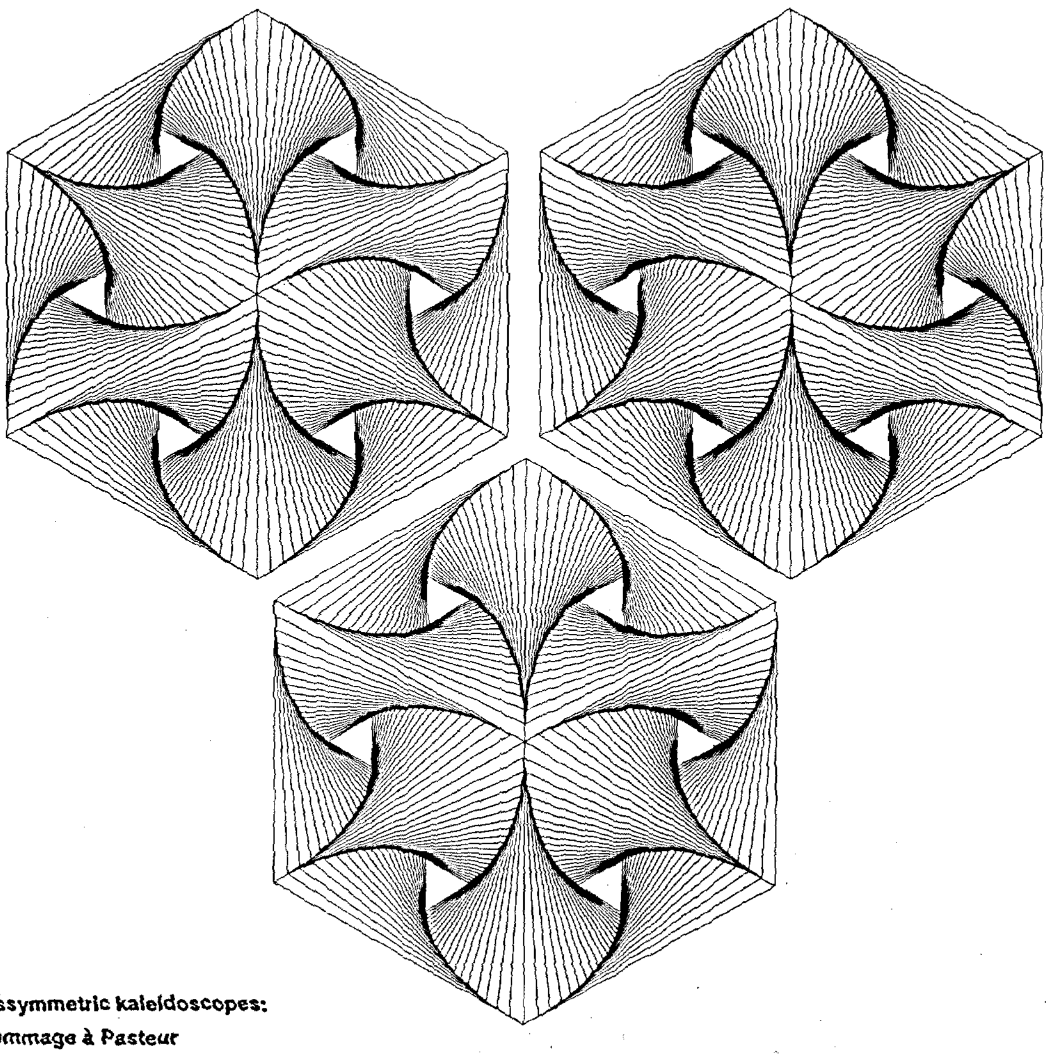
Symmetry: Culture and Science

SPECIAL ISSUE
Symmetry in a Kaleidoscope 2

The Quarterly of the
International Society for the
Interdisciplinary Study of Symmetry
(ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

Volume 1, Number 2, 1990



Dissymmetric kaleidoscopes:
Hommage à Pasteur

INTERNATIONAL SOCIETY FOR THE INTERDISCIPLINARY STUDY OF SYMMETRY (ISIS-SYMMETRY)

continued inside I

Re: 2nd Interdisciplinary Symmetry Symposium and Exhibition
August 17-24, 1992 - at the Synergetics Institute of Japan

This is a brief report on the symposium. Despite the financial difficulties, there were participants from 14 countries of four continents. Vol. 3, No. 1 of the quarterly only includes Part 1 of the extended abstracts (those ones which were received until the final deadline), while a forthcoming issue will be Part 2, inluding the papers by the Past-President of the Japanese NSF, who was a student of Tomonaga (and he is elected as a new Honorary Member of the Society); the inventor of the 3D Penrose tilings; the leading Japanese expert of experimental quasicrystallography; the Japanese "star" of the buckminster-fullerine; and the American Director of a related NATO ASI Program. Although we had less people than in Budapest, but the quality was much higher (in this sense the symposium was even better!). The Editor of the Japanese Sci. Am. was also present, and he will write a survey of the symposium. The most exciting parts were: the minisymposium on quasicrystals and the Japanese arts and crafts festival with workshops at the exhibition. There are seven candidates to host forthcoming ISIS-Symmetry events: Germany, Holland, Hungary, India, Portugal, Arizona/Washington/USA. We will also coorganize a "floating univesity" program "Equilibrium, Symmetry, and Similitude: Modelling in Natural, Social and Technical Systems" travelling with a flaoting hotel on the river Danube en route Budapest-Bratislava-Vienna-Linz-Regensburg (four countries), two-week program in late June 1993; and the ARS+ Symposium, Science City Tsukuba, Japan, November 1994. We will distribute detailed information later (let Gyorgy Darvas know if you are interested: h492dar@huella.bitnet <-- this is now the best address). After the symposium, I attended some other meetings in Tokyo, and spent a longer period in Australia, delivering some lectures and discussing the details of a possible conference in Australia.

Yours very "symmetrely" - Denes Nagy

Information in brief for the Board Members of ISIS-Symmetry

The Second Interdisciplinary Symmetry Symposium and Exhibition of the Society: *Symmetry of Patterns* was held at the Synergetics Institute, in Hiroshima, Japan, August 17-23, 1992.

Participants from 14 countries of four continents gave lectures and/or exhibited their works during the six days' program, or organized workshops at the evenings.

Kodi Husimi, one the most distinguished Japanese scientists (see details on the next page) was elected as a new honorary member of the Society. The Sympsoium discussed, among others, Ted Goranson's proposed *Taxonomy Project* (see in. Vol. 1, No. 2, p.208 of *Symmetry: Culture and Science*), the report of Director Ron Goforth on the preparation of the forthcoming NATO *Advanced Study Institute* program (see below), and the report of Tohru Ogawa (ARS$^+$ and Katachi Society) on the preparations of the 1994 Conference *Katachi no Chi, Chi no Katachi* (see below)

Several *board meetings* were included in the program, however many board members couldn't take part at the Symposium, so formal decisions were not made. The President and the Executive Secretary gave account on the three years' activity as well as financial affairs of the Society. They reported the (reasons of the) establishment of the International Symmetry Foundation, which should facilitate the better financial conditions (tax advantages for the Foundation and for its sponsors) of the Society; as well as the reforming of the Society's Budapest Office into the *Symmetrion - The Institute for Advanced Symmetry Studies*, emphasizing the shift from administrative functions to professional activity: organization of lectures, courses, exhibitions etc. In the lack of the majority of board members, the mandate of ISIS-Symmetry's officials was prolonged for the next three years. The board should be revised. The Symposium participants remembered two board members: Andrew Duff-Cooper and Jarek Woloszyn, who passed away last year. The invitation of some new members to the Board was decided.

Many members of our Society, including several Board members, took part at the *Art and Mathematics 92 Conference* in Albany, N.Y., June 8-12, 1992. Therefore another partial Board meeting was held in Albany, June 10, 1992, where all main issues of ISIS-Symmetry's past and future activity were discussed.

Summarizing the two above occasions the following main items can be mentioned:

The journal of the Society, *Symmetry: Culture and Science* should be published on a more regular basis. The publication of the 3rd (1992) volume has been started, and the missing 1991 volume - which was reserved for the documentation of the symmetry exhibitions in Budapest - will be available later, parallelly. (Copies of Vol. 1 No. 4 will be also distributed later.) The journal cannot survive without a relevant increase of subscribers. For decreasing the financial uncertainty of the publication (now it depends on sponsors) it needs more subscriptions. Therefore all Board members are kindly asked to strengthen their activity in their region and scientific, artistic circles to recruit new subscribers. Over new individual readers, the need for more institutional subscribers, first of all libraries, is emphasized. Please, use in your argumentation that *Symmetry: Culture and Science* is a referred journal (by the Mathematical Review, which is not a narrow disciplinary review journal, but focuses on all sciences where mathematics plays an important role). Note, that the editors have less personal contacts and local experience than you to contact and convince individuals and librarians in your environment.

A special issue on *Origami* was decided during the Hiroshima Symposium. Please, contact the editors with proposals.

Coming events: ISIS-Symmetry will start its educational program in 1993. A two weeks Summer School under the title *Equilibrium, Symmetry, and Similitude: Modelling in Natural, Social and Technical Systems* is planned for the second half of June 1993, on the board of a ship, moving on the Danube from Budapest through Bratislava (Slovakia), Vienna, Linz (Austria) to Passau, Regensburg (Germany) and back, therefore it has got the (nick)name *floating university*.

Parallel to the *floating university, exhibitions* are planned on the banks of the Danube. It will start with the *Ars (Dis)Symmetrica* in a distinguished museum of Budapest.

ISIS-Symmetry is invited by the ARS+ to act as a co-organizer together with the Japanese Katachi (*Form*) Society of the international symposium *Katachi no Chi, Chi no Katachi* (Form of knowledge, knowledge of form), planned in Tsukuba, late November, 1994.

ISIS-Symmetry plans its 3rd International Interdisciplinary Symmetry Symposium and Exhibition in 1995. There are invitations to the following places to organize forthcoming events: Germany (Ulm and other proposals), Hungary (Szeged) India (Bombay), the Netherlands (Delft), Portugal (Lisbon), USA (Seattle, Washington and Tempe, Arizona). The logistical and financial conditions will motivate the final decisions.

Local branches of the Society organize special meetings. All our regional chairpersons are encouraged to initiate specialized regional meetings between the Society's three years symposia: in 1993, 1994.
Regular local forums are also welcome. E.g. such meetings were the inaugural lecture of honorary member István Hargittai in Budapest in February 1992, the special dinner of the ISIS-Symmetry members and invited guests in Albany, N.Y. in June 1992; and also the monthly meetings of the Hungarian Symmetry Club with cca. 50 participants each, as well as the regular Symmetry Seminars organized by our Polish Branch in Wroclaw and the regular meetings organized by Hahn Art & Science Research Institute in Germany on the Evolutionary Symmetry Theory (EST). Where will the next one start?

An *E-mail journal* (Bulletin Board) is under organization by George Lugosi
(Howard Florey Institute, University of Melbourne, Parkville, Vic. 3052 Australia,
E-mail: george@hfivax.hfi.unimelb.edu.au or x9902975@ucsvc.ucs.unimelb.edu.au)
which will be available for everybody who joins the "Club" and has an access to use E-mail. The E-mail journal will be updated continuously and appear on your screen whenever you contact the E-mail address which will be identified soon.

Sponsorship for the Society:
The Symmetry Foundation makes it easier to make potential sponsors interested to contribute to the activity of the Society. There are several forms of sponsorship: one can contribute to the general fund of the foundation, one can sponsor certain events or projects and one can advertise in the journal *Symmetry: Culture and Science*, at our symposia, conferences, exhibitions, summer courses (inside and outside) or in special TV broadcasts shot at ISIS-Symmetry meetings, or popularizing symmetry studies and the activity of the Society, etc.
We are submitting applications for different announced programs. They are mainly regional programs, therefore all regional chairpersons are kindly asked to identify local and/or regional resources and to submit applications on behalf of ISIS-Symmetry, *notifying* of course at the same time *the officials of the Society*.
Examples in the European region are the TEMPUS program (our 1992 application was rejected, we are submitting a new application for 1993), a NATO Advanced Study Institute program application for the 1993 *floating university* on the Danube (see above), an application to the European Communities for a Pan-European Symmetry Network, and another application to the EC to sponsor three years of a Postgraduate Summer School (training seminar). Local (national) foundations like the National Science Foundation in the U.S., the Volkswagen Foundation in Germany, the Gulbenkian Foundation in Portugal, the Hungarian National Research Fund, and so on, have to be aimed at too, of course by the local chairpersons of the Society, and emphasizing the local activity of the Society, but facing higher success with an international intellectual background provided by ISIS-Symmetry behind you. You may act on behalf of the Society, when submitting applications, and use your position in the Board to increase your chances. You may turn both to local and international foundations, private companies, banks, and individuals. In this way we could repeat our summer (winter) schools at different continents, regions and could support the participation of local members at meetings in distant regions, etc.

Symmetry: Culture & Science

The Quarterly of the International Society for the Interdisciplinary Study of Symmetry (ISIS-Symmetry)

Editors:
György Darvas and Dénes Nagy

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The section *SFS: Symmetric Forum of the Society* has an E-Journal Supplement.

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Student Members, US$63.00 or US$15.00, respectively;
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EDITO-SYMMET-RIAL

FROM TAXONOMY TO INTERDISCIPLINARY SYMPOSIUM

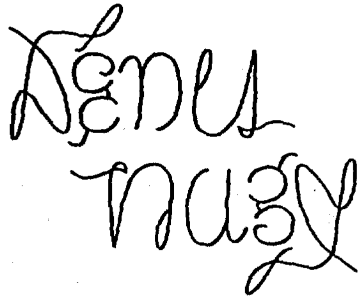
Let us start with "symmetric" news: this quarterly will be indexed and reviewed by the monthly periodical *Mathematical Reviews*, which is available practically everywhere in the world in the major university libraries. The *Mathematical Reviews* considers not only strictly mathematical papers, but also has regular sections about many related fields. For example the section "History and bibliography" deals with papers associated with the historic and cultural dimensions of the subject. Occasionally papers on mathematics and art are also feature there. Many papers about symmetry in crystallography are reviewed in another section, "Structure of matter", while works about biological symmetries, including phyllotaxis, are usually available in the section "Biology and behavioral sciences". Papers on symmetry can be found, of course, in various sections about specialized fields of mathematics, physics, and computer science. The list of reviewers include many pioneers of the interdisciplinary approach to symmetry, from the geometer H. S. M. Coxeter to the physicist Joe Rosen.

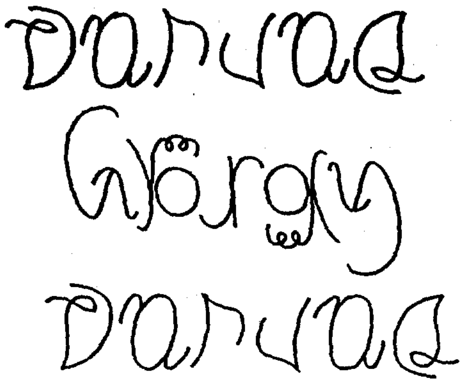
Although the *Mathematical Reviews* is very useful source of information about symmetry, it cannot consider those works which are published in periodicals, proceedings, and collections with foci far from mathematics. This fact gave us the idea that, beyond cooperating with review journals, we should start to work out our own classification system and information network. The last issue announced the "Bibliographic Project of ISIS-Symmetry", while this issue launches a new "Taxonomy Project" proposed by Ted Goranson. The section *Symmetro-graphy* will regularly review the symmetry-related literature (instead of giving just the list of publications of members as it did in the first issue). All interested members and non-members are encouraged to send their symmetry-related publications (reprints, photocopies, etc.) to the section editor.

Apropos, the rather awkward name Budapest Office of the Society — which clearly misses the kiss of the muses — will disappear and a new institute will take over its responsibilities from 1991 with a much broader profile. We had very many administrative difficulties starting the series of symposia and this quarterly, but we hope that a major part of this "asymmetric" paperwork is solved with an "invariant" result. Consequently, the new institute should focus — beyond coordinating the programs of the society and helping with fundraising — strongly on scholarly questions, educational programs, and research projects. To find a good name for this new institute was not easy, but luckily goddess Isis, using her Hellenistic connections, sent us some muses from the Mouseion. They understood that we are interested in symmetry in various fields of art, science, and technology, as well as in organizing related symposia and exhibitions, and suggested the name Symmetrion. Of course the institutionalization (symmetrionization) does not make any change in the policy of informality; indeed, this more poetical name should reflect this goal. We will inform our readers regularly about the "symmetrionical" developments.

Last, but not least, let us turn to a very concrete point. The main goal of ISIS-Symmetry is to help the personal interaction of interested scholars and artists, educators and students. The printed quarterly and the electronic journal supplement of ISIS-Symmetry — together with the related correspondence — are very useful in maintaining the connection at some level, but cannot substitute for real discussions. The *hot* atmosphere in a lecture room, the strong debate at a round table, or a quiet chat at a remote corner of the conference building (all of them are frequent visiting places for the muses!) may give inspirations and ideas much more quick than the readings. The central forum of the Society is the triennial symposium. This issue includes — supplementing the preliminary announcement in the previous issue — the *Call for papers, workshop topics, and exhibition items* for the Second Interdisciplinary Symmetry Symposium and Exhibition in 1992.

Let there be (dis)symmetry!





Symmetry: Culture and Science
Vol. 1, No. 2, 1990

SPECIAL ISSUE

SYMMETRY IN A KALEIDOSCOPE, 2

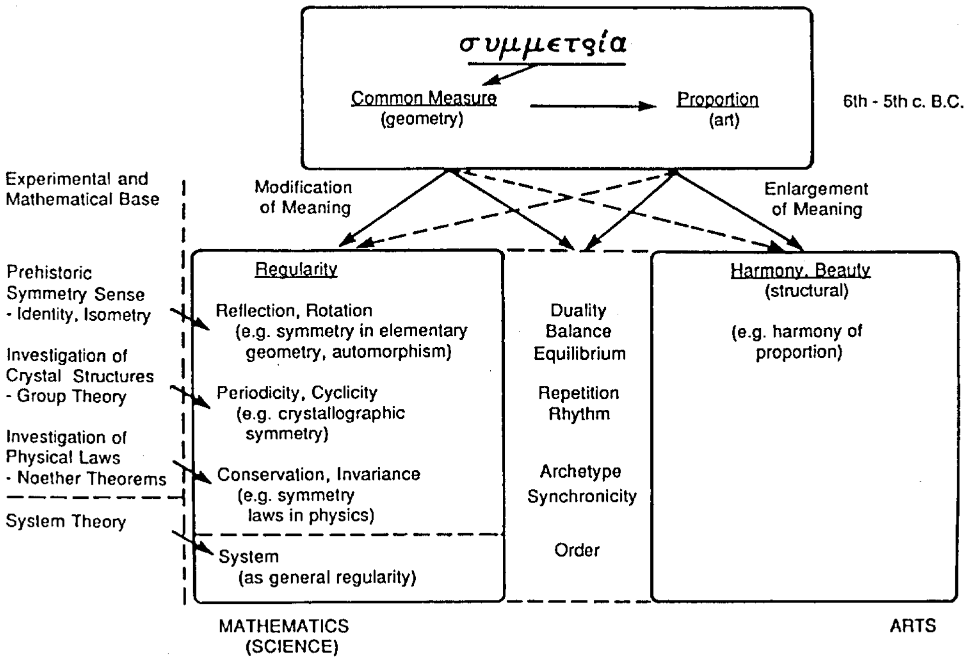
Edited by
György Darvas and Dénes Nagy

The collection presents the "kaleidoscopic" views of many scholars and artists on symmetry, and introduces the contributors in a sort of *Who's Who* in the field. Thus this "kaleidoscope" can be turned symmetrically in both directions: counter-clockwise to see some foci of the earlier symposium *Symmetry of Structure* (Budapest, August 1989) and clockwise to see the perspectives of future cooperations...

