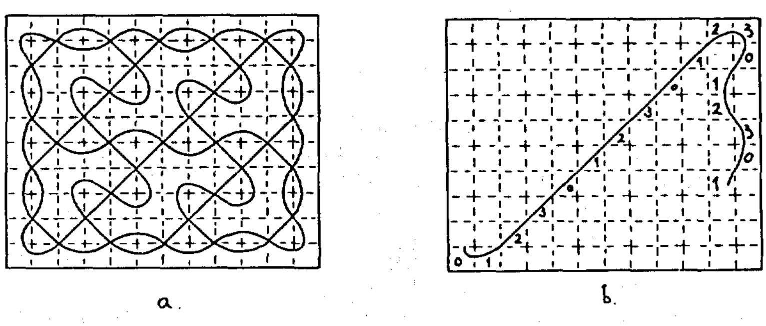

Figure 20.

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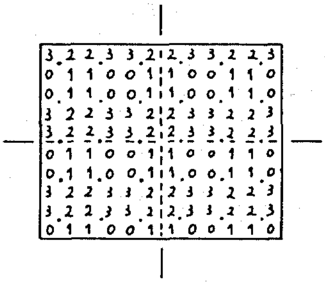


Figure 21.

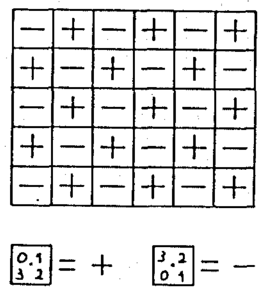
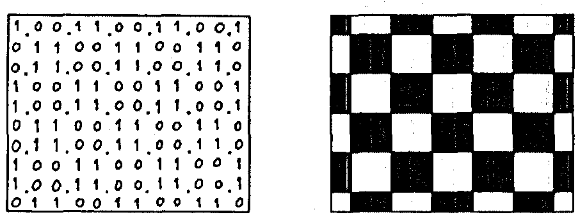


Figure 22.

Moreover the four squares around a reference point are always numbered clockwise (positive rotation) or counter-clockwise (negative rotation) 0, 1, 2, 3. Positive and negative rotations alternate as the checkers of a chessboard (see Fig. 22). When one counts the squares modulo 2, i.e. 0, 1, 0, 1, 0, 1, ... one also obtains a chessboard structure with two semi-symmetries (see Fig. 23). The involved theorems are not difficult to prove (Gerdes, 1988h).



a. b.

Figure 23.

These results stimulated the research of other classes of related curves. For instance, what happens if one also admits vertical mirrors in the middle between two vertical-neighbour points and horizontal mirrors in the middle between two horizontal neighbour points. Figures 24a and 25a give two examples. In the first

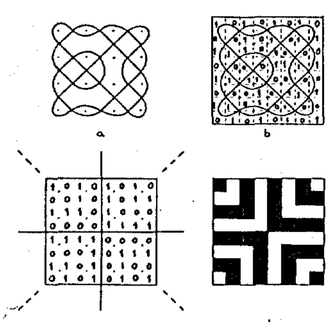


Figure 24.

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case the curve displays only one (geometrical) symmetry. Its underlying arithmetical scheme modulo 2, however, has two line symmetries (diagonals) and two axial semi-symmetries (Figs. 24b, c). Figure 24d shows the geometric structure (1=black, 0=white) of the underlying arithmetical scheme. Although in the second case the curve itself displays a rotational symmetry of order 2, its underlying arithmetical scheme presents (rotationally) only semi-symmetry, but, on the other hand has a line symmetry, that was not present in the curve itself (Figs. 25b, c, d). It seems worthwhile to investigate further the inconsistencies between the symmetries of these types of curves and their underlying arithmetical structures.

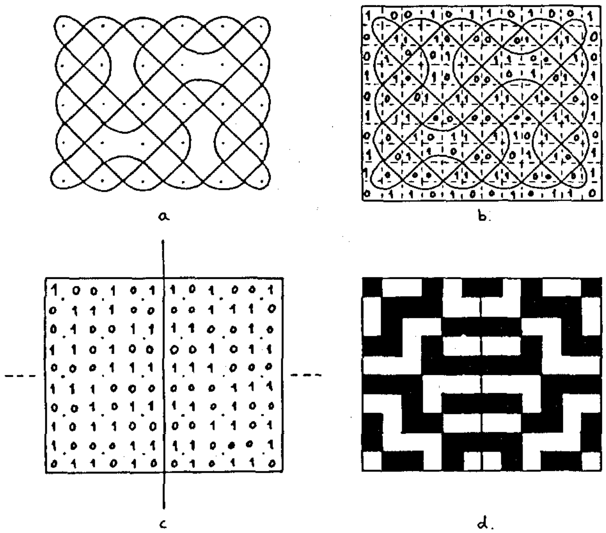

Figure 25.

Not all traditional Tchokwe sand drawings are made out of only one closed curve. Figure 26a represents the unity of a married couple. Its is composed of two overlapping curves (see Fig. 26b for one curve). The symmetry in social relations reflects itself in the symmetry of the design; the symmetry of the design expresses a "social symmetry".

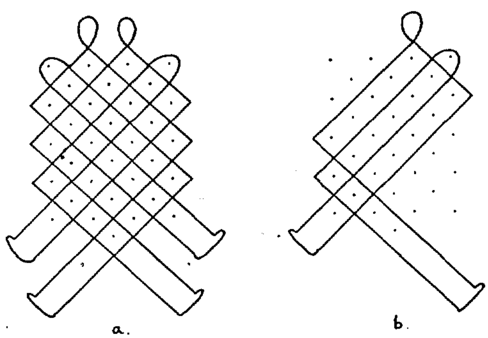

Figure 26.

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The below diagram (see Fig. 27) may summarize the interdisciplinary impact of the study of symmetries in the Tchokwe sand drawings.

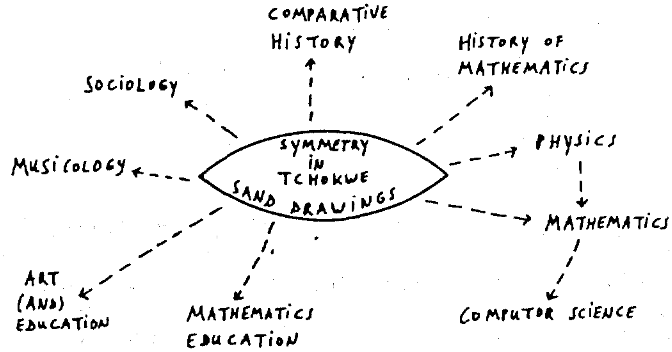
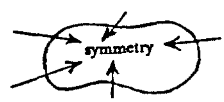


Figure 27.

QUESTION 3



In traditional Mozambican handicraft, dance, and art, symmetries play an important role (cf. Figure 1). In basketry rotational, line, and point symmetries and also strip or frieze patterns (see Fig. 29 for an example) are common. In traditional Makonde tattooings (northern Mozambique), bilateral symmetry (Fig. 28) and sometimes rotational symmetry of order 2 are used (Fig. 2). The frequency and social-cultural value of these symmetries stimulated my research in the following directions: *why* do these symmetries occur? *Why* are they culturally valued? *How* can they be incorporated in the *teaching* of symmetry in particular and of geometry in general? *How* can their mathematical potential be explored?

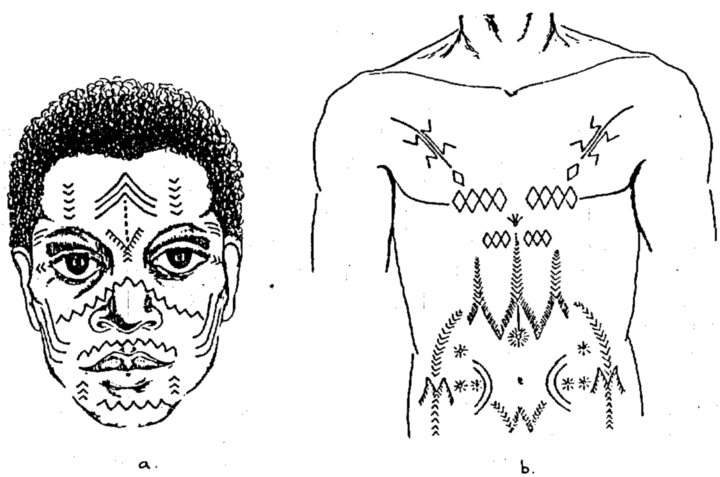


Figure 28.

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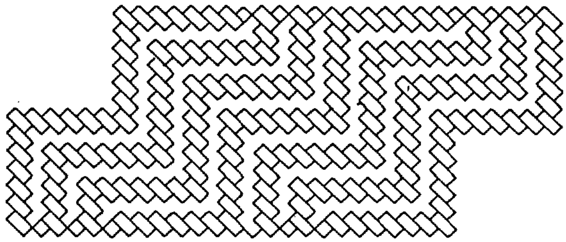

Figure 29.