The editors would like to thank the contri-symmetro-butors for their cooperation and patience.

THREE QUESTIONS: THREE AXES OF SYMMETRY

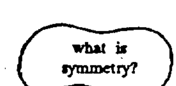
The concept of symmetry, having its roots in both ancient geometry and aesthetics, has become a central concept in modern science and art. The meaning of symmetry is broad enough to serve as a "pillar" of some "bridges" between the two sides of our split culture.



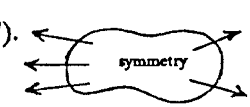
The roots and the modern meanings of symmetry
(after Dénes Nagy, 1988)

Aiming to help the discussion between various fields of art, sciences, and technology, we asked some representatives to answer the same three questions in short essays.

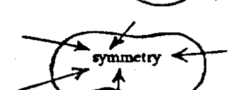
1. The methodological and heuristic role played by symmetry (and asymmetry) in your field of study and cultural circle.



2. The interdisciplinary impact of symmetry -- as used in your field -- on other scientific and/or cultural spheres ("output").



3. The special meaning of symmetry influenced by your cultural background (external effects, "input").



These three axes of symmetry direct all the contributions, except the first paper.

Symmetry: Culture and Science
Vol. 1, No. 2, 1990, 119-138

SYMMETROSPECTIVE: A HISTORIC VIEW

Introductory essay to the Special Issue

THE KALEIDOSCOPE AND SYMMETRY (OR, A SYMMETROSCOPE)

PART 2: FROM SCIENCE TO ART (20TH CENTURY)

Dénes Nagy

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(on leave from Eötvös Loránd University, Budapest, Hungary)

In the first part of this paper we have seen that the kaleidoscope, contrary to the original intention of its discoverer, Sir David Brewster, had little success in the arts. More sophisticated devices gained, however, some importance in the mathematical and crystallographic works of Möbius and Fedorov, respectively. In this Part 2 we continue the history of the kaleidoscope in the 20th century which leads, in same sense, back to artistic problems. After surveying various new kinds of kaleidoscopes we provide a "periodic table" of all the possible devices formed by plane mirrors.

A NEW TURN OF THE KALEIDOSCOPE (MORE SYMMETRICALLY...)

After the "golden age" of geometric crystallography in the late 19th century, and the parallel kaleidoscope-related activities of E.S. Fedorov, as well as of Edmund Hess and others, this optical device disappeared from the scholarly interest. It did return, albeit at first metaphorically, in the 1930s. The book *Mathematical Kaleidoscope* by Hugo Steinhaus, a leading Polish mathematician, offered not only "colorful pictures" on the topic, but many interesting recreational questions and serious problems in connection with symmetrical figures, including regular tilings, models of polyhedra, shapes of crystals, close packings of equal balls, and so on. The original Polish edition *Kalejdoskop matematyczny* (Lwów, 1938) was followed by an English version in the same year, and later by a large number of other translations and enlarged editions. Interestingly, this book, which deals with various devices useful in mathematics, does not consider the kaleidoscope itself, even the very word disappeared from the English title: *Mathematical Snapshots* (New York, 1938, 1950, etc.). Many other translations, however, preserved the "kaleidoscope" in the title, including those in Russian, German, and Hungarian.

It was not necessary, however, to wait too long for the application of the kaleidoscope in forefront of mathematical literature. Notably, related topics appeared in two other pearls of the field, in W.W. Rouse Ball's book *Mathematical Recreations and Essays* (London, 1939), more precisely in a new chapter of its revised edition written by H.S.M. Coxeter (1939), and in Richard Courant and Herbert Robbins's book *What is Mathematics?* (London, 1941). In the first work, Coxeter — giving credit to Hess (1889) and also referring to Möbius — discusses those kaleidoscopes which we called cylindrical and trihedral. Moreover he also remarks on the possibility of further devices: three types of tetrahedra as well as certain triangular and rectangular prisms with internal reflecting surfaces. These kaleidoscopes, shaped by four, five, or six mirrors, respectively, lead to 3-dimensional solid tessellations. Later we will return to this topic. The book by Courant and Robbins (1941), which is not a "kaleidoscopic" work, but a systematic introduction to mathematics for both students and interested laymen, includes a short chapter on repeated reflections in cases very similar to the arrangement in the cylindrical kaleidoscopes, without referring to these instruments by name. They use a very simple, purely geometric approach focusing on rectangles and those kinds of triangles which are able to generate just by reflections non-overlapping coverings of the plane (we may call them mirror tilings or mirror tessellations). It would be interesting to go further with this approach and to show that there are only those four possibilities which are illustrated in the book (cf., the four types of cylindrical kaleidoscopes in Part 1 of this article, p. 34), but Courant and Robbins move to another interesting topic. They consider two circular mirrors, or two reflecting columns based on circles, illustrating the repeated inversion of circles (the inversion is a kind of angle-preserving transformation; here it is not associated with central symmetry which is the conventional meaning of the expression in geometric crystallography). We will deal, however, with simpler cases: in our discussion, except when indicated otherwise, the kaleidoscopes are formed by plane mirrors.

The real return of the kaleidoscope to geometric questions and its systematic application in this field came with the monographs by H.S.M. Coxeter (1948; 1961). Indeed, the Canadian mathematician not only revitalized synthetic geometry as a university subject, but also gave a new meaning to Brewster's old optical instrument. He analyzes the dihedral reflection groups using the kaleidoscope; moreover his results inspired an educational film *Dihedral Kaleidoscopes* (Coxeter, 1965). These devices are useful to present regular polygons and star-polygons by fixing appropriate line-segments in the object box. The possibility of kaleidoscopic representation of some regular arrangements in the plane, specifically rosette-, frieze-, and wallpaper-like patterns (symmetry groups), was also remarked by another geometer, the Hungarian László Fejes Tóth (1964). Considering analogous questions in 3-dimensional space, Coxeter discusses, independently of Fedorov, but referring to Möbius, the trihedral kaleidoscopes which can represent some regular polyhedra, or, in other words, Platonic solids. We have seen in Part 1 of this article (pp. 35-36) how to make such a kaleidoscope by three mirrors forming a "reflecting corner", and we considered all the possible types: there are infinitely many variable

(Part 1, Fig. 1) and three further types of trihedral kaleidoscopes. While the variable types are easy to imagine by standing a hinged dihedral (two-mirror) kaleidoscope on a third horizontal mirror, the form of the additional three types are more complicated. Coxeter (1961) describes an interesting device which shows the connections between these three types. He only uses rectangular mirrors, by means of which he is able to form just a part of the trihedral surface, not the entire corner, but still one can see the kaleidoscopic reflections if the position of the object is carefully chosen. The advantage of this partial kaleidoscope, whose surface has gaps at the corner and sides, is its variability: by rotating one of the mirrors we are able to simulate all of the above-mentioned three types of trihedral kaleidoscopes. Probably the easiest way to imagine this variable device, slightly modifying the original description, is to start with a cylindrical kaleidoscope of (90°, 60°, 30°), standing upright on a table, then to turn that face down — rotating around the edge on the table — which lies opposite to the angle of 60° (Fig. 3).

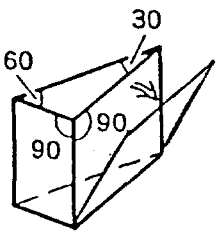

Figure 3: The partial (or "gappy") kaleidoscope: a device for simulating the reflecting surfaces of three trihedral kaleidoscopes. The idea of starting with the indicated cylindrical kaleidoscope is ours, while the main trick of rotating a mirror-face is due to Coxeter (1961, p. 280). Originally, he refers to a dihedral kaleidoscope with an angle of 60°, standing upright on a table, while the third mirror is in an appropriate horizontal position. The latter should be turned up, inversely to the direction indicated here. The present version of the device has a minor advantage: it makes clear the connection with the referred to cylindrical kaleidoscope.

The rotated face during the opening-out process reaches all those three positions where the three mirrors together, disregarding the gaps between them, represent the required trihedral kaleidoscopes. In those cases the device, with a suitable object, presents "spatial rosettes", namely icosahedral, octahedral and tetrahedral patterns, without ambiguity. Note that these three trihedral kaleidoscopes, considering in each case the entire surface without gaps, are not flexible: these three cannot be transformed into each other with simple rotations.

Later Coxeter went on to develop the most exciting type of the trihedral kaleidoscopes as a useful educational tool for demonstrating a set of polyhedra. It is the first one reached when opening out the cylindrical kaleidoscope (Fig. 3), and the last one in our earlier list (Part 1, p. 35). An interesting property of this type is that the sum of the three angles of the faces at the corner (the rounded data are 37°, 32°, 21°) gives a right angle. This means that the surface of the instrument can be obtained from a square by simple paper-folding, with the three inner faces lined with a reflective tape, or replaced by mirrors. In a version of this instrument we may make a small hole at the corner (or vertex) of the three mirrors. In this case, the device can be turned in the direction of the light as with the classical kaleidoscope. Coxeter does not use liquid as an object in the kaleidoscope, which was Fedorov's suggestion, but his method of construction shows some special line-segments on the original square. Marking these lines, by inserting thin strips at the corresponding place on the mirrors, we may see the image of an icosahedron and a dodecahedron, as well as some star-polyhedra (Fig. 4). The typical activity with this kaleidoscope is not the simple turning of the device, as in the case of Brewster's original one, but the fixing of the appropriate line-segments. Coxeter's foldable kaleidoscope really requires some geometrical creativity from the user, and it may contribute to a better understanding of the theory of polyhedra. A very accurate version of this trihedral kaleidoscope was completed during the above-mentioned educational film project in the mid 1960s. Unfortunately the second film about this advanced kaleidoscope was never completed, but the mathematical discussion of this instrument is available in a later monograph by Coxeter (1974). Incidentally, Coxeter speaks about icosahedral or polyhedral kaleidoscopes, emphasizing the presented image, not the actual trihedral surface of the instrument. We prefer to use the term trihedral kaleidoscope; the advantage of this will become clear later. Another useful name was coined by the British crystallographer Alan Mackay (1967) who speaks about "corner kaleidoscopes". He uses these devices not only for demonstrating various polyhedra, but also for presenting models for polyhedral viruses. Indeed, inserting spheres we may present icosahedral packings and other related structures.

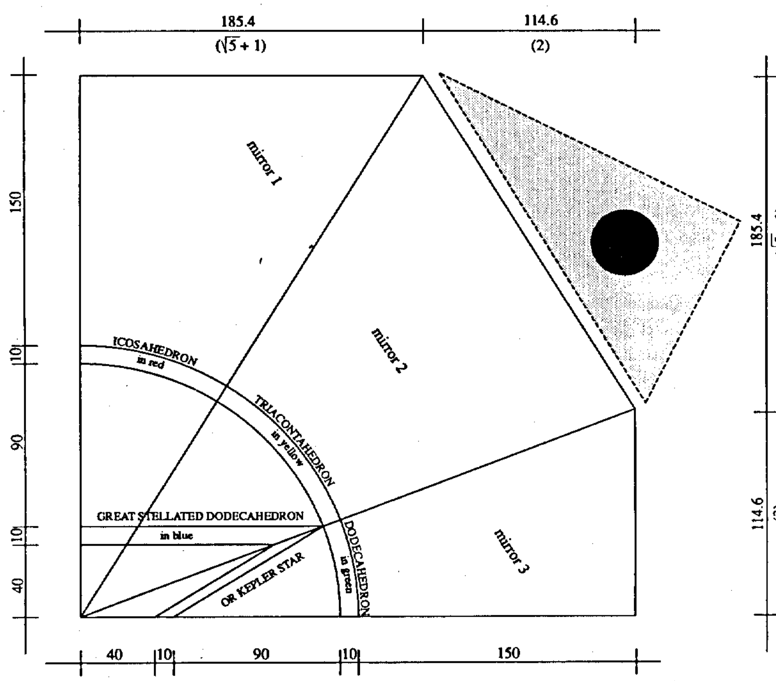

Figure 4: The foldable kaleidoscope of Coxeter was also rediscovered by Nikolaus and Caspar Schwabe. It is given here in their layout: starting with a given square one may easily form the surface of the corresponding kaleidoscope by simple folding, while the fourth triangle (top right) should be cut and reoriented to close the trihedral surface with the exception of an eyehole for viewing (black circle). The Schwabes mark, except in one case, arcs — instead of line-segments — by which they may present the images of spherical models of the identified polyhedra. Moreover the line-segments and arcs have different colors, thus one may see the colored images together. Originally all the data are in millimeters. Note that both the top and the right sides of the square are divided, as the data in parentheses show, according to the ratio of the golden section. Caspar Schwabe calls it the "golden section kaleidoscope" (copyright, 1986). In the case of the earlier version of Coxeter (1974, p. 24) the smallest mirror, which is marked here as No. 3, is in the middle, and thus his layout does not make possible to utilize the fourth corner for closing the trihedral surface. Coxeter describes the possibility of shaping the image of some further polyhedra by inserting appropriate line-segments: the small stellated dodecahedron, the great dodecahedron, and the icosidodecahedron. (The layout given here is made following a sketch by Caspar Schwabe; the computer drawing is courtesy of Mihály Szoboszlai.)

While Coxeter started in the Brewsterian tradition, A.V. Shubnikov and V.A. Koptsik (1972) continued the Fedorovian one. Although the new interest of Russian crystallographers in kaleidoscopes was indicated much earlier when Shubnikov (1939) published a paper on the subject and also discussed this topic in a comprehensive monograph on symmetry (1940), the international recognition of these results had to wait until the publishing of the English version of the cited book *Symmetry in Science and Art*, written jointly with Koptsik. The authors discuss kaleidoscopes not in a separate chapter, but ever more advanced versions appear in parallel with the generalization of the concept of symmetry. This "invariant" presence of kaleidoscopes in the corresponding chapters demonstrates clearly that the instrument is not only a toy, but also a useful tool in education. The original Russian edition of 1972 (p. 32), not Koptsik's revised English version, makes the same mistake about the classical dihedral kaleidoscopes that was in the center of the Kircher-Brewster controversy (the allowed angles are 360°/*n*, instead of restricting them to 180°/*n*). This fact supports the view that the two traditions were really independent. Shubnikov and Koptsik survey, however, not only the classical kaleidoscopes, but they go far beyond.

Discussing the symmetries of friezes, linear ornaments on one side of a band, Shubnikov and Koptsik describe those two kaleidoscopes which present such patterns (cf., also Fejes Tóth, 1964, p. 41): the first one has two parallel mirrors facing each other (a special case of the dihedral kaleidoscope; see Part 1 of this article, p. 34); while the second one has three mirrors where the former arrangement of two parallel mirrors is extended by a third perpendicular one giving a U-shape (all the mirrors facing inwards of the U). We should add that the latter arrangement can be interpreted as a special case of the infinite class of trihedral kaleidoscopes (see Fig. 1 in Part 1 of this article). Namely, if n is infinitely large, then the angle of 180°/n between the corresponding two mirrors is 0°, so that they are parallel. The common property of both special cases, the dihedral one discussed earlier and the new trihedral one, is that they present frieze patterns. We may also obtain these two special kaleidoscopes by removing either two opposite mirrors or just one, respectively, from a cylindrical kaleidoscope of rectangular cross-section.

It has been mentioned that the trihedral kaleidoscopes, Shubnikov and Koptsik call them Fedorov kaleidoscopes, are able to represent local arrangements of crystallographic structures, namely, symmetrical systems of points around a singular lattice-point (point groups). A further step in the book is the discussion of those kaleidoscopes that are able to represent global crystallographic structures, specifically 3-dimensional periodic systems (space groups), more precisely a part of them (cf., Coxeter, 1939, pp. 159-160). These instruments are based on the internal surface of various polyhedra, such as a rectangular prism, for example. The function of this instrument can be easily understood by the analogy of a mirror hall, where we, as an object, are standing in a closed space surrounded by mirrors (suppose that the floor and the ceiling are also covered by mirrors). We need a mirror hall where we may occupy any position inside without causing an ambiguity of images. Although mirror halls were popular in earlier periods, in the most cases the spectators could enjoy the phenomenon only from a few marked spots (cf., Greenslade, 1982; Walker, 1986). Now we require total freedom in selecting our viewing position. In the case of a mirror hall formed as a rectangular prism we have around us four mirrored walls as in the cylindrical kaleidoscope of rectangular cross-section. This part of the mirrors presents a periodic pattern of our images as standing on an extended floor. Two additional mirrors at the top and the bottom multiply the horizontal layer of our images to further layers in both directions up and down. Finally, we see a finite part of an infinite 3-dimensional pattern.

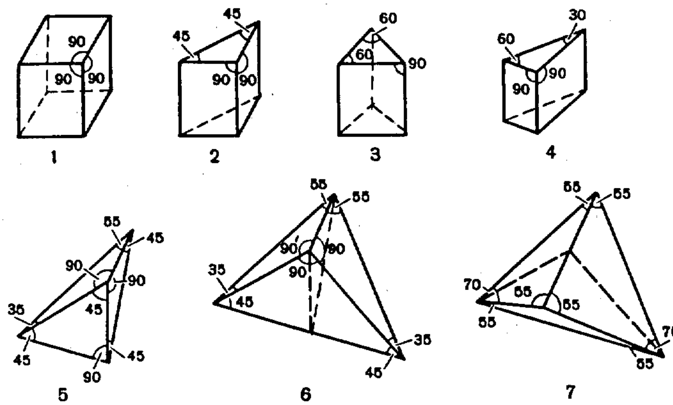

Figure 5: Shubnikov and Koptsik's polyhedral kaleidoscopes. The first four with prismatic shapes can be obtained from the four types of cylindrical kaleidoscopes by closing them at both open ends (adding a reflecting top and a similar bottom). The other three have tetrahedral shapes. The first tetrahedron (type 5) can be derived by cutting a cube into 48 congruent parts along all the symmetry planes; the second one (type 6) can be constructed by joining the earlier tetrahedron and its mirror image as the dashed lines show; finally the last one (type 7) can be obtained in a similar way by joining two tetrahedra of type 6. Each of these seven polyhedral kaleidoscopes should have a dihedral kaleidoscope at each of the edges, and a trihedral one at each of the vertices, always with the allowed angles which were discussed earlier in Part 1 of this article, pp. 33 and 35. Note that 35°, 55°, and 70° are rounded. (After Shubnikov, 1940; Shubnikov and Koptsik, 1972, Fig. 188, with a minor modification.)

Shubnikov and Koptsik find that there are altogether seven kaleidoscopes for presenting spatial patterns (Fig. 5). The proof of the completeness of their list is not given, but all of the types are described in detail. We suggest naming them, according to the arrangement of the mirrors, polyhedral kaleidoscopes (not in Coxeter's sense, but consistent with the terminology of dihedral, cylindrical, and trihedral kaleidoscopes). The polyhedral kaleidoscopes are surprising from many points of view. The classical literature on the subject often emphasized that it is impossible to make a kaleidoscope with more than four mirrors. In the case of this new kind of kaleidoscopes the cited statement is not true: one type has six mirrors (rectangular prisms), three other types have five mirrors each (triangular prisms). We are facing, however, new technical problems: the mirrors form a closed polyhedron, therefore it is not obvious how to see the images from outside, and how to illuminate the object (supposing that it is not a mirror hall, but a small polyhedral kaleidoscope). The first problem can be solved easily by drilling a small eyehole on the polyhedral surface. The second one is more complicated, but Shubnikov and Koptsik have an interesting suggestion. By using an electric bulb or a flashlight as an object inside the kaleidoscope, the problem of illumination will be also solved. Of course this method strongly limits the image; we will always obtain just a set of illuminated points. We have, however, another suggestion: to make small holes at the vertices, which will not disturb the reflections too much, and to illuminate via these holes an arbitrary object placed inside the kaleidoscope. We may use one of the holes at the corners as an eyehole. A further possibility is to form the surface of the kaleidoscope from one-way mirrored glass, like that one used in some sunglasses, which makes the illumination easy. It would be interesting to combine the polyhedral kaleidoscopes with various light effects, including laser illumination, or even making holographic objects inside. We may also project objects from outside into the kaleidoscope by classical optical methods.

Sophisticated systems of mirrors are also crucial for the recent attempt at building optical computers operated by laser light. The polyhedral kaleidoscopes — it is just a fuzzy idea — could be useful in constructing optical memory, that is, to keep some visual information inside the closed space of the device, then to retrieve it by opening a shutter-like hole. It also gives a simple tool to demonstrate the "swallowing up" of light by black holes, now a favorite topic in astrophysics and cosmology, where the light is not lost, but kept inside. Devices with curved mirrors may demonstrate curved spaces and their symmetry operations which are important in the theory of relativity and astrophysics.

To make our list of kaleidoscopes formed by plane mirrors complete, we should add some additional kinds that are not mentioned by Shubnikov and Koptsik. Consider those that are between the kaleidoscopes providing 2-dimensional and 3-dimensional patterns (assuming that we have a flat enough object, thus the dimensionality of the observed pattern is determined by the reflections and not by the shape of the original object). In some sense, we are moving towards a two-and-half-dimensional case. Adding a reflecting bottom to each of the four cylindrical kaleidoscopes (it is the same as removing the top of each of the four prismatic kaleidoscopes in Fig. 5) we obtain kaleidoscopes in the shape of an open box. Its cylindrical part presents a wallpaper pattern, while the bottom will double the whole image. Similarly we may add to the U-shaped kaleidoscope a perpendicular mirror at the back (a box without the front and top), where the new mirror doubles the frieze. It is not necessary to deal with the doubling of rosettes, because the variable trihedral kaleidoscopes (see Fig. 1 in Part 1) do precisely that. We may add, however, a horizontal top to these types which would ultimately give us a hinged dihedral device between two parallel mirrors (trough-like shape). Each of the dihedral kaleidoscopes presents a rosette, while the two parallel mirrors, as in the case of frieze patterns discussed earlier, line up copies of this rosette, giving finally a pattern similar to pearls on a stretched string or carvings on a rod. It is easy to see that we have obtained by this derivation all the possible types of kaleidoscopes of the corresponding kinds. The patterns discussed here (two-sided rosettes, two-sided friezes, two-sided wallpapers, and rods) are also considered in mathematical crystallography. For example, the two-sided wallpaper patterns are useful when studying various layers, including those of liquid crystals, thin films, membranes, and biological tissues. It is interesting to note that the terms frieze and wallpaper refer, similarly to the everyday meaning, to such ornamentations that are available only on one side of a surface, while the back is fixed to the wall. Now we have spatial generalizations of those planar patterns where the ornamentation is visible from both sides as, for example, in a fence or stained glass. Among textiles we find examples of both one-sided and two-sided designs.

And what about the presentation of colored symmetries, where the equivalent elements change their colors periodically as in the case of a chessboard? Of course the mirrors of a kaleidoscope cannot alternate the colors of the images. Although we may start with a colored object — the commercial dihedral kaleidoscope uses pieces of colored glass or beads — we obtain just symmetric patterns with colors, not colored symmetries. In other words, the repeated images, if we disregard the reduced brightness, have exactly the same color composition, and there is no periodic change of colors. The coloring of some mirrors, however, may help a bit in the cases of those devices which represent two-sided rosettes, friezes, and wallpapers. The two-sided patterns are analogous to their one-sided, but two-colored versions. This statement is not obvious, but one can imagine that the new opportunity of reversing the pattern to the other side is equivalent to using a second color on the same side. More precisely, we should accept a generalized understanding of the two-colored patterns to make this statement valid. The 17 types of one-sided wallpaper patterns (groups), discussed very briefly in Part 1 of this paper (pp. 34-35), can be generalized to 80 types of two-sided wallpaper patterns (this set also includes the 17 one-sided cases as those where we do not make use of the bottom side). These 80 types were first enumerated by Carl Hermann in 1929 and, slightly later in the same year, by Leonhard Weber. The latter author illustrated all of them on one side of the plane by projecting the patterns from the bottom side to the top, and marking them black. Specifically, Weber's one-sided patterns consist equal black and white triangles that represent, respectively, the original arrangements on the bottom and the top of the two-sided plane. This step clearly belongs to the preliminaries of the theory of two-colored (black-and-white) patterns, which are considered on one side of the plane. Moreover, it could have inspired Heinrich Heesch, who first proposed this approach (later Shubnikov, independently from Heesch, also introduced the concept of black-and-white symmetries). Indeed, the 80 patterns of Weber consist of 46 truly two-colored types of wallpaper patterns, where there are separate black and white triangles; 17 gray types, where the bottom and the top patterns were originally mirror images, therefore the projection led to coinciding black and white triangles; and the 17 one-colored types, where the bottom side was not utilized, therefore there is no second color. The latter two sets of 17 gray types and 17 one-colored types are geometrically the same. The doubling of this set as gray and one-colored types has, however, an appropriate interpretation: if the black and white colors represent two different physical properties (e.g., negative and positive magnetism), the gray types are neutral, while the one-colored types are associated with one of the properties. Thus, the crystallographers usually speak about 80 types of black-and-white wallpaper patterns, including the degenerate gray and one-colored cases (46 + 17 + 17 = 80). These 80 types really correspond to the 80 types of two-sided, but one-colored wallpaper patterns. We saw earlier in Part 1 (p. 34) that exactly four of the 17 one-sided wallpaper patterns can be represented by kaleidoscopes, specifically by cylindrical devices. If we extend a cylindrical kaleidoscope by a colored bottom mirror, we see both the original wallpaper pattern and its colored mirror image. Even the "projecting" of the colored figures to the other side can be simulated by standing a flat object on the colored mirror (suppose that it has no thick glass part, that is, the object is lying directly on the reflecting surface). In this case, however, the original object and its mirror image would coincide, therefore we cannot have truly two-colored patterns, only gray ones. Indeed, using a flat transparent object and a gray colored mirror, we may turn the object gray (as the result of the mixture of the colors of the transparent object and of the coinciding gray image), and all the repeated images would have the same mixed gray color. In this way we are able to represent four types of the gray wallpaper patterns: the gray versions of the previously discussed four one-colored patterns. This means that we may present altogether eight degenerate types of the total 80 types of two-colored (black-and-white) wallpaper patterns: four gray and four one-colored types. Similarly, we may represent four degenerate types — two gray and two one-colored — of the total 31 types of two-colored frieze patterns (17 truly two-colored, 7 gray, and 7 one-colored). Interestingly, when we extend the kaleidoscope made of two parallel mirrors by a third one to form a U-shaped arrangement we have two choices: to use a usual uncolored mirror to present another type of the 7 one-colored frieze patterns, or to use a colored mirror to present the gray version of the original frieze pattern. In the case of two-colored rosettes, again we may represent only degenerate cases, moreover all of them: the infinitely many gray ones by the variable class of trihedral kaleidoscopes, discussed in Part 1 (Fig. 1), with a colored horizontal mirror, and the infinitely many one-colored ones using simple dihedral kaleidoscopes. Note that all of the represented cases are almost trivial, thus the kaleidoscopes could not help us to go deeper into the theory of colored symmetries. Probably the idea of coloring the mirrors, as well as of using various illuminations, may have more importance for aesthetical experiments than for the study of crystallographic symmetries. There is, however, a rather artificial possibility for representing real colored symmetries: we could start with a colored symmetric (i.e., symmetrically colored) object and fit it into a kaleidoscope, if it is possible, in such a way that produces the periodic repetition of colors. For example, if we cut the usual black-and-white chessboard into four triangular parts along the diagonals, we may reconstruct the whole board again using just one quarter and a dihedral kaleidoscope with the angle of 90°. The similar diagonal cuts are always suitable for similar purposes if the chessboard has $n \times n$ squares, where n is an arbitrary integer, greater than 1. Moreover, in the case of odd numbers we may cut the chessboard into four squares along the median lines (this does not work, however, for even numbers). Strictly speaking, the kaleidoscope does not make colored symmetric patterns, only continues an initial one. Of course, not all patterns are suitable for similar kaleidoscopic continuation even if we select special positions for them. This remark may raise further questions about the classification of colored symmetric patterns.

Of course the group theoretic equivalent of all the kaleidoscopes, including the listed new ones, are well known in mathematics, not as optical devices, but as symmetry groups generated just by reflections. Coxeter (1948) gave a survey of the group theoretic aspects of the topic, including the n-dimensional analogues ("generalized kaleidoscopes"), when dealing with regular polyhedra and politopes. The pioneers of the geometrical theory of groups include Dyck, Klein, Poincaré and Schwarz, who were very active in this field in the late 19th century. We should also mention the name of Edmund Hess (1879; 1882; 1883; 1889) who, contributing to the theory of spherical tilings and the related polyhedra, rediscovered Möbius's trihedral kaleidoscopes. He worked independently of Fedorov (1883; 1885; 1890; 1891), therefore we should give more credit to him in connection with the trihedral devices (Möbius-Hess-Fedorov kaleidoscopes). The concept of reflection gained new importance in the approach of Friedrich Bachmann (1959) who proposed to use this to build the foundations of geometry. Interestingly, referring to a group theoretic paper by Vinberg (1981), there are no crystallographic reflection groups, as well as no generalized polyhedral kaleidoscopes, in non-Euclidean hyperbolic (or Bolyai-Lobachevskii) spaces of large dimension (cf., Stewart, 1986). The theory of black-and-white and multicolored symmetry, pioneered by Heesch's paper in

1929, Shubnikov's monograph in 1952, and Belov's coauthored papers in the mid 1950s, became a dynamic field on the border of mathematics and crystallography which led to some individual monographs (Jawson and Rose, Loeb, Koptsik, MacGillavry, Wieting, Zamorzaev, Zamorzaev et al.) and many important papers (Coxeter, Donnay, Schattschneider, Senechal, Schwarzenberger, van der Waerden and Burckhardt, and others). Interestingly, but perhaps not surprisingly, the topic of colored symmetry was also considered in the arts, not only on the level of intuitive construction of related patterns, but also in the systematic classification of them. Specifically, the Dutch graphic artist M.C. Escher and the British textile engineer H.J. Woods, who published a series of papers in the mid 1930s, significantly contributed to the field. The graphics by Escher became widely used for illustrating colored symmetries in the mathematical-crystallographic literature, while the approach of Woods was less known — with the rare exception of the monograph by Shubnikov (1940) — until the rediscovery by the crystallographer Donnay, and more recently by the mathematician Crowe and the anthropologist Washburn.

THE MYTH OF THE KALEIDOSCOPE QUICKENS

There were many mathematical congresses in the early 1980s where the participants twisted Rubik cubes during intermissions, but there were also geometrical conferences in the 1970s where the mathematicians handed over to one another Coxeter kaleidoscopes. Moreover, the interest in kaleidoscopes also reached the circles of artists and designers, making the myth of such applications real. A group of students around Keith Critchlow, who authored artistic-geometrical monographs on polyhedra and patterns, also became interested in kaleidoscopes, focussing on regular polyhedra (these kaleidoscopes were available in the Rudolf Steiner Bookstore, behind the British Museum in London). Later an architect designed a more advanced version of such devices for presenting icosahedral structures. It is marketed as *KaleidoSphere* by a company of the same name (Wilkes-Barre, Pennsylvania). After examining the polyhedral kaleidoscopes for presenting 3-dimensional symmetries, our fired-up imagination may suggest further steps. It would be interesting to connect internal mirroring surfaces of cubes, following the well-known models of the hypercube, to visualize 4-dimensional symmetry groups, or at least to present associated images. Note that the topic of 4-dimensional polyhedra gained some importance in various movements of abstract art. One of the leaders of the cubist movement, Marcel Duchamp, even made experiments with mirrors in connection with his interest in 4-dimensional space. Moreover he mentions in his notes a "3-sided mirror" (cf., Henderson, 1983, p. 154).

The advanced kaleidoscopes can probably be applied in visual psychology, where both the use of mirrors and of group theory have some tradition. Thus, mirror tracing — following a star-polygon with a pencil when the figure cannot be seen directly, but only in a mirror — is a well-known experiment in psychology from the 1910s. The concept of groups was introduced into the theory of perception relatively early by Ernst Cassirer's paper of 1938, while the school of Jean Piaget adapted it as a reference for analyzing the structure of cognition in developmental psychology. In the 1960s William Hoffman suggested describing various perceptual invariances by Lie transformation groups. Kaleidoscopes, connecting visual effects and symmetry groups, may help the studies in these directions, or at least may provide an additional domain in experimental psychology. For example, it is interesting to consider the orientation of individuals not only in a maze, but also in mirror halls or their smaller kaleidoscopic versions. The left-right orientation can also be investigated on a broad scale. Moreover, by applying polyhedral kaleidoscopes we may leave the "flatland" — using the title of Edwin Abbott's book — of everyday life, where we can rarely see spatial structures. (Even in the case of skyscrapers the outside view of the walls hides the structure of the building; while being inside we either recognize the "flatlands" in various levels, or move between them just seeing the elevator shaft.) The observations in the "mirror world" may also have some didactic advantage. Thus the use of mirrors for illustrating left and right pairs of particles, molecules, and biological shapes became customary in popular literature. Lewis Carroll's novel *Through the Looking-Glass and What Alice Found There* in 1872 gave not only an artistic interpretation of a mirror world, but the mathematician-author anticipated some later developments in science (this work also contributed to the preservation of the expression "looking-glass", which was earlier a much more popular alternative to mirror than today). Robert Routledge (1891), in his survey on the discoveries and inventions of the 19th century, discusses the contrast between the mirrors of the kaleidoscope, producing harmlessly symmetrical images, and of the *polemoscope* (from the Greek *polemos*, war), a periscope like device serving the artillery to observe the effect of their shots. Note that systems of mirrors are often used to display objects or events from many points of view concurrently, as in the case of security systems of supermarkets, some operating theaters, and museums (see also the use of various optical devices including telescopes, periscopes, prisms, and more recently optical fibres in many fields). Similar methods also gained some importance in arts. For example, many photo artists have made experiments with kaleidoscopic views of everyday objects. Some video clips apply kaleidoscopic effects, which is obviously the cheapest way to have a *corps de ballet* by multiplying a single performing dancer and automatically "coordinating" the parallel or mirror motions of the "group". The most practical applications of systems of mirrors, or mirroring surfaces, however require the opposite "anti-kaleidoscopic" principle: the goal is not the multiplication of images, but the exclusion of the kaleidoscopic effects which could disturb the observation or orientation. Thus the mirrors in a dance studio serve the checking and correction of gestures in the 3-dimensional space individually, without any desire to multiply the dancer. Similarly, it is better if the surface of a revolving door are non-reflecting; otherwise we may find ourselves in a rotating kaleidoscope. The problem of non-reversing single mirrors, which can be solved using curved mirrors, goes back to Plato, but it has gained a new importance for contemporary inventors (cf., for example, David Emil Thomas's U.S. Patent No. 4,116,540 in 1978). The same problem can also be solved by a dihedral arrangement of plane mirrors, which may present the mirror image of the mirror image, that is the original view of an object.

We discussed earlier, very briefly, the use of circular mirrors, or reflecting columns, in the book by Courant and Robbins (1941). This topic received a new emphasis when the British mathematician Caroline Series (1990a; 1990b) connected this topic with non-Euclidean geometry and the theory of fractals and chaos. The repeated reflections in circular mirrors — Series speaks about non-Euclidean kaleidoscopes — produce smaller and smaller circles and we may present in this way various non-Euclidean patterns. (Indeed, this system is able to produce, by successive inversions of circles, congruences of non-Euclidean hyperbolic planes). Moreover, the ever smaller circles lead to some questions on the geometry of fractals. Note that Helmholtz was probably the first scholar who suggested that convex mirrors offer an image of non-Euclidean space, moreover this idea could also have inspired Poincaré to construct his models of non-Euclidean geometry (cf., Henderson, 1983, p. 154). It is interesting to add that reflecting circular cylinders are also used in anamorphic art (from the Greek *ana*, again, and *morphê*, shape), an idea going back to Leonardo and other artists in the Renaissance. The distorted and hardly recognizable picture can be seen correctly only by seeing its reflection in an appropriate cylinder stood perpendicularly (cf., Gardner, 1975; Leeman et al., 1976; Moscovich, 1988). Interestingly, this general idea, considered mostly as an artistic curiosity, received a serious technical application when the Twentieth Century Fox used anamorphic lenses to produce the wide screen motion picture *The Robe* (1953). The lenses projected the wide image onto the standard film, which was projected, again by anamorphic lenses, to a wide screen. In addition to this, the traditional form of anamorphic art with cylindrical mirrors also survived, and it was reintroduced to the galleries of modern painting. For example, a Hungarian graphic artist, István Orosz, uses this idea to present visual ambiguity: in his case a recognizable picture is the input, not a meaningless distorted figure, which is transformed by the cylindrical mirror into another picture. Turning back to special mirrors, we should mention the artistic-photographic experiments with various curved mirrors made by some members of the *Bauhaus* movement in the 1920s, especially by László Moholy-Nagy (cf., his monograph *Malerei, Fotographie, Film*, München, 1925, which is translated as *Painting, Photography, Film*, London, 1968). Mirroring effects were also used in the kinetic sculptures and light mobiles, pioneered by the same group of artists. Escher's lithographs *Still Life with Reflecting Sphere* (1934) and *Hand with Reflecting Sphere* (1935) demonstrate the interest of another artist in related questions. Curved mirrors are also used for entertainment in various fun houses and science exhibitions, where we may see our transformed image. The behavior of animals and small children in the front of a plane mirror, or a system of mirrors, is often studied in ethology and developmental psychology. (It is used in experiments testing the recognition of the self). The mirror even became a metaphor in the title of a book about the natural history of human knowledge, written by Konrad Lorenz, one of the pioneers of ethology (*Die Rückseite des Spiegels*, München, 1973; English translation, *Behind the Mirror*, London, 1978). The idea of "mirroring" is also well-known in art therapy, especially in those cases when the clients miss someone. This relatively simple method provided good results in the case of children who were separated from their mothers: the client is asked to draw an abstract image while the therapist simultaneously makes the mirror image shaping together a nonseparable symmetric picture. It is also interesting to observe that phenomena associated with mirrors often feature in children's adventure novels. For example, the frequent motif of a mysterious car in the night whose light suddenly appears and disappears opposite to the young heroes' car, and which is produced by a simple mirror at a curve, can be interpreted as an exercise for the recognition of the self in a more complicated situation. Mostly the heroes, after some exiting pages, solve this problem, realizing that they are frightened by the mirror image of their own car. Illusionists also use various mirror systems. The "ghost illusion", especially the method patented by Pepper and Dricks, was a very popular trick in late 19th century (Routledge, 1891). The more complex systems of mirrors and their kaleidoscopic effects in mirror halls are also impressive attractions in some scientific and educational exhibitions.