My studies have been much influenced by the *changes in symmetry* that occur during the production *process* of traditional Mozambican artifacts and during the construction of houses. The transition from a parallelogram with only a rotational symmetry of 180° to a rectangle with its double bilateral symmetry, as occurs in the construction of the base of a traditional rectangular house, has been explored to formulate an alternate axiom for Euclidean geometry (for details, see Gerdes, 1988c, pp. 141-144). The transition from a square with its rotational symmetry of order 4 to equilateral triangle with its rotational symmetry of order 3 (Fig. 30), as

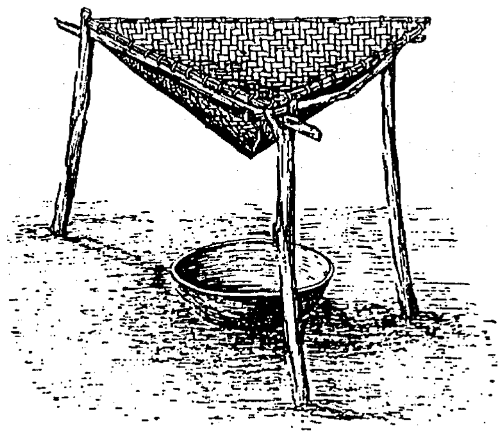

Figure 30.

occurs in the weaving of some Mozambican funnels, has inspired me to analyze similar transitions from order 2^k to lower orders (2^k-1, 2^k-2, ..., 2^{k-1}+1). A general method for the construction of regular polygons (n-gons) was the result (see Gerdes, 1986; 1988c, pp. 144-149). By joining four of those (congruent) funnels, one obtains a square pyramid. On the basis of the corresponding relationship between the volumes of the funnels and the square pyramid and some cultural-historical considerations, a new hypothesis on the origin of the Ancient Egyptian (!) formula for the volume of a truncated pyramid has been formulated (see Gerdes, 1985; 1988a, chap. 6).

Maybe the most surprising and beautiful result I arrived at in this context, was the invention of an infinite (!) series of new proofs of the so-called Pythagorean theorem. This discovery was "provoked" by changing the double bilateral symmetry

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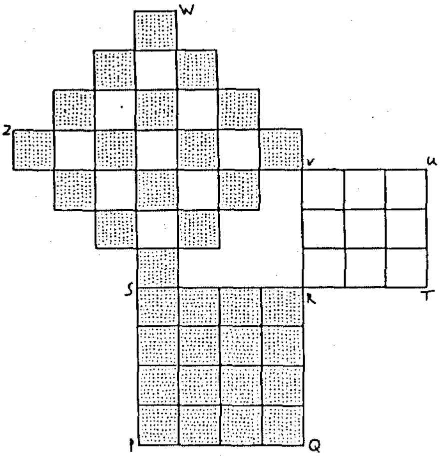

Figure 31.

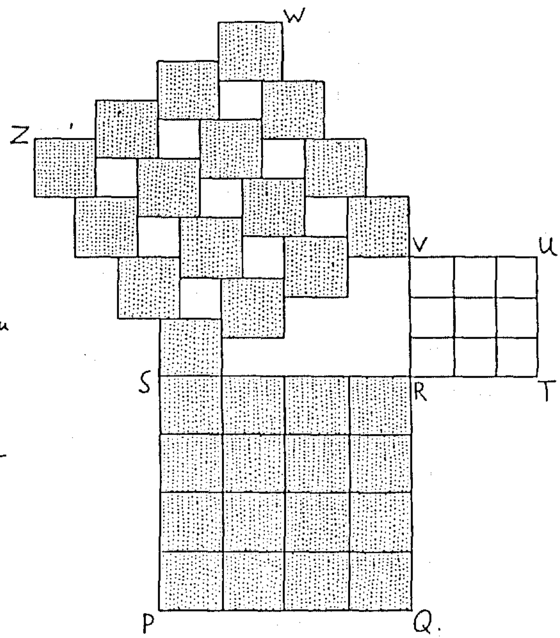

Figure 32.

of a traditional pattern (see the dented square *VWZS* with chessboard pattern in Fig. 31) into a "less strong' one, i.e. into a rotational symmetry of order 4 (see Fig. 32): sum of the areas of the squares *PQRS* and *RTUW* = area of the dented square *VWZS* = area of the real square *VWZS* (for details, see Gerdes, 1988a; 1988d; 1988e).

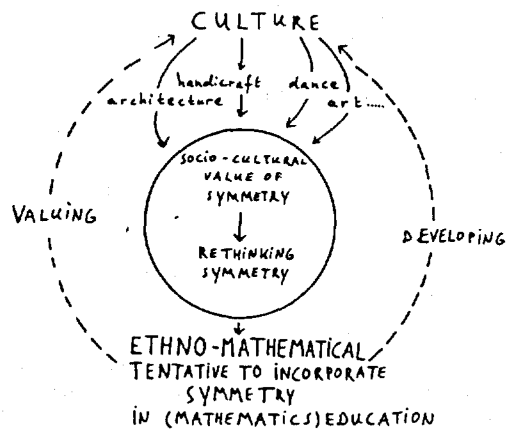

Figure 33.

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