The kaleidoscope is also useful in explaining the composition principles of Bach or Schoenberg. Both of them desired to use not only the root position of a basic theme, but also its symmetric images: *rectus* (or crab), *inversus* (mirror), and *recto-inversus* (crab-mirror). All of these symmetric variations can be seen using a dihedral kaleidoscope with the angle of 90°. It should be stood perpendicularly on the scoresheet of the basic theme, preferably in such a way that one of the mirrors is parallel to the lines, while the other one is perpendicular to them. In the latter mirror we see the *rectus* of the theme, in the parallel one the *inversus*, finally, the fourth pattern in the virtual mirrors would give the *recto-inversus*. Of course, it is one thing to see the scores of a basic theme and its symmetric variations thus derived and another to actually listen to the music. While the described symmetry transformations are almost obvious by visual perception (symmetries in plane), the most people will not realize them using just acoustic perception (symmetries in time). The physicist-philosopher Ernst Mach delivered a popular scientific lecture in 1871 in which he discussed and performed the "mirror music" (this German paper is also available in English: "On symmetry", see in his book *Popular Scientific Lectures*, La Salle, Ill., 1893; reprint, 1986, 89-106). Similar mirrorings also appear in language games and experimental poetry. Such a verse even inspired the composer Webern. The history of palindromes (reversible words or sentences) goes back to ancient times; often these were associated with magical meanings. It is not surprising that mirrors are also useful in children's interdisciplinary music education, where the basics are taught not only acoustically, but visually as well. Moreover, sometimes music, poetry, drawing, and geometry classes can be conducted together, as some Hungarian programs impressively demonstrated (Budapest, Kecskemét, Pécs). In the German-speaking countries Hugo Kükelhaus was a promoter of the complex use of the sense organs in education. He organized various travelling exhibitions, where the kaleidoscope also featured as an educational tool (see, e.g., Kükelhaus, 1975; cf., Baltrusaitis, 1986; Arn, 1990).

It has become clear that even the geometrical-optical aspect of the simple kaleidoscope is not a totally closed chapter. For example, to calculate in general the number of images of an object between two plane mirrors at an arbitrary angle is not as easy as it has been believed. An-Ti Chai (1971) found that some textbooks give erroneous formula even in simple special cases. Chai's short paper reconsidered the problem and gave a correct formula. If two mirrors are hinged, as in the case of the polyangular dihedral kaleidoscope, we may use this device as a teaching aid or a tool in geometrical construction. A paper by Jack Robertson (1986) investigates this possibility, replacing the ruler and the compass of the classical Euclidean construction by this instrument. There is relevant literature in mathematical education on how to use mirrors (plastic ones are available!) in order to understand the geometric properties of various figures; see *The Mirror Puzzle Book* by Marion Walter (1985) and many other publications and educational kits on *Mira*, *Kaleidoscope Maths* and related topics. Dale Seymour Publications in California and Tarquin Publications in England have fine selections about this approach. Related questions are often discussed by various journals on mathematical education. Returning to optics, Thomas Greenslade (1982) described how it is possible to use multiple mirror images in present-day physics education. Interestingly, most of his illustrations are adopted from 19th century textbooks. In another case the observation of a child led a physicist-father to investigate an attractive phenomenon, and finally to publish a joint paper about the reflection in a polished tube (Laurence Marshall and Emma Marshall, 1983). This device can be considered to be a cylindrical kaleidoscope with circular cross-section, which is formed by infinitely many plane mirrors (here we use the word cylinder in the everyday sense). True, it presents patterns without ambiguity only in special cases. The authors describe that case where the object is a central spot (it is a central hole in a covered end); here we see a set of concentric rings. Jearl Walker (1985) in his series in the *Scientific American* reconsidered the classical kaleidoscopes, and raised the question of those we call cylindrical kaleidoscopes with polygonal cross-sections. The readers had one month to think about the problem, after which Walker supplied the solution. Obviously the author very carefully studied the modern literature on kaleidoscopes and geometrical optics, but still could not find out whether the four arrangements in question had already been discovered. The truth is, the knowledge he was after was well known in the late 19th century, but it vanished in our age, except in such non-trivial sources as Coxeter's chapter on polyhedra in Rouse Ball's book, Shubnikov and Koptsik's monograph on symmetry, and Courant and Robbins's introduction to mathematics. We recommend that interested readers have a look at Walker's paper because of his impressive illustrations and clear explanations.

Walker's paper reflects, however, an entirely new phenomenon in connection with the kaleidoscopes, namely the interest of contemporary artists and inventors in the topic. There are several new versions of the classical kaleidoscope, and even interesting marriages with modern technology, using systems of lenses, flashing diodes, and other devices. Of course the kaleidoscope is not a universal tool for high-tech art, but it has gained some application. For example, the *karascope*, which has a curved reflecting sheet inside the tube and uses a polarizing filter, was patented not long ago by Judith Karelitz. It was commissioned by the New York Museum of Modern Art, where the device is on sale. Even Brewster's idea of using the kaleidoscope for color-musical performance has got a new impact with a special *projection kaleidoscope*, or kaleidograph, designed by Don Anderson (1975), the director of the industrial laboratory of the Eastman Kodak Company. In this system the classical device is connected to a loudspeaker in such a way that the sound waves are able to jiggle the light colored fragments in the object box of the kaleidoscope. Furthermore Anderson upgrades the system by polarizing filters and other tricky ideas. Finally, using a projection lense, the patterns will "dance" synchronously with the music, changing both form and color. May we add that the most light organs, used for example by the Russian composer Skryabin in early 20th century, focus just on the color association of the music. There was, however, a movement pioneered by Alexander Laszlo in the 1920s, during his performance at the Festival of Kiel, Germany, which aimed to find the harmony of form, color, and music (see his monograph *Die Farblichtmusik*, Leipzig, 1925). Although the entertainment industry, from disco bars to rock concerts, rediscovered the colored music, but the theoretical and aesthetical studies are missing (the interdisciplinary works of the scholars and artists around the Prometei [Prometheus] Studio led by Bulat Galeev in Kazan', U.S.S.R., are the rare exceptions). Returning to kaleidoscopes, the idea of using various lenses and polarization effects also appeared in commercially marketed kaleidoscopes. Visiting some shops we saw, for example, the *bubblescope* the *polarizescope*, and other Hong Kong and Taiwan made devices. Interestingly, most of these technical kaleidoscopes do not go far beyond Brewster's classical arrangement of mirrors, but still provide many new phenomena. Obviously the kaleidoscopes of Möbius, Fedorov, Hess, Coxeter, Shubnikov and Koptsik would give further possibilities in a marriage with technology. The "ars mathematica" of Nikolaus and Caspar Schwabe (1986) represents an interesting chapter in the history of the kaleidoscope, because they also deal with trihedral kaleidoscopes (cf., Fig. 4 of this paper). They constructed various large-scale models and walk-in-kaleidoscopes (Darmstadt Symmetry Exhibition, 1986; Farnborough, 1988; Frankfurt, 1989). The Gallery and Shop AHA in Zürich, Switzerland, co-founded by the Schwabes, also deals in kaleidoscopes. Another interesting project was worked out by Vedder Wright (1989), an American specialist in spatial design, who gave classes for elementary school students about the symmetry of Platonic solids using trihedral kaleidoscopes. In Japan the artist-inventor Naoki Yoshimoto, the discoverer of the Yoshimoto cube, also deals with kaleidoscopes. The designer Naomi Asakura (1990) makes experiments with both plane and curved mirrors. It is interesting to note that the American inventor Buckminster Fuller (1979) also deals with the same foldable layout which is associated with the Coxeter kaleidoscope (Fig. 4), but he uses it for a different purpose as one of his quanta modules. In other cases he refers to kaleidoscopes metaphorically or, at least, semi-metaphorically.

Another analogy between the old and the recent period can be seen in the book market. Brewster wrote an entire monograph on the kaleidoscope soon after its invention. Moreover, in old age, almost 40 years later, he returned to the same topic, revising and extending the earlier work under a new title (Brewster, 1819; 1858). Although this second book had a further edition in 1870, later the kaleidoscope, as an optical device, disappeared from the titles of books (but not in a metaphoric sense in the title of very many collections, anthologies, and some periodicals). More recently, however, the kaleidoscope again became a topic for books. We are aware of two recent monographic surveys of the optical device, although we do not have access to them (Baker, 1985; Yoder, 1988). Another book discusses how to create kaleidoscopic designs (Finkel and Finkel, 1980). There are also other items available on the topic, including a thesis on the use of kaleidoscopic patterns in elementary art education (Costello, 1988), another one using the kaleidoscope for discussing culture-theoretic questions (Gray, 1989), as well as a great number of popular articles for children and amateurs. Even the expression "kaleidoscope" inspired further technical terms in both Brewster's time and more recently. Let us start with the early period. Charles Wheatstone, who was 15-year old when Brewster patented his kaleidoscope in 1817, invented a decade later an instrument which he called *kaleidophone*, or *phonic kaleidoscope* (Wheatstone, 1827). The young Charles was also inspired by his family's profession of making musical instruments which directed his interest to acoustical experiments. The kaleidophone is a device for visual display of sound-waves: it is based on the illumination of the free end of a vibrating rod. Wheatstone became also interested in Chladni's sand pattern technique which produces symmetric patterns on solid plates vibrated, for example, by a violin bow at their edges. This approach gained some prominence in more recent experiments (cf., Mary Walker's book *Chladni Figures: A Study in Symmetry*, London, 1961). Wheatstone's name is, however, better known for his later studies in electronics, in particular in connection with the Wheatstone bridge for measuring resistances (originally invented by Samuel Christie). The second flowering of kaleidoscope-related terms is associated with the more recent period. On the border of art and mathematics such expressions were coined as *kaleidocycles*, that is kinetic polyhedra made from chains of tetrahedra (Schattschneider and Walker, 1977), and *kaleidometrics*, which is a method of drawing kaleidoscopic patterns with special grided circles (Shaw, 1982). Interestingly, the term *kaleidophone* was also reintroduced in a new sense in 1984 by the Prometei Studio in Kazan'. Their kaleidophone is an optical device for enriching musical performances by light effects. After seeing so many modern artistic applications, we can no longer reject so strongly Brewster's dream and the related myth about the artistic usefulness of the kaleidoscope. Sometimes a myth is a herald of a later chapter...

Kaleidoscopes, mostly the well-known classical ones, are also mentioned in some interdisciplinary works on symmetry. Here we refer only to the recent monograph by Werner Hahn (1989) which has an especially broad spectrum with an unbelievably large number of illustrations. During his study of symmetrization in space Hahn uses dihedral kaleidoscopes to multiply some basic shapes. This chapter, which also emphasizes the importance of light during the experiments, includes photographs. In a later chapter the artist-biologist author characterizes his *ars evolutoria* as "kaleidoscope-related genesis-processes". Hahn remarks that in this sense his works give a counter-example to the negative views expressed by Gombrich about the exploration of kaleidoscopic shapes. Probably Gombrich is right from the geometrical point of view: these shapes, as we discussed in Part 1 of this paper, are so simple that we would have little joy in exploring their structures. However Hahn's approach of looking for an analogy to evolutionary transformations of organic shapes is more biological. A classical chapter of embryology focused exactly on the splitting of fertilized eggs and their connections with the later symmetry of the organisms. This topic also aroused the interest of the mathematician Hermann Weyl who discusses related questions in his book *Symmetry* (Princeton, 1952, pp. 33-38). A new impact came in the mid-1970s when Richard Gordon and Antone Jacobson suggested a computer simulation technique, based on the distortion of a geometric grid, which "performs" experiments that cannot be done in biological laboratories (see their article in *The Journal of Experimental Zoology*, 197 [1976], 191-246 and the survey in *Scientific American*, 238 [1978], No. 6, 106-113). Some of the simulated images even have aesthetical value. Returning to Hahn's monograph, he also refers to computer graphics as a field where kaleidoscopic images may have an importance. Moreover, he is not the only one who combined artistic and biological interests in connection with the kaleidoscope: Thomas H. Huxley, as we have already mentioned in Part 1, also referred to the optical device.

THE PERIODIC TABLE OF KALEIDOSCOPES AND KALEIDOSCOPIC GROUPS (KALEIDOSCOPIANA SYMMETRICA)

During our kaleidoscopic excursion around the kaleidoscope we have seen many kinds of instruments. It is probably useful to summarize them, moreover, to present a classification that would be in "harmony" with the presented symmetries (symmetry groups). This list - or, with an analogy to chemistry, periodic table - considers all the possible devices formed with plane mirrors and the related groups, which we may call kaleidoscopic groups (the conventional name is reflection groups). In the right column the terms variable versus stable, that is the possibility versus impossibility of a hinged edge, refer only to the types of kaleidoscopes, while the number indicates both the number of types of kaleidoscopes and the number of symmetry groups (see more details later):

Kaleidoscopes according to the arrangement of mirrors (number of mirrors)	**Presented patterns (symmetry groups)**	**Number of types of kaleidoscopes (symmetry groups)**
— dihedral ($m = 2$)	rosettes (planar point groups)	infinitely many variable
special case (two parallel)	frieze (frieze group)	one stable
— cylindrical ($m = 3, 4$)	wallpaper (plane groups)	four stable
— trihedral ($m = 3$)	spatial rosettes/polyhedra (spatial point groups)	infinitely many variable and three stable
special case of variable (U-shaped)	frieze (frieze group)	one stable
— trough ($m = 4$)	strung pearls (rod groups)	infinitely many variable
— box without front and top ($m = 4$)	two-sided frieze (two-sided frieze group)	one stable
— open box ($m = 4, 5$)	two-sided wallpaper (two-sided plane group)	four stable
— polyhedral ($m = 4, 5, 6$)	spatial periodicity (space groups)	seven stable
— variable trihedral, box without front and top, and open box — with colored bottom	rosettes, friezes, and wallpapers doubled in another color (gray groups)	[as above in the corresponding cases]
— determined by appropriate colored symmetric object	color change periodically (colored symmetry)	[usually just one stable arrangement]

In the case of determining the types of kaleidoscopes (right column) of a given kind (horizontal row), we do not distinguish between those arrangements of mirrors where the corresponding angles are the same. In other words, two kaleidoscopes are considered to be of the same type if their arrangements of mirrors are equivalent by an appropriate angle-preserving transformation. Such a transformation is called conformal; here we also require that straight lines should not be curved (i.e., the transformation is also affine), because all the kaleidoscopes discussed here are formed by plane mirrors with straight edges. Although this classification may seem too sophisticated, it is really a natural convention, which is used directly or intuitively in many fields. For example, in the terminology of elementary geometry, moreover in colloquial language, such words as rectangle and rectangular right prism ("brick") refer to infinitely many figures with various sizes and proportions; only the corresponding angles remain invariant. One of the earliest laws of crystallography, which was discovered by Niels Stensen (Nicolaus Steno) in 1669, states the constancy of angles for the crystals of the same material. The biological growth of individual bodies often can be characterized by conformal transformations, where the angles are invariant, but the lines can be curved. The fact that the proportion of the human body changes significantly from childhood to adulthood — that is the transformation is more general than just a mere similarity (enlargement) — has been investigated by many artists since the Renaissance. Albrecht Dürer worked out an entire set of canons following this process. D'Arcy Thompson in the 1910s generalized and systematized this idea in theoretical biology, also discussing botanical and zoological examples. This work inspired, among others, the aforementioned simulation model for embryology by Gordon and Jacobson. The pursuit curves in the three-bug problem, which were discussed in Part 1 in connection with the image on the cover of this volume, are derived by an angle-preserving process. Sergei Petukhov investigated conformal symmetries — properties which are invariant under conformal transformations — in the case of growth and motion of biological objects, and he applied the findings in mechanical engineering. More general conformal symmetries also play significant roles in particle physics.

Returning to our kaleidoscopes, we can clearly see two sorts of types. In the cases of all the dihedral kaleidoscopes, a class of trihedral ones, and all the trough-shaped ones, the types differ only in one angle ($180°/n$, where n is a variable). In these cases the arrangement of mirrors has some variability, so a specific type can be easily transformed into other ones just by rotating a mirror around an edge (cf., the idea of the polyangular kaleidoscope and our earlier note in the caption of Fig. 1 in Part 1). Other types of kaleidoscopes are stable in some sense: changing one angle is not enough to form another type. Note that in case of the partial kaleidoscope (Fig. 3), which is a simplified version of the stable trihedral kaleidoscopes where a part of the surface is missing, there is variability; this is precisely the advantage of the device over the complete trihedral versions. Obviously, the property of variability or stability only refers to the types of kaleidoscopes, while in the case of symmetry groups, or repetition types of patterns, just the numbers are important. The table gives in each row the specific number of kaleidoscopic groups, that is, of those symmetry groups which can be generated just by reflections. Among the 17 wallpaper groups and 230 space groups there are four and seven kaleidoscopic groups, respectively; these can be fully represented by kaleidoscopes. In many other cases, where at least one reflection is involved, we may represent the local arrangements (point-groups) of ornamental or crystal structures.

An important advantage of this classification of kaleidoscopes is its correspondence with the group theoretic characterization of the presented patterns. It is manifested by the double function of the right column. Specifically, all of the kaleidoscopes which belong to a given type present patterns of the same symmetry group (allowing any kind of asymmetric object), while different types represent different groups. This further supports the argument that - contrary to Shubnikov and Koptsik - we do not consider the trihedral kaleidoscopes of (90°, 90°, 180°/*n*) as one type, but as infinitely many variable ones (cf., Fig. 1 in Part 1). Of course this theoretical number is severely limited by the imperfections of the mirrors. We should also remark that not all of the obtained types of rosettes (point groups) can occur in ideal crystal-structures: the periodicity (lattice-likeness) allows only 1-, 2-, 3-, 4-, and 6-fold rotational symmetries. Real crystals, however, may depart from this strict rule; see the recent developments in connection with the 5-fold symmetric quasicrystal. Some dihedral, trihedral, and trough-shaped kaleidoscopes can present 5-fold symmetric finite patterns. Now these structures can be associated not only with the organic world, but also with special crystal-structures.

Of course we should again emphasize, self-critically, that the simple descriptions cannot replace the real joy of looking into these kaleidoscopes. If a picture is worth a thousand words, as the ancient Chinese proverb claims, the kaleidoscope presents a multiplication of a thousand words: almost a whole booklet on geometric crystallography and related subjects. To avoid further frustration, it should be remarked that we plan a more detailed survey of these kaleidoscopes, as well as the organization of a workshop on the topic with an exhibition mounted during the forthcoming *Interdisciplinary Symmetry Symposium and Exhibition* (1992); see also the "Call for kaleidoscopes" in this quarterly (Vol. 1, No. 1, p. 93).

We have visited many fields, from optics to design, from geometry to crystallography, emphasizing the educational aspects, during this rather kaleidoscopic and incomplete survey of the kaleidoscope. Moreover, we have also touched on linguistic, poetic, aesthetic, psychological, as well as practical questions, and also covered some relevant events in the history of science. At this point we should pay tribute to the early pioneer of the topic, Sir David Brewster (1781-1868). He was a very productive man, not only as a kaleidoscope-maker and popularizer of science, but also as a researcher, author of monographs, editor of encyclopaedias and scientific journals, and finally as a father (he was in his 80th year when his younger daughter was born of his second marriage). His major scholarly interest was in optics (cf., the terms Brewster angle of polarization, Brewster fringes), but he also contributed to the theories of magnetism and of hydrodynamics. He has a strong record of works on the history of science, including an often quoted biography of Newton, another one on the three pioneers of modern astronomy Galileo, Brahe, and Kepler, as well as some publications on the history of optics. Other works by him deal with religious questions, the history of Freemasonry, and numerous other topics. Brewster kept in close contact with many eminent scholars and artists of his period, including the astronomer Herschel, the pioneer of photography Talbot, and the poet Sir Walter Scott; he visited in Paris, among others, Arago, Berthollet, Biot, Laplace, and Poisson. Brewster's father-in-law was the poet "Ossian" Macpherson. While the real authorship of the "ancient" Ossian-poems remained secret for a long time, Brewster solved a similar problem in science: he proved that the "discovery" of the correspondence between Newton and Pascal was a simple case of forgery.

From the point of view of our subject it is also interesting to mention Brewster's studies of the forms of crystals and of the figures of equilibrium in liquid films, which are strongly connected with the later interdisciplinary studies of symmetry. One of the earliest comprehensive monographs on the subject, F.M. Jaeger's *Lectures on the Principle of Symmetry* (Amsterdam, 1917), refers to him in connection with the optical anomalies in crystals, but surprisingly misses the subject of the kaleidoscope. Brewster contributed significantly to the modern approach of symmetry and to the spread of the term "symmetry". We should learn from the interdisciplinary "excursions" of Sir David and many other pioneers in this survey. In their cases the interdisciplinarity was not only a slogan, but real practice. Unfortunately this component is often missing in our age, especially in the broader spectrum of arts and sciences.

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SYMMETRY: CULTURE & SCIENCE

MATHEMATICS AND SYMMETRY A PERSONAL REPORT

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QUESTION 1

what is symmetry?

Within mathematics the concept of symmetry is ubiquitous.
Perhaps more so in algebra and geometry than elsewhere, but in all branches of mathematics there have been practitioners who were particularly guided by the symmetry principle. In its simplest form, "twofold symmetry", this principle is expressed throughout mathematics by such words as "duality" or "complement", or even "if and only if". All mathematicians are familiar with the duality, and complementation, expressed in the laws of Boolean algebra which govern, for example, the unions and intersections of sets:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C),$$
$$(A \cap B)' = A' \cup B',$$
$$(A \cup B)' = A' \cap B'.$$

The appeal of these simple laws is certainly in large part due to their symmetry.

As has been pointed out by H. S. M. Coxeter (1948, pp. 162-163, 258-259) there can be little doubt that this instinctive search for, and response to, symmetry was inherited from George Boole by his daughter, Alicia Boole Stott. Coxeter tells us that as a young woman Alicia Boole was taught the rudiments of four-dimensional geometry by that Howard Hinton who later became known for his mystical books on higher space. In later years she determined, using only synthetic methods, the entire sequence of cross-sections of the regular four-dimensional polytopes. This led to a collaboration with P. H. Schoute, a skilled professional mathematician who had determined only the middle one of her sequence of sections by more orthodox analytic methods. At the age of 70 she was introduced to Coxeter with whom she collaborated on the study of a four-dimensional polytope he was investigating at the time. It is unimaginable that she accomplished all this work on regular (i.e. highly symmetric) polytopes, with no formal training in mathematics, except by the use of a powerful instinctive sense of symmetry.

In the foundations of geometry, the incomplete duality between the Euclidean axioms of plane geometry,

> Two points determine a unique line,

and

> Two lines determine a unique point, *except when they are parallel*,

leads to a formulation of the completely dual axioms for projective geometry. Projective geometry has been recognized since its invention as a particularly beautiful branch of mathematics, exactly because this duality, i.e. symmetry, between point and line (point and plane in space) does not need to be qualified by the exceptions which render it imperfect in Euclidean geometry.

Also, in the foundations of geometry, it has been suggested (Heath, 1956, vol. 1, p. 202) that the millenia-long search for a proof of Euclid's "parallel postulate" was motivated in part by the expectation of a symmetry between a theorem and its converse in geometry. The parallel postulate is equivalent to Euclid's Proposition I.29, "If two parallel lines are cut by a transversal, then the alternate interior angles are equal". Now, the converse to this proposition, "If two lines are cut by a transversal in such a way that the alternate interior angles are equal, then the two lines are parallel," was known to be true. (It is essentially the content of Proposition I.28, which immediately precedes I.29 in Euclid.) In Euclid's treatise, many facts occurred in *theorem — converse theorem* pairs. For example,

> I.5: If two sides of a triangle are equal then their opposite angles are equal,

is followed immediately by its converse,

> I.6: If two angles of a triangle are equal then their opposite sides are equal.

Such examples, combined with the complexity of the statement of the parallel postulate, led some commentators on Euclid's *Elements* to urge that it should be "struck out of the Postulates altogether; for it is a theorem involving many difficulties" (Heath, 1956). In this way the expected theorem—converse symmetry contributed to the persistent search for a proof of I.29, a search which culminated in the discovery of non-Euclidean geometries by Bolyai and Lobachevskii in the early nineteenth century. It is a commonplace now that this discovery is one of the most revolutionary in the history of thought because of its revelation of the independence of our mathematical model-making from any actual physical universe. This freedom led to the possibility of new models, such as relativity, to describe aspects of the universe unthought of by minds tied to the view that the human brain is constrained by its very construction to think only in the Euclidean way. It is interesting to speculate that this revolutionary development stems in part from the (misguided?) search for theorem—converse symmetry in Euclid's *Elements!*

Turning to more intricate manifestations of symmetry in geometry than the simple twofold symmetry of the preceding influential examples, the most familiar instance is probably that of the regular polyhedra and related highly symmetric structures. Recall that Euclid begins his *Elements* with the construction of the simplest of regular figures, the equilateral triangle (Proposition I.1), and concludes, nearly at the end of Book XIII, with the construction of the most complicated of the regular polyhedra, the icosahedron and dodecahedron (Propositions XIII.16,17). Apparently Proclus already suggested that this showed that the geometric purpose of the *Elements* was to provide a treatise on the construction of regular figures. In his own treatise on *Regular Polytopes*, Coxeter (1948, p. 13) reports that D'Arcy Thompson repeated this opinion to him.

Without believing that this is an entirely accurate characterization of Euclid's work, we still recognize the fundamental significance of these most symmetric figures throughout mathematics and science. As Marjorie Senechal recently put it: "Today we believe that it is not the classical form of the regular polyhedra that is significant: instead it is the high degree of order which they represent" (Senechal and Fleck, 1988). It is only necessary to mention Felix Klein's book *Lectures on the Icosahedron* to remind a mathematician of the unifying role of polyhedral symmetry, in the guise of group theory, in treating problems in analysis.

Group theory itself is, in one of its main aspects, the study of permutation groups or transformation groups, that is, the study of the symmetries of various mathematical or real-world objects. Even in "abstract" group theory, where the groups studied are not initially groups of symmetries of any particular mathematical objects, one of the main problems is often to find some such object on which the abstract group acts in a natural way as the group of all symmetries. In particular, now that the enumeration of the finite simple groups has been completed, much effort is being spent to understand them by creating more or less natural geometric objects whose symmetries are described by the new simple groups.

At a still deeper level, an interesting program was suggested, and partially carried out, by L. Fejes Tóth in his book *Regular Figures* (1964). He started from the observation that "extremum postulates often involve regularity". (Here by "regularity" he means "symmetry".) That is, "regular arrangements are generated from unarranged, chaotic sets by the ordering effect of an *economy principle*, in the widest sense of the word". Several such results are well-known. For example, among all polygons having a given perimeter and a given number of edges, the regular (that is, the most symmetric) polygon has the maximum area. Likewise, if we ask for the densest packing of congruent circles in the plane, the answer is the regular arrangement of circles at the centers of the cells of a hexagonal honeycomb.

In space such problems are much harder. If we ask for the convex polyhedron of maximum volume, having a given surface area and given number f of faces, the answer is known to be the regular polyhedron with f faces if $f = 4, 6$, or 12. However, it has apparently not even yet been proved that the analogous question for polyhedra having 6 vertices is answered by the regular octahedron. An intriguing problem of this type, introduced by the biologist Tammes, is that of the optimal arrangement of orifices, or spines, on pollen grains. A proposed arrangement of 122 points on a sphere, derived from the regular icosahedron, was shown by Fejes Tóth (1964, pp. 232-233) not be optimal. Could it be that the (approximately) 122 spines on the pollen grain in Figure 1.1 constitute a solution to the optimization problem for 122 points? A brief discussion, with recent references, of these and similar problems is given by Fejes Tóth (1986).

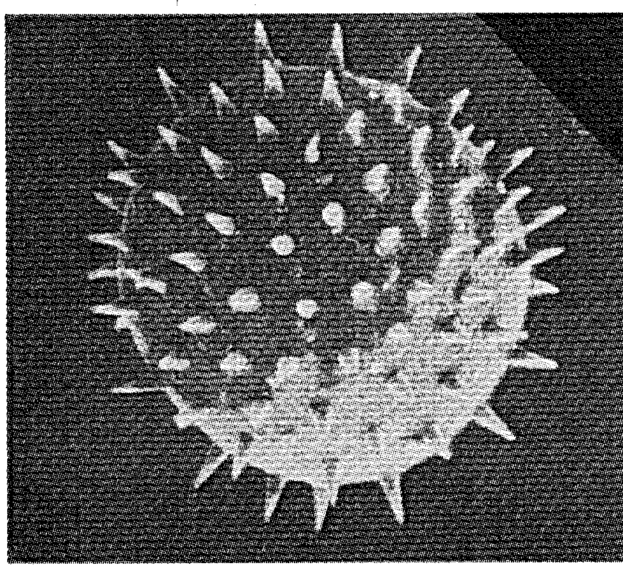


Figure 1.1: Pollen grain of *Hibiscus*.
(Scanning Electron Micrograph by Joan W. Nowicke, SmithsonianInstitution.)

To conclude this section I describe a famous example of a possible application of Fejes Tóth's program (not mentioned explicitly by him). This is the attempt, over more than a hundred years, to explain the regular arrangement of leaves around a stem in growing plants, or the apparently similar regular spiral arrangement of the florets in the heads of daisy-like flowers. Such an arrangement is most easily seen on a giant sunflower, where the numbers of spirals visible in two directions on the head are very often adjacent numbers of the Fibonacci sequence

$$\{f_i\} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

The photograph in Figure 1.2 shows a sunflower in which the numbers of spirals in the two directions are 55 and 89, a typical pair for large sunflowers. The purely mathematical properties of this sequence have been extensively studied since its introduction by Fibonacci in the 13th Century. The sequence is connected to aesthetic theory by the fact that the ratio of adjacent terms approaches the "golden ratio", $\varphi = (1 + \sqrt{5})/2 \approx 1.618\ldots$, which has been widely considered to be especially attractive. (Hence the prevalence of 3×5 and 5×8 notecards in America.)

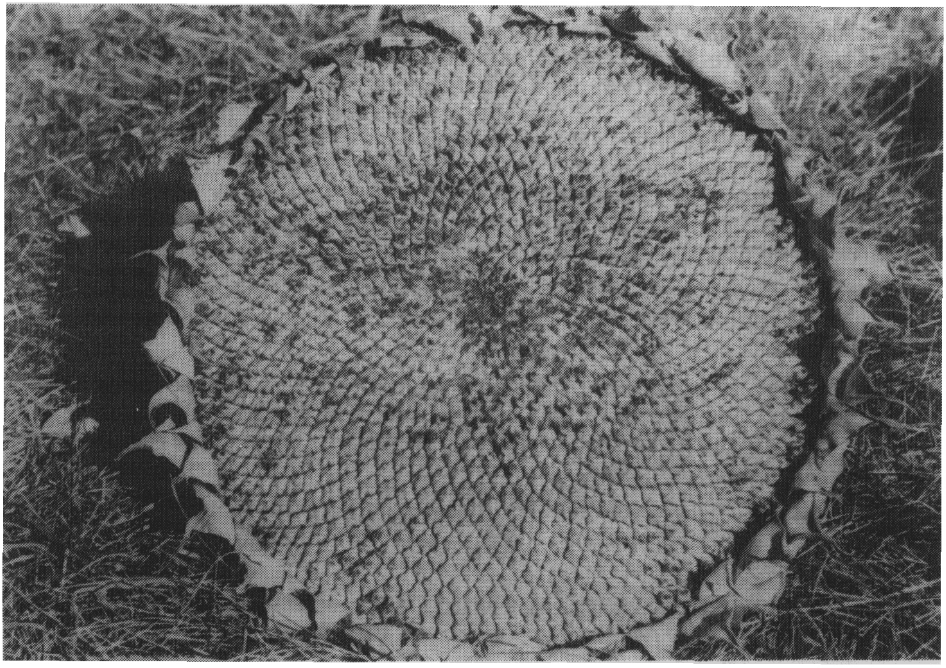


Figure 1.2: Sunflower. The number of spirals in one direction is 55; in the other it is 89. These constitute a Fibonacci pair.

Over the years many explanations have been proposed for the occurrence of such number pairs in plant growth, none of them completely satisfactory. The most recent ones imagine the plant is "trying" to apply some *economy principle*, such as to maximize the share of light received by each leaf, or to minimize the physical crowding of one floret by the next. Although other, perhaps chemical, mechanisms have been proposed as the means by which the plant carries out its "desires", there are aspects of the theory in which purely mathematical consequences of suitable economy principles lead to the observed symmetric spiral arrangements. A brief description, which owes much to the text and illustrations of Jean (1984), Marzec and Kappraff (1983), Dixon (1981), and Stevens (1974), of such a mathematical model which yields the observed shape and number of spirals with convincing accuracy follows. [See also the article "Symmetry in phyllotaxis" by Irving Adler in this issue — Eds.]

It has been well documented that the florets (which later become the seeds) are laid down successively along a logarithmic "growth spiral" whose equation in polar coordinates (r,θ) is $r = e^{k\theta}$ (Marzec and Kappraff, 1983, p. 205-207). This is also called an "equiangular spiral" because it is the locus of points (r,θ) such that at any point the radius vector makes a constant angle with the tangent vector at that point. The constant angle is A, where $\cot A = k$. Thus if $A = 90°$ we have $k = 0$ and the equation becomes $r = 1$, which is a circle of radius 1. If $k > 0$ is very near to 0 then A is very near to 90°; in this case the spiral is tightly wound around the origin — almost a circle.

It also seems that the florets are laid down at equal intervals along such a spiral, i.e. at points (r,θ) where θ successively takes on the values $\alpha, 2\alpha, 3\alpha, \ldots,$ for some fixed angle α. We want to apply a suitable *economy principle* to determine α. Since the angles $\alpha, 2\alpha, 3\alpha, \ldots$ can be represented as points $(1,\alpha)$, $(1,2\alpha)$, $(1,3\alpha), \ldots,$ on the unit circle, such an economy principle is one which will ensure that the first n of these points are "equally distributed" around this circle. (This is a mathematical version of the biological requirement that all florets should have equal access to the sunlight, or that later ones should not physically crowd any of those already laid down.) At first glance, a reasonable such principle seems to be:

First Attempt: For each n, the number of different lengths among the n arcs into which the n points $(1,\alpha)$, $(1,2\alpha)$, $(1,3\alpha)$, $\ldots$ $(1,n\alpha)$ divide the circle is minimized.

For if, among the n arcs, there are many different lengths, then some points are bunched together and others are not. However, it is a remarkable, though simple, fact (the truth of which the reader can readily verify experimentally by moving a paper angle around a circle) that:

For each choice of angle α the number of different lengths among the n arcs into which the points $(1,\alpha)$, $(1,2\alpha)$, $(1,3\alpha), \ldots,$ $(1,n\alpha)$ divide their circle is *never greater than three.*

Thus this first attempt is useless as a criterion for selecting one α instead of another. We discard it.

A slight modification is much more successful. Note first that if $n-1$ of the points have already been placed around the circle then the nth point divides one of the $n-1$ arcs into two parts. (Of course we are assuming that α is an irrational multiple of 360°; otherwise two points will eventually coincide.) It would seem desirable that it should divide this arc into nearly equal parts. We say that the point $(1,n\alpha)$ causes a "bad break" if one of the two arcs it creates is more than twice the length of the other. The appropriate economy principle is that *no bad break occurs,* that is, NBB Economy Principle: For each n, the point $(1,n\alpha)$ does not cause a "bad break" in the arc in which it appears.

An astonishing combination of beauty and function is expressed by the unexpected mathematical fact,

Theorem (Knuth, 1973, p. 543): The only value of α for which NBB is satisfied is $360°/\varphi^2 \approx 137.5°$ (or its complement, $360° - 360°/\varphi^2$, which is just $360°/\varphi$), where φ is the golden ratio, $(1+\sqrt{5})/2$.

Thus, for purely mathematical reasons, a plant might "choose" the angle 137.5° for laying down successive florets, or, in the case of leaf growth, for laying down successive leaves around a stem. There is considerable documentary evidence for the actual occurrence of just this angle.

Now, assuming that points are placed on a tight logarithmic spiral at equal intervals of $360°/\varphi^2$, what would be seen by the human eye? The eye does not see the original spiral, because it is so tightly wound, but sees certain "secondary" spirals. These are the ones seen in the sunflower photo in Figure 1.2. A simple computer drawing shows vividly how this can happen. In the example of Figure 1.3 some 750 points have been plotted at intervals of $360°/\varphi^2$ along the curve $r = e^{k\theta}$, with $k = 1/800$. It is a simple matter to count the spirals and see that there are 21 in one direction and 34 in the other direction. To obtain the numbers 55 and 89 seen in Figure 1.2 it is only necessary to choose a still smaller value of k, i.e. a still more tightly wound logarithmic spiral.

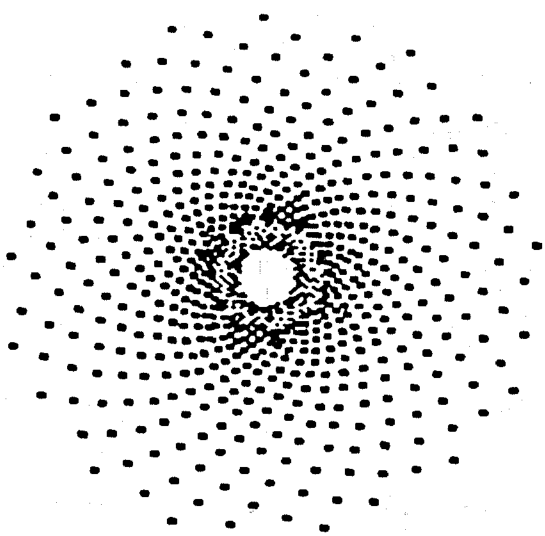
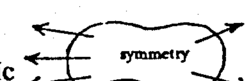


Figure 1.3: Points equally spaced along the logarithmic spiral $r = e^{k\theta}$ ($k = 1/800$) at intervals of $360°/\varphi^2$ appear to form 21 secondary spirals in one direction and 34 in the other.

QUESTION 2



Many examples of the interdisciplinary impact of geometric symmetry are comparatively well-known. Of these, one of the most striking is the story of the artist M. C. Escher who found, in an illustration from a paper by the geometer H. S. M. Coxeter, the solution to an artistic problem. As he explained in a 1958 letter to Coxeter (Coxeter, 1979), he had for a long time been interested in patterns containing motifs which kept getting smaller and smaller. He had solved the problem of creating such patterns when the small motifs approached a single point at the center of the pattern, as in his 1939 print *Development II* and the 1956 print *Smaller and Smaller* (Ernst, 1976, pp. 102-103). He had even made such patterns where the motifs became smaller and smaller toward a line limit. However, he had never been able to make a pattern whose motifs grew smaller towards an outside circle which would form a natural artistic boundary for his print. But in the copy of Coxeter's paper for a Canadian Symmetry Symposium (Coxeter, 1957), sent to him by Coxeter, he saw an illustration of the symmetry group [4,6] in the non-Euclidean hyperbolic plane which was exactly the inspiration he needed. He enclosed with the letter a copy of his *Circle Limit I*, based on this [4,6], and later made the famous *Angels and Devils* (*Circle Limit IV*) based on the same illustration. When he learned that there were infinitely many other symmetry groups which should serve the same purpose he used one of them, [3,8], as the basis for the two other *Circle Limit* prints, *II* and *III*.

An even better-known example of such an interplay between geometric symmetry and the rest of the world is the famous Rubik's Cube. Here was a toy, originally devised by Ernő Rubik to illustrate symmetry principles in a particularly concrete fashion, whose popularity swept the world in a way not seen since the "15 Puzzle", also based on symmetries (odd and even permutations) many decades earlier.

Of the hundreds of other, less well-known, examples, let me choose two which are quite different from each other. The first is a result of the researches of the archaeologist, Dorothy Washburn. Early in her career she was confronted with the problem of analyzing a collection of pottery in the Peabody museum of Harvard University. This pottery had been collected from well-documented sites in the Upper Gila River in what is now New Mexico. As she reported it to me, she began by using one of the conventional tools of pottery analysis, "typology", which had been particularly carefully developed for this heavily studied area of the Southwest USA. However, each morning when she returned to work on the classifications she had produced in the preceding day she found that she could no longer remember why pot *x* had been assigned type *y*. Not only would it be difficult for subsequent investigators to duplicate her "scientific" conclusions, but she herself could not confirm her own previous day's work after a 12 hour time lapse!

In this context she asked herself whether some truly objective attribute could be found for pottery so that today's classification would still stand up to tomorrow's scrutiny. *Symmetry* turned out to be just such an attribute. Most of the pottery in question was decorated, either with finite designs (often with rotational symmetry), one-dimensional designs ("bands"), or two-dimensional designs having a variety of symmetries. She extended the earlier ideas of Brainerd (1942) and Shepard (1948) by incorporating the two-color symmetry classification of the textile scientist H. J. Woods (1935-36). In this way she developed a suitable tool for the study not only of patterned pottery, but of patterned material of any type (Washburn, 1977).

Without knowing it at the time, Washburn had thereby repeated some of the work of crystallographers such as Belov and Tarkhova (1964), but in a context more directly suitable for use by archaeologists, anthropologists, and art historians. A systematic treatment of this method of symmetry analysis, incorporating the relatively standardized crystallographic notation, presented specifically for the study of patterned material of any kind is found in a more recent monograph (Washburn and Crowe, 1988).

To see how this symmetry tool could be applied, we look at an example suggested by Washburn herself. In Chaco Canyon, in what is now New Mexico, USA, there are dozens of impressive ruins, one of which is shown in Figure 2.1. At first glance these ruins, each containing many separate rooms, appear to have been inhabited by a large number of people. However, Chaco Canyon is extremely hot and dry, and evidence suggests that it was not much different when it was still inhabited, some 700-800 years ago. When archaeologists first became aware of these ruins, about 100 years ago, the apartment houses and "skyscrapers" of New York were the talk of the world. Thus it is not surprising that archaeological opinion of that time, and for many years afterward, estimated a huge population for Chaco Canyon, in spite of the obvious difficulties of living in such inhospitable country.

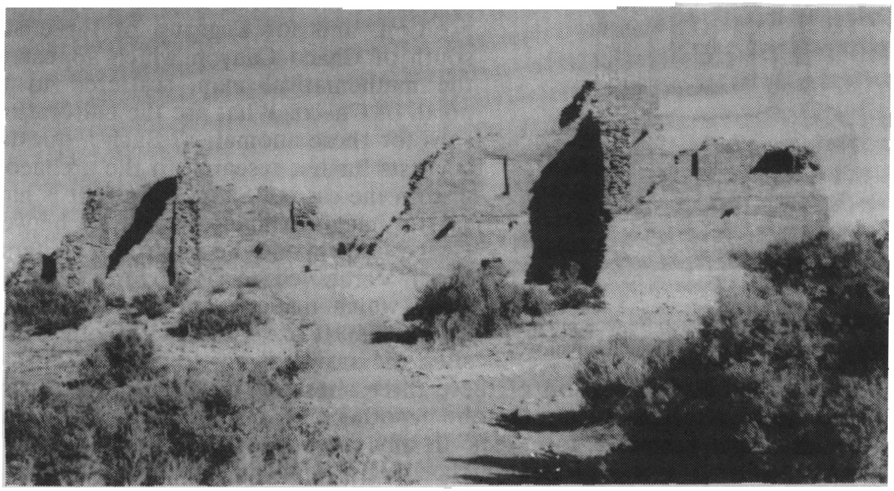


Figure 2.1: A portion of the ruins of Chaco Canyon as they appear today.

As archaeology became aware of the fact that there are many other, smaller, sites in the general vicinity of Chaco Canyon (and as the marvels of American culture came to be the "shopping malls" instead of skyscrapers) a new possibility presented itself. Perhaps the buildings of Chaco Canyon were not really densely inhabited, but were mainly the store rooms of a vast "shopping center", which was the center of a broad trade area. The fact that the pottery found in the Chaco outliers was much like the pottery of Chaco itself (see Figure 2.2) was consistent with this idea.

In collaboration with a statistician (Washburn and Mattson, 1985) Washburn compiled information about the relative frequency of occurrence of symmetry types (which she had already developed for her earlier study) of patterns found on pottery at the various sites in and around Chaco Canyon. If indeed these sites were in constant communication and trade with each other, it is a reasonable hypothesis that two sites with similar distributions of pattern types were close together, whereas those which have dissimilar distributions of pattern types were farther apart geographically. Using this hypothesis, she used "multidimensional scaling" to make a map of the various sites so that their distances from each other best agreed with the percentage correspondences between pattern types. And, indeed, this

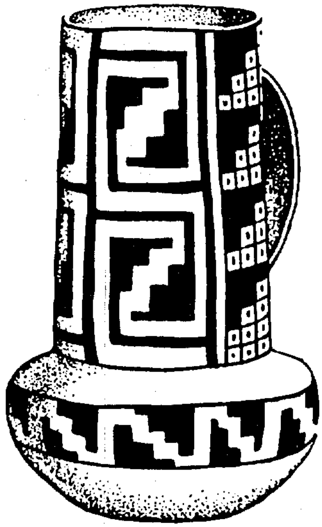


Figure 2.2: Typical Chaco Canyon pitcher (from Pueblo Bonito, Chaco Canyon).

mathematical map agreed in a general way with the actual known location of the sites round Chaco Canyon.

However, it is the *disagreements* which are the most instructive, since they suggest directions for further investigation. The two most notable disagreements are for Salmon Ruin, which appears on the mathematical map far to the east of its true geographical position; and for a group of three sites south of Chaco Canyon which appear, on the mathematical map, scattered to the north of Chaco. What are the cultural reasons for these anomalies? Such a question suggests further research to the archaeologist. In the case of Salmon Ruin, it is natural to imagine that its location between Chaco Canyon and the important centers at Mesa Verde led to a mixture of design styles which made it appear farther from Chaco than it really is. For the other three sites, the answer is not so obvious. Perhaps the fact that the actual location of these three sites was quite separate (farther south) from the other sites meant that the hypothesis of constant cultural trade and communication is not valid for them. In any case, the study of symmetry has suggested specific sites to which the archaeologist might give further attention to determine what particular aspects of culture contributed to these discrepancies.

My second example is very different. It represents the combined work of a mathematician, W. F. Orr, and a professor of French, C. W. Carroll. It began with the observation that one of the oldest of French verse forms, the Provençal sestina, was based not on *rhyme*, but on a symmetrical rearrangement of the final words of each line. More specifically, the sestina form consists of six stanzas of six lines each, followed by three final lines. The six final words of the six lines of stanza one are permuted to reappear as the final words of the six lines of stanza two. Applying the same permutation yields the final words of stanza three, and so on, in such a way that the same permutation takes the final words of stanza six back to the original order of stanza one. (Moreover these same six words occur in the three culminating lines of the sestina, in the same order as in the first stanza. In the present description, however, these three lines will be ignored.)

The oldest known sestina, and the one whose analysis led to the formulation of the general rule of construction, is the late 12th Century "Lo ferm voler", by Arnaut Daniel. It is reprinted in Carroll and Orr (1975). Two modern sestinas, "Paysage moralisé" and "Have a good time", were composed by the English poet W. H. Auden. ("Paysage moralisé" appears in the collection by Williams, 1951, pp. 750-751.)

The actual permutation used in Arnaut Daniel's sestina is $P = (163542)$. The literary name for it is *retrogradatio cruciata* ("crossed-reverse"); it leads to the spiral pattern in Figure 2.3 when $P(1), P(2), \ldots, P(6)$ are connected in order. As a generalization of the sestina, the French author Raymond Queneau, known for his applications of mathematics to literature, suggested the problem: "For which n can an n-ina exist?", and apparently answered that question for $n < 100$.

Orr and Carroll made this more precise by defining a spiral permutation P on the n symbols $1, 2, \ldots, n$ by

$$P(2r) = r, \qquad 1 \le 2r \le n,$$
$$P(2r+1) = n - r, \qquad 1 \le 2r+1 \le n.$$

An n-ina exists, by definition, if this permutation is *cyclic*. Their complete result, which contains Queneau's previously obtained results, is the Theorem (Carroll and Orr, 1975): An n-ina exists if and only if

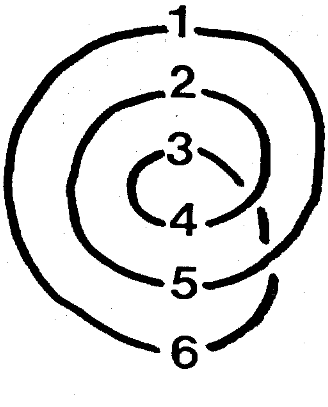


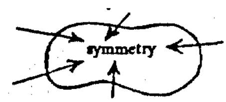
Figure 2.3: The spiral symmetry of the sestina.

(i) $2n + 1$ is a prime, p, and
(ii) either $+2$ or -2 is a generator of the multiplicative group of the finite field of p elements.

For example, in the range $3 \le n \le 20$, the values 3, 5, 6, 8, 9, 11, 14, 15, 18, 20 satisfy condition (i). Of these, 8, 15, and 20 do not satisfy condition (ii). Thus for $3 \le n \le 20$, an n-ina can exist exactly when $n = 3, 5, 6, 9, 11, 14$, or 18.

Surely this is a remarkable instance of an occurrence of symmetry in the world of literature which inspired a purely mathematical investigation and theorem!

QUESTION 3



What is the origin, in my own life and cultural background, of a continuing interest in symmetry and pattern? Are there any particular events or experiences which formed this interest?

Kh. S. Mamedov (1986, pp. 512-514) has related what in retrospect seems a dramatic contrast that aroused his curiosity about the prevalence of symmetry. He observed that in his early nomadic life the decorative objects of that culture were geometric (i.e. symmetric); on the other hand the nomads' physical surroundings were "a wonderful kingdom of various curved lines and forms". But when he moved to town to go to school he found the opposite to be true. The townspeople's physical environment was predominantly straight-line geometry; in contrast the decorative objects they chose were less symmetric, more ornate and curvilinear.

I can claim no such picturesque background to my later personal perceptions of symmetry and duality. Perhaps the austerity of a Nebraska childhood in the Great Depression, and the dryness of the climate in the Dust Bowl of the 1930's had an influence, but the connection is not quite apparent to me now, and certainly was not apparent to me then. I didn't know until later that I was living through a

Depression, and thought the hot dry winds were just a constant feature of Nebraska life.

Four particular memories from my 1930's childhood suggest the beginnings of a career in geometry and symmetry. The first was a sort of game which someone bought me, consisting of a 10 inch square board with some 400 round indentations (the integer points of a Cartesian coordinate system?), and several hundred small clay "marbles" in a variety of solid colors. With these marbles placed in the indentations a colored design could be produced. Today this seems like an exact precursor of a color computer monitor screen with its pixels which can be lit up to make colored designs on the screen, but of course this was long before the advent of electronic computers, or even ordinary television screens. In the 1960's I worked for a time on finite geometries, and at that time I always felt a direct connection between the kinds of pictures one draws to illustrate finite affine geometries and the pictures I plotted, as a child, on this marble picture board.

I also had a small loom on which I made beadwork "watchfobs", belt decorations, and the like in the style of some American Indian beadwork. In retrospect these beads were also the points of a finite geometry. Other visual images of this same type came from the game boards for the game of "Chinese checkers", played with marbles on an indented board much like my "pixel board", but placed in a hexagonal, rather than square, array whose boundary is a star hexagon. Of course the universal game of tic-tac-toe was also played in Nebraska. Its nine cells naturally correspond to the nine points of the finite affine geometry having three points per line, and the much desired three in a row is one of the lines of that geometry.

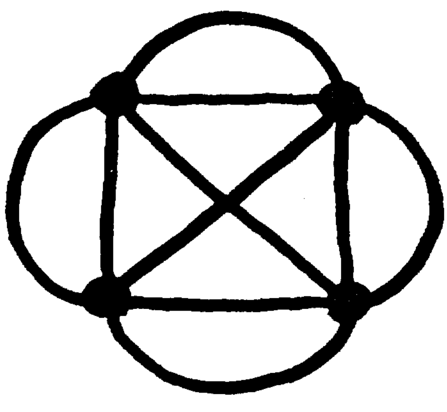


Figure 3.1: A problem from childhood: Draw this figure without lifting the pencil from the paper, or redrawing any line.

My second example is a simpler one. No elaborate commercial aids were required to while away the time in elementary school classrooms by trying to trace out an Euler circuit on the network shown in Figure 3.1! (An *Euler path* is a path containing each edge of the network exactly once. It is an *Euler circuit* if it ends at its starting point.) Most of us discovered fairly soon that if one of the diagonals was left out, or one of the outside loops, then we could find an Euler path. But so far as I know none of us ever realized just why. We were a long way from discovering Euler's result that such a path could be traversed if and only if there were exactly two (or no) vertices of odd degree. Indeed, the whole idea of "generalization" was foreign to us. We were only interested in this particular network, and never considered the possibility of devising other similar puzzles which might lead us to a general solution. Certainly we never imagined such beautiful and elaborate networks as the sand drawings of the Tchokwe people of Angola (Gerdes, 1988, 1990; Ascher, 1988b), or the *nitus* from the island of Malekula in Vanuatu (Ascher, 1988a). Figures 3.2 and 3.3 show turtles drawn by the Tchokwe and on Malekula, respectively.

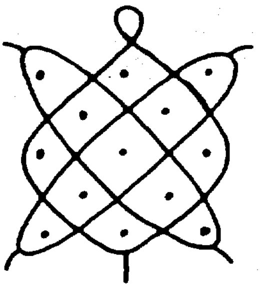


Figure 3.2: Tchokwe sand drawing of a turtle (Gerdes, 1988, p. 8)

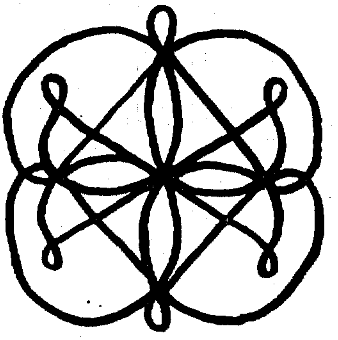


Figure 3.3: Turtle, as drawn on the island of Malekula, Vanuatu (Ascher, 1988a, p. 216)

There was, however, something about the symmetry of Figure 3.1 which appealed to us. Although the deletion of one arc to make it traversable might have made it still more appealing because of the subsequent availability of an Euler path, the resulting decrease in symmetry made it entirely unacceptable. Incidentally, I don't recall that we ever distinguished between an Euler path and an Euler circuit. Since it was only a pencil and paper problem it was a matter of indifference to us whether the path returned "home" to complete a circuit or not.

The third of these early symmetry influences was the image of long rows of upright cornstalks in cornfields on the flat Nebraska steppes. These were a common sight as we passed by in an automobile following the Platte River to visit relatives in Colorado. Of course, the cornfields had been *planted* in straight rows by tractors, these rows often being perpendicular to the direction of the highway. But I always noticed that "rows" were visible in many directions, not just at right angles to the highway, but at 45° angles, and others as well, with the angles varying depending on my line of sight through the field. In fact, I had a certain reputation for "squinting" when I thought no one was looking; I was only sighting along those unplanned rows of corn to confirm that the cornstalks really lined up.

The effect is the same as that of the regularly placed marbles on their board (in my first example), except that now the scene is viewed not *vertically* from above the board (cornfield), but *horizontally*, from the same level as the board. This experience, thus, was not a precursor of a study of finite geometries, but of the problem known as "Sylvester's Problem". This problem was originally formulated by Sylvester (1893) in terms of orchards (a comparative rarity in Nebraska at that time), not cornfields. In the interest of symmetry, he asked whether it is possible to plant the trees in an orchard so that they are all in rows, that is, so that any straight line ("row") containing two trees also contains at least one other tree. Sylvester'sproblem fascinated me from the first time I heard of it, and the mental picture I used to describe it was always in terms of cornfields. After Gallai proved that the answer to the original question in "no", one revision of the problem was to determine the minimum number of "short rows" (containing only two trees) in an orchard of *n* trees. My small contribution to that problem was made in a paper (Crowe and McKee, 1968).

When I was about 13, I found instructions in a boys' *How To Do It* book for making a "Tower of Hanoi" puzzle. This, my fourth example, is known to everyone nowadays, because it is a favorite recursion exercise in beginning computer programming courses. It consists of three pegs, on one of which is placed a "tower" formed of n disks of decreasing size. The puzzle is to move the disks from one peg to another, one at a time, and thus transfer the tower to another peg, never placing a larger disk on a smaller in the process.

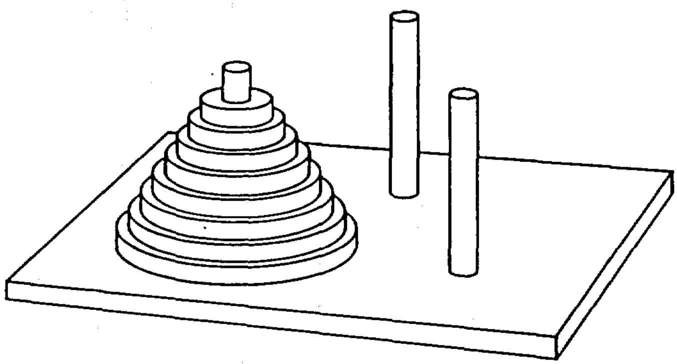


Figure 3.4 The Tower of Hanoi puzzle.

I don't think I realized then that the minimum number of moves required for a solution was $2^n - 1$, but I certainly realized that the kinetic symmetry and hypnotic monotony of the *process* of solution was very soothing and relaxing. This feature of the Tower still makes it more appealing to me than the "Chinese Rings" puzzle, which is almost equivalent from the purely mathematical point of view.

The fact that there was a direct connection between a simplest solution to the Tower and a particularly symmetric Hamilton circuit (that is, a circuit which contains each vertex exactly once) on the n-dimensional cube only became obvious to me when I was a beginning graduate student. That discovery became my first published paper (Crowe, 1956), and the wooden Tower I built at age 13 is used for lecture demonstrations to this day.

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ON ETHNOMATHEMATICAL RESEARCH AND SYMMETRY

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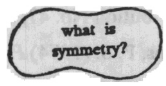
QUESTION 1



Symmetry is such an overwhelming phenomenon both in nature and in culture, that it is easy to forget to question: *why*? Why do fans all over the world have an axial symmetry (Fig. 1a)? Why are fire drills always twirled at a right angle to the drill-stick (Fig. 1b)? Why do combs normally display a bilateral symmetry (Fig. 1c)? Why do most cooking pots have a rotational symmetry? Why do many baskets, when seen from above, show a double bilateral symmetry pattern (Fig. 1d)? Why do most string figures have a line symmetry (Fig. 1e)? Why are bellows symmetrical (Fig. 1f)? Why do boats, shields, musical instruments, boomerangs take on a symmetrical shape?

At first sight one might think that symmetry arose in human culture as a blind copy of symmetry in nature. In reality however, e.g. hand axes were initially *not* symmetrical, but they became so as the result of the production traditions of thousands of generations (Frolov, 1977-78, p. 151). Rotational symmetry of order 2 is (almost) absent in nature (cf. Brew, 1946, p. 269), but frequent in human culture (see the example in Figure 2).

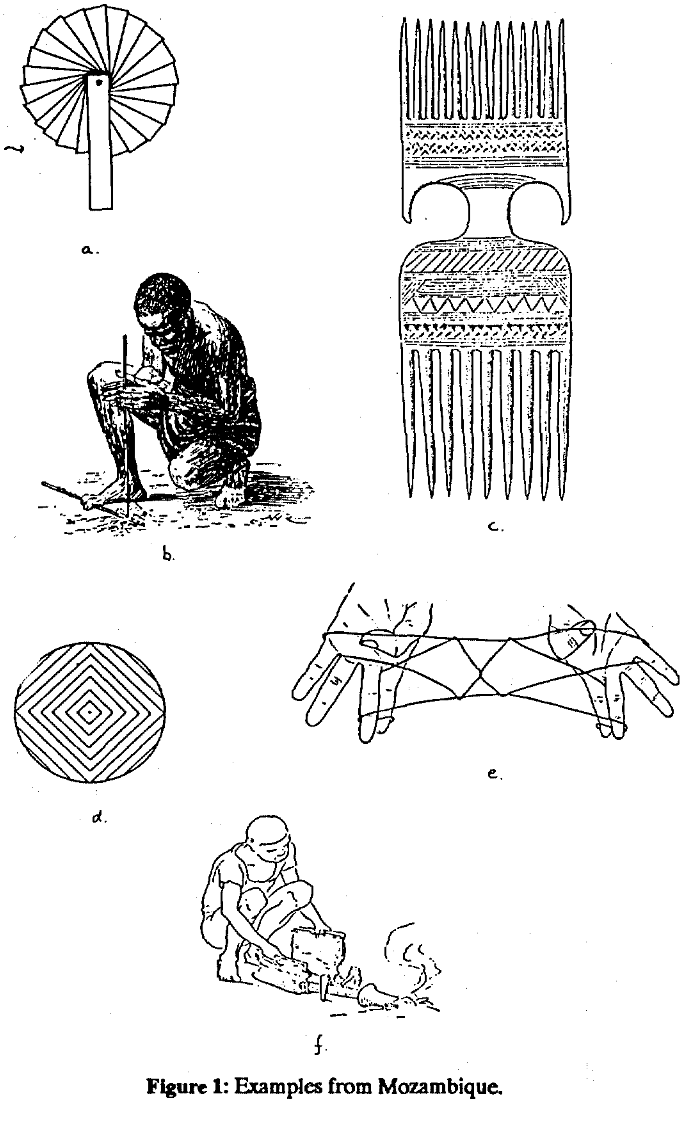


Figure 1: Examples from Mozambique.

So far, we may conclude that the question "*What is symmetry*?" cannot be answered completely in an abstract way, detached from history or from any cultural context.

The answer is changing over time. "*Thinking in terms of symmetry*" is human and it is a cultural-historical product: human beings *learn*t to think in terms of symmetry, learnt to see symmetry in their artifacts and in nature, and learnt to esteem symmetry as an esthetical value.

As there are so many different forms in nature, it has to be explained *why* man gradually became capable of observing certain forms in nature. There are no forms in nature that are *a priori* destined for human observation. The capacity of man to recognize geometrical forms in nature and in his own products has been developed through his *labour activity* ["Tätigkeit", in German] (cf. Gerdes 1985; 1988a).

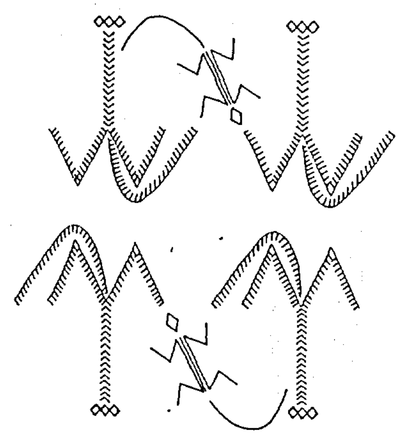


Figure 2: Traditional Makonde tattooing (northern Mozambique).

Regularity and symmetry of man-made objects are the result of creative human labour. The real, practical advantages of an invented regular and symmetrical form for an artifact lead to a growing consciousness of this regularity and symmetry. The same advantages stimulate man to compare this artifact with other labour products and with natural phenomena. The regularity and symmetry of a product generally simplifies its reproduction and in this way both the consciousness of its form and the interest in it are reinforced. The growing consciousness and interest develops at the same time a positive valuing of the invented form, and this is also used where it is *not necessary* for material, objective reasons; this form becomes experienced as *beautiful*.

The cylinder, cone, and other symmetrical forms of recipients, the regular hexagonal hole pattern of baskets, fishtraps, hats, snowshoes, etc. may *appear* at first sight as the result of instincts or of an innate feeling for these forms, or, mechanically, as the imitation of natural phenomena. In reality, however, these forms have been *created* by man in order to satisfy his daily needs. Working with the materials at his disposal, man learned to understand which were the necessary forms in order to produce something useful. Some examples will clarify my ideas.

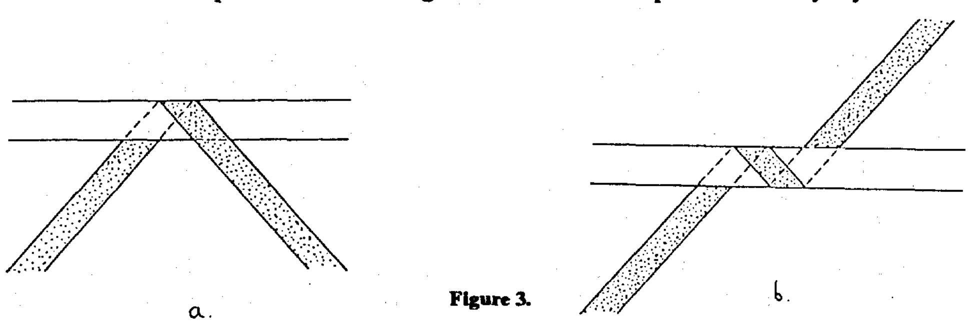


Figure 3.

Folding one strand once over a second strand, automatically an axial symmetry is obtained (Fig. 3a). Folding it twice over the second strand, one arrives immediately, and independent from human will, at a rotational symmetry of order 2 (Fig. 3b).

In the coastal zones of Mozambique fish is dried to be sold in the interior. How should the fish be dried? What happens if you place the fish around the fire like in Figure 4a. Some fish will be grilled, while others remain moist. Through experience, the fishermen discovered that it is necessary, when the wind does not blow, to place the fish *equidistant* from the fire, i.e. in a *circle* (Fig. 4b).

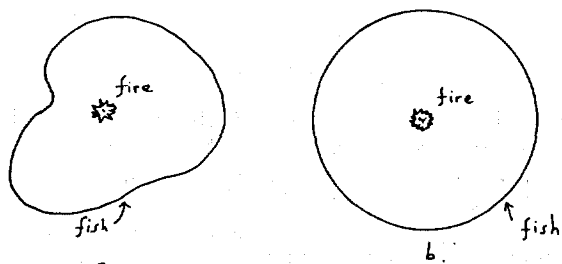


Figure 4.

The Makonde artisans in North-Mozambique weave their *chelo*-basket in the following way. In order to make its border, they bend a rectangular strip of wood and bind its ends firmly together. Automatically (independent of human will), the border becomes *circular* (=symmetrical), as the homogeneous material of the strip forces it to become so. The artisan weaves a rectangular mat and binds its sides to the border of the basket (Fig. 5a). He wets the mat and then presses it uniformly downwards with one of his feet (Fig. 5b). Finally he trims the end pieces and binds the rest of the bottom to the border. Experience has shown the artisans that the mat has to be a *square* (rotational symmetry of order 4). If it were not a square, then, for lack of equilibrium, the basket would easily fall over to one side. In order to press the mat downwards, it is necessary to bind first all its four sides to the border, and not only two or three of them. This has to be done exactly at their *midpoints*, otherwise the right angles between the strands of the mat would be unevenly distorted under the pressure of the foot which would result once more in a lack of equilibrium.

Figure 5.

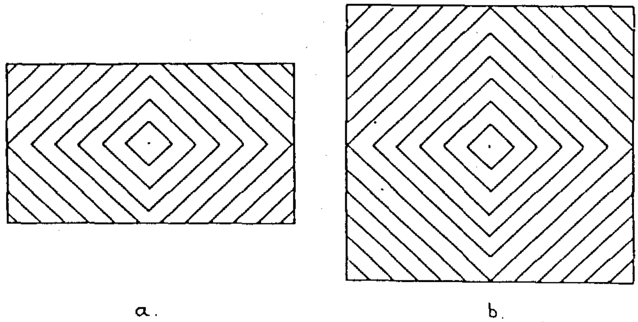


Figure 6.

How may one know if the mat is square and where are the midpoints of its sides? The Makonde artisans use a solution invented by many people. The solution avoids measuring. One weaves the mat in such a way, starting from its future center, that the weaving pattern *immediately* shows if the mat is square or not and where are the midpoints of its sides. Figure 6 gives examples. The weaving patterns have to display almost automatically a double bilateral symmetry or a rotational symmetry of order 2 or 4 (see Figure 7, cf. Figure 1d).

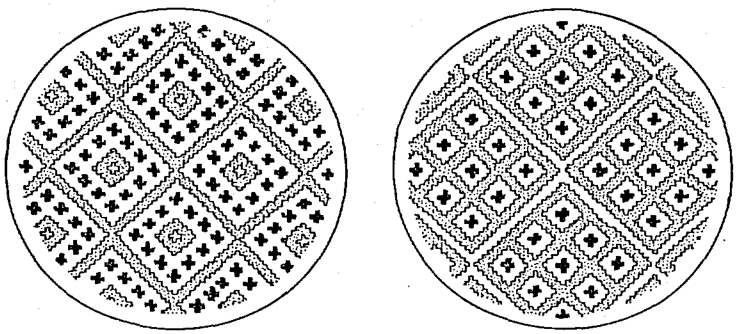


Figure 7: Examples from Guyana.

In my research I use symmetry/dissymmetry of patterns in the reconstruction of possible original patterns that have been lost with time. Some examples will clarify this heuristic role of symmetry.

During the harvest month, Tamil women in South India draw designs in front of the thresholds of their houses. In order to prepare their drawings, they set out a rectangular reference frame of equidistant points. Then curves are drawn in such a way that they surround the dots without touching them. The (culturally) ideal design is composed of a single closed line. However some of the reported threshold designs do not conform to the standard as they are composed of two, three or more superimposed closed paths. Figure 8a shows an example, made out of three separate closed lines (Figs. 8b, c, and d). The outer part of this design displays a rotational symmetry of 90° (Fig. 9a), but, on the other hand, the inner part displays only a rotational symmetry of 180° (Fig. 9b). Is this asymmetry (90°-180°) related to

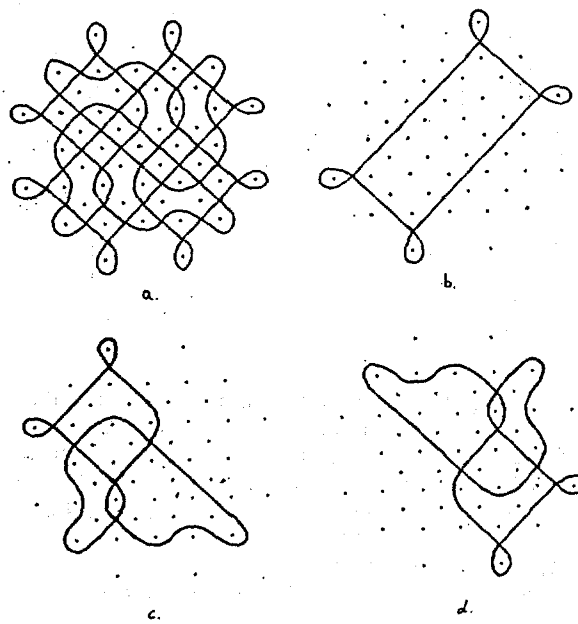


Figure 8.

the fact that the design does not conform to the cultural standard? If we adapt the inner part in such a way that it displays also a rotational symmetry of 90° just as the outer part (Fig. 10a), then we arrive at a design (Fig. 10b) rather similar to the reported design, and it satisfies at the same time the norm, as it turns out, to be composed of only one closed, smooth path. The reported plural closed line pattern (Fig. 8a) is probably a "degradation" of the reconstructed original design (Fig. 10b), a consequence of deficient transmission from one generation to another (for more details and related examples, see Gerdes, 1989).

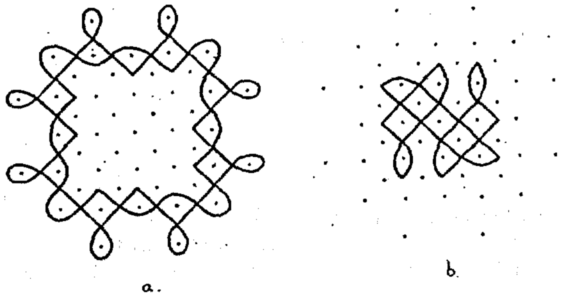


Figure 9.

Many cylindrical baskets with a square bottom display a wall decoration with exactly or almost a rotational symmetry. When such a basket has only *almost* a rotational symmetry, I pose the question "What went wrong?" For instance, during

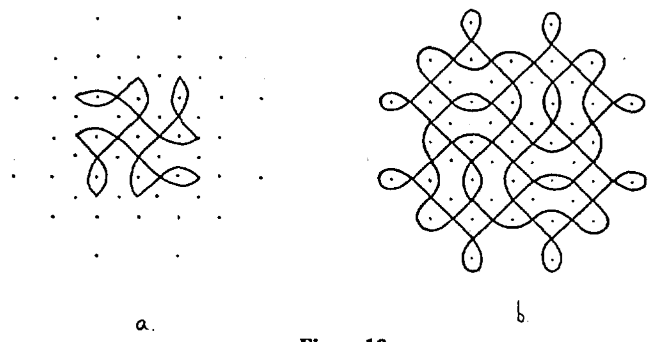


Figure 10.

my visit to Brazil, I encountered an Indian basket, that displayed on each horizontal layer of its wall 16 woven dented squares (Fig. 11a) and one dented rectangle (Fig. 11b). The dented rectangle that breaks the rotational symmetry of the basket had probably been caused by an "error of calculation", as each quarter of the bottom had two strands too much to generate the exact rotational symmetry of order 16 (for more details and analysis, see Gerdes, 1988f). In the concrete case of the basket under consideration, in order to achieve the rotational symmetry of order 16, the inventor of the ornamentation had to know that $4\times6 = 6\times4$ (*arithmetical symmetry* or commutativity).

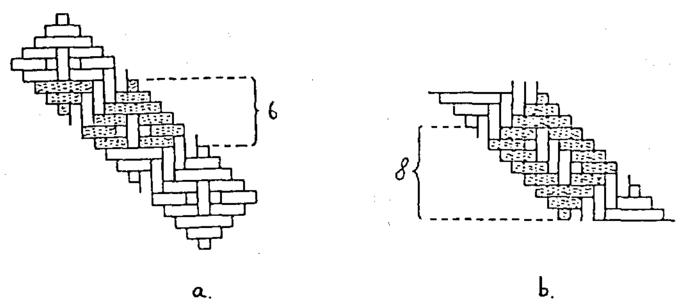
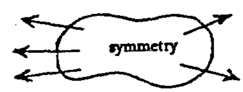


Figure 11.

QUESTION 2



In order to give an idea of the many possible and multiple links between the ethnomathematical research of symmetry and other scientific and/or cultural spheres, I will present a concrete example.

The Tchokwe people of northeast Angola are well known for their beautiful decorative art. When they meet, they illustrate their conversations by drawings on the ground. Most of these drawings belong to a long tradition. They refer to proverbs, fables, games, riddles etc. and play an important role in the transmission of knowledge from one generation to another. Just like the Tamils of South India, the Tchokwe people invented a similar mnemonic device to facilitate the memorization of their standardized drawings. After cleaning and smoothing the ground, they first set out with their fingertips an orthogonal net of equidistant points. The number of rows and columns depends on the motif to be represented. By applying their method, the Tchokwe drawing experts reduce the memorization of a whole design to that of mostly two numbers and a geometric algorithm. Most of their drawings display bilateral and/or rotational (90° or 180°) symmetries (see Fig. 12). The symmetry of their pictograms facilitates the execution of a drawing. This is important as the drawings have to be executed smoothly and continuously. Any hesitation or stopping on the part of the drawer is interpreted by the audience as an imperfection and lack of knowledge, and assented with an ironic smile. A beautiful collection of 287 different Tchokwe sand drawings has been published by Fontinha (1983).

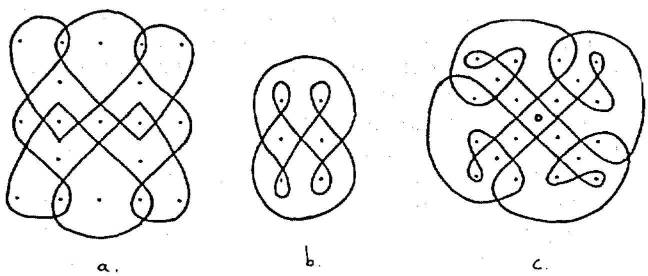


Figure 12.

The ethnographer and musicologist Kubik was very impressed by the order, regularity, and symmetry of the Tchokwe graphic tradition. He analyzed the "deep structure" of the drawings and saw a parallel with the shaping and composition of music in some regions of Africa (e.g. the Kiganda musical system in Uganda): the same type of symmetry and regularity that enable one to deduce nearly automatically the whole composition, as soon as a basic part is composed (Kubik, 1987).

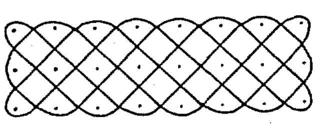


Figure 13.

In my own research on the Tchokwe sand drawings, I was initially mostly interested in the reconstruction of the mathematical knowledge that had been present at the invention of some (types) of these designs. Among other results, it came out that in order to draw one type of design (see the example in Fig. 13), made out of only one closed curve, both numbers (length and width) have to be relatively prime. This led me to the formulation of a didactical, geometrical model for the determination of the greatest common divisor of two natural numbers (see Gerdes, 1988b) and of a physical model for the determination of prime numbers (see Gerdes, 1987; 1988a). Symmetry considerations speeded up the velocity of the second model.

In the same way as in the example from the Tamils given before, symmetry considerations helped me also to find possible original designs of reported sand

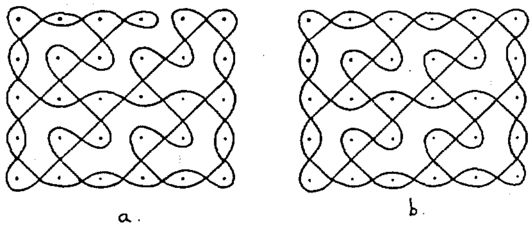


Figure 14.

drawings (Gerdes, 1988i). Figure 14a gives an example. The reported drawing lacks rotational symmetry and is furthermore composed of two closed curves instead of only one, as generally preferred in Tchokwe culture. The reconstructed design (Fig. 14b), which is made out of only one closed curve, displays a rotational symmetry of order 2 and is relatively easy to draw as the underlying geometrical algorithm facilitates the execution of the drawing. This reconstructed design is similar in structure to Figure 15, reported by Vergani (1986, p. 286), that represents the marks on the ground left by a chicken when it is chased (cf. also Fig. 15 in Ascher, 1988). Both conform to the same geometrical algorithm, only the dimensions of the reference frame have changed. In this sense, one might say that Figure 15 constitutes an *extension* of Figure 14b. In the same way, Figure 16 is an extension of the reconstructed Tamil threshold design (without border ornamentation), shown in Figure 10b. The extensions of such Tchokwe or Tamil designs display the same symmetries as the 'mother" figures and, for appropriate dimensions of the reference frame, they are composed of only one closed curve.

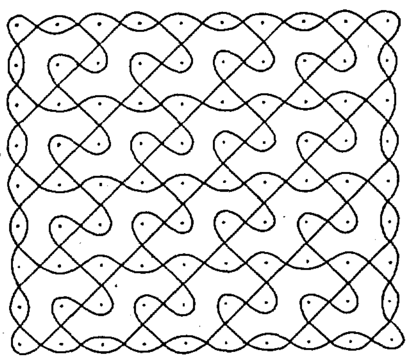


Figure 15.

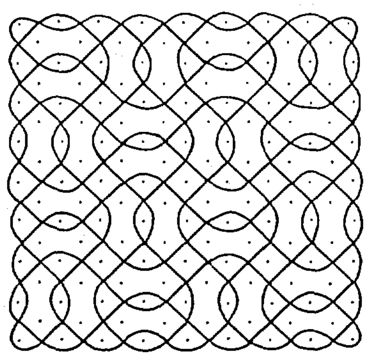


Figure 16.

The first use of these extended Tchokwe and Tamil patterns I arrived at, was a didactical one. Well known as a pedagogical tool, there are arithmetical problems of the type "*Find the missing numbers*". For instance: 2, 2, 6, 10, 22, .., 86, .., ... Which are the missing numbers? As a variant on this theme, a series of geometric problems has been elaborated: "*Find the missing figures*" (Gerdes, 1988g). Figure 17 gives an example. These problems have the objective to develop a sense for geometric algorithms and symmetry.

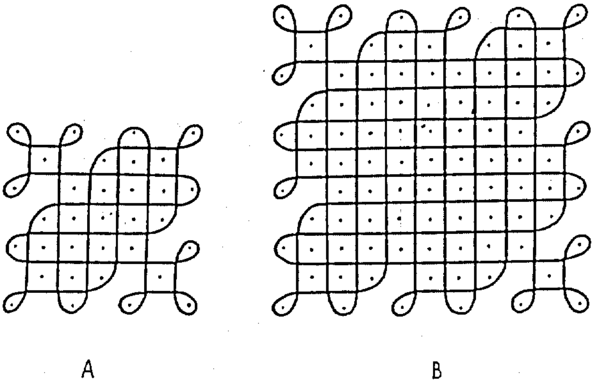


Figure 17.

As I found this type of original (reconstructed) Tchokwe and Tamil pictograms and their extended versions *beautiful*, I looked for a common *construction principle*. The curves of the type considered may be generated in the following way. Each of them is the smooth version of the poligonal path described by a lightray emitted from point *A* (see Fig. 18a). The ray is reflected in the sides of the "circumscribed" rectangle of a (basic) point reference frame. It encounters on its way through the point reference frame double-sided mirrors which are placed, at regular intervals, horizontally in the middle between two vertical-neighbour frame points and vertically in the middle between two horizontal-neighbour frame points (see Fig. 18 for an example). The underlying "mirror-patterns" of the designs display the

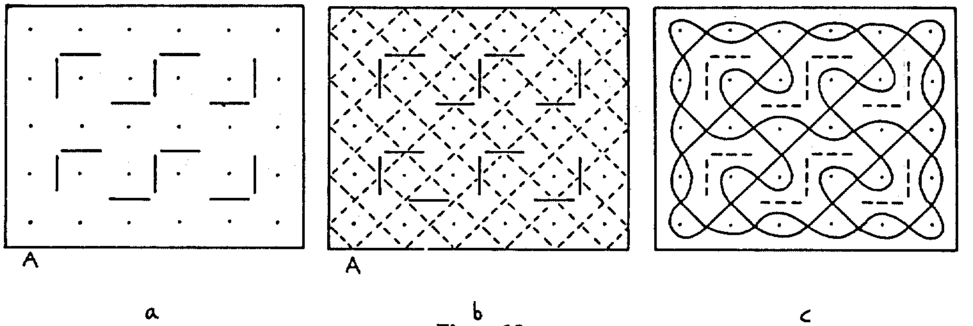


Figure 18.

same axial and rotational symmetries as the designs themselves. Once was this common construction principle formulated, it became possible to find a whole class of single closed curves that satisfy the same principle. Figure 19 gives examples. The class of curves I found in this way is attractive and interesting for many reasons. The curves are esthetically appealing. They may be used for instance in textile design. By filming them, starting the curve at one point, one sees a geometrical algorithm at work. Possibly they may be applied in the codification of information, in the development of laser memory circuits for optical computers, in the study of the topology of large scale integration chips etc., as suggested by my colleague Petrossiuk (1988).

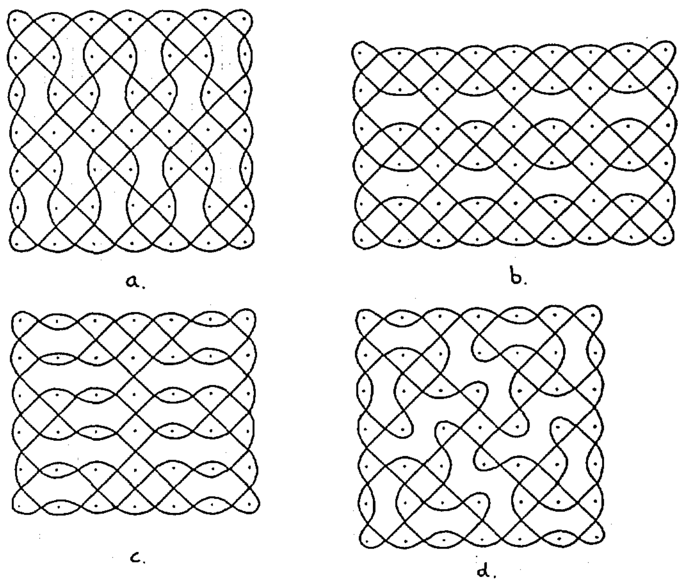


Figure 19.

Such applications become more probable if one takes into account the underlying *arithmetical symmetries* of the designs that belong to the considered class of curves. If one draws such a design on squared paper (Fig. 20a gives an example) and enumerates the squares through which the curve successively passes, modulo 4, i.e. 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, ... (see Fig. 20b) one always obtains a scheme like the one in Figure 21. These underlying numerical schemes display interesting symmetries. In the example there is a vertical line symmetry and a horizontal "semi"-symmetry: a "0" on the one side of an axis always corresponds to a "3" on the other side and vice versa; to a "1" on the one side of an axis corresponds always a "2" on the other and vice versa.

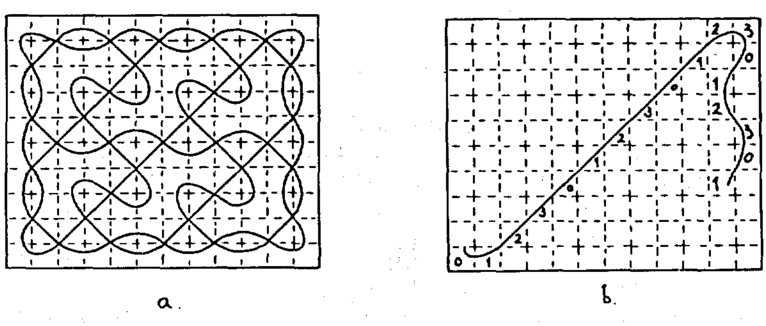


Figure 20.

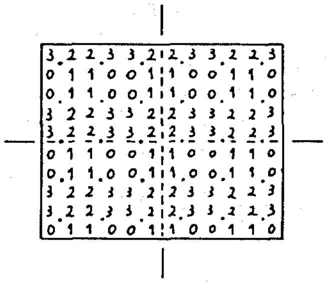


Figure 21.

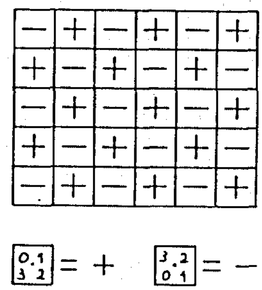


Figure 22.

Moreover the four squares around a reference point are always numbered clockwise (positive rotation) or counter-clockwise (negative rotation) 0, 1, 2, 3. Positive and negative rotations alternate as the checkers of a chessboard (see Fig. 22). When one counts the squares modulo 2, i.e. 0, 1, 0, 1, 0, 1, ... one also obtains a chessboard structure with two semi-symmetries (see Fig. 23). The involved theorems are not difficult to prove (Gerdes, 1988h).

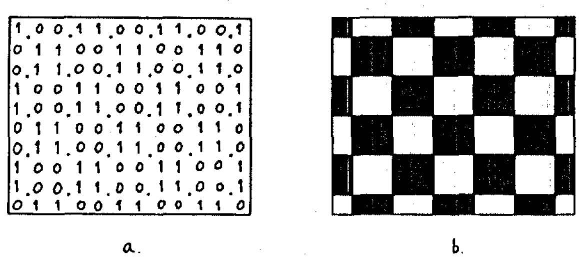


Figure 23.

These results stimulated the research of other classes of related curves. For instance, what happens if one also admits vertical mirrors in the middle between two vertical-neighbour points and horizontal mirrors in the middle between two horizontal neighbour points. Figures 24a and 25a give two examples. In the first

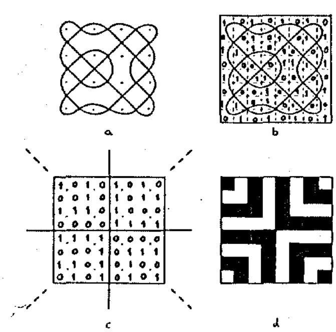


Figure 24.

case the curve displays only one (geometrical) symmetry. Its underlying arithmetical scheme modulo 2, however, has two line symmetries (diagonals) and two axial semi-symmetries (Figs. 24b, c). Figure 24d shows the geometric structure (1=black, 0=white) of the underlying arithmetical scheme. Although in the second case the curve itself displays a rotational symmetry of order 2, its underlying arithmetical scheme presents (rotationally) only semi-symmetry, but, on the other hand has a line symmetry, that was not present in the curve itself (Figs. 25b, c, d). It seems worthwhile to investigate further the inconsistencies between the symmetries of these types of curves and their underlying arithmetical structures.

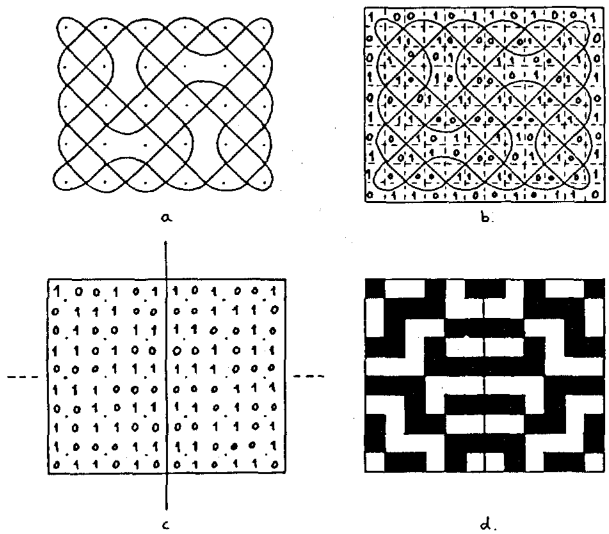


Figure 25.

Not all traditional Tchokwe sand drawings are made out of only one closed curve. Figure 26a represents the unity of a married couple. Its is composed of two overlapping curves (see Fig. 26b for one curve). The symmetry in social relations reflects itself in the symmetry of the design; the symmetry of the design expresses a "social symmetry".

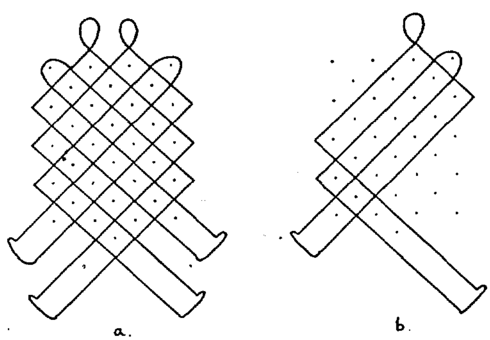


Figure 26.

The below diagram (see Fig. 27) may summarize the interdisciplinary impact of the study of symmetries in the Tchokwe sand drawings.

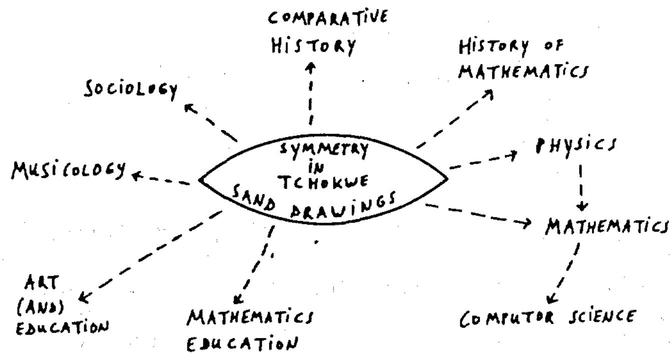
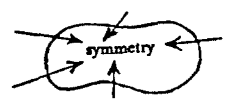
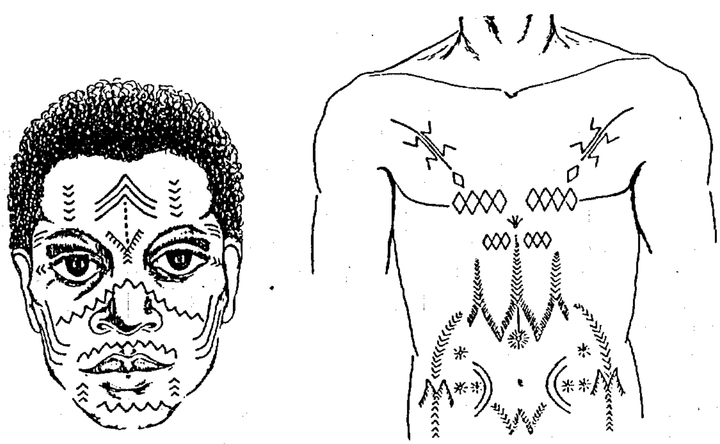


Figure 27.

QUESTION 3



In traditional Mozambican handicraft, dance, and art, symmetries play an important role (cf. Figure 1). In basketry rotational, line, and point symmetries and also strip or frieze patterns (see Fig. 29 for an example) are common. In traditional Makonde tattooings (northern Mozambique), bilateral symmetry (Fig. 28) and sometimes rotational symmetry of order 2 are used (Fig. 2). The frequency and social-cultural value of these symmetries stimulated my research in the following directions: *why* do these symmetries occur? *Why* are they culturally valued? *How* can they be incorporated in the *teaching* of symmetry in particular and of geometry in general? *How* can their mathematical potential be explored?



a. b.

Figure 28.

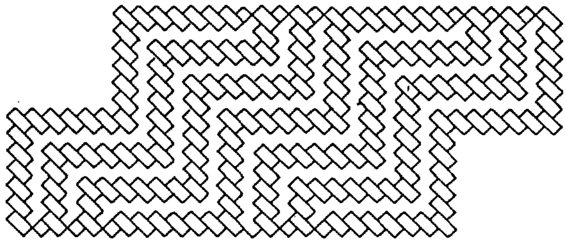


Figure 29.

My studies have been much influenced by the *changes in symmetry* that occur during the production *process* of traditional Mozambican artifacts and during the construction of houses. The transition from a parallelogram with only a rotational symmetry of 180° to a rectangle with its double bilateral symmetry, as occurs in the construction of the base of a traditional rectangular house, has been explored to formulate an alternate axiom for Euclidean geometry (for details, see Gerdes, 1988c, pp. 141-144). The transition from a square with its rotational symmetry of order 4 to equilateral triangle with its rotational symmetry of order 3 (Fig. 30), as

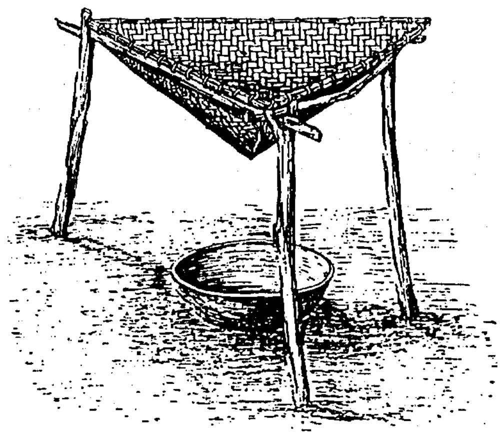


Figure 30.

occurs in the weaving of some Mozambican funnels, has inspired me to analyze similar transitions from order 2^k to lower orders (2^k-1, 2^k-2, ..., 2^{k-1}+1). A general method for the construction of regular polygons (*n*-gons) was the result (see Gerdes, 1986; 1988c, pp. 144-149). By joining four of those (congruent) funnels, one obtains a square pyramid. On the basis of the corresponding relationship between the volumes of the funnels and the square pyramid and some cultural-historical considerations, a new hypothesis on the origin of the Ancient Egyptian (!) formula for the volume of a truncated pyramid has been formulated (see Gerdes, 1985; 1988a, chap. 6).

Maybe the most surprising and beautiful result I arrived at in this context, was the invention of an infinite (!) series of new proofs of the so-called Pythagorean theorem. This discovery was "provoked" by changing the double bilateral symmetry

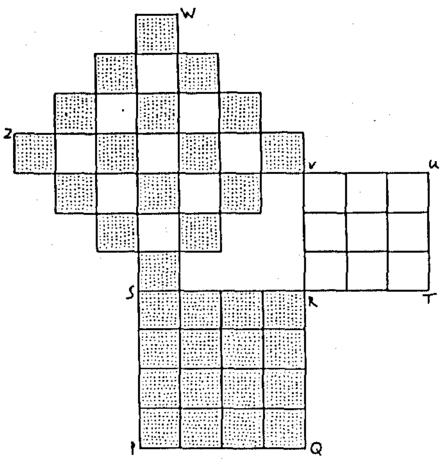


Figure 31.

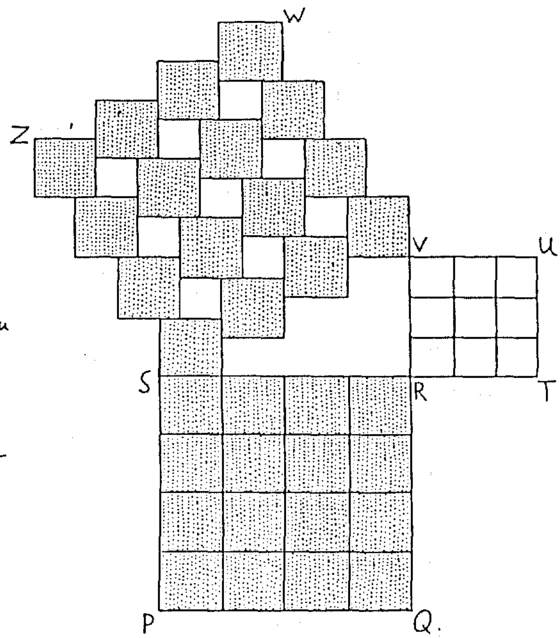


Figure 32.

of a traditional pattern (see the dented square *VWZS* with chessboard pattern in Fig. 31) into a "less strong' one, i.e. into a rotational symmetry of order 4 (see Fig. 32): sum of the areas of the squares *PQRS* and *RTUW* = area of the dented square *VWZS* = area of the real square *VWZS* (for details, see Gerdes, 1988a; 1988d; 1988e).

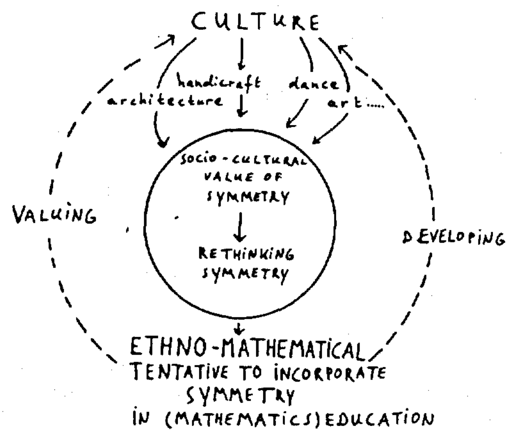


Figure 33.

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SYMMETRY: SCIENCE & CULTURE

SYMMETRY IN PHYLLOTAXIS

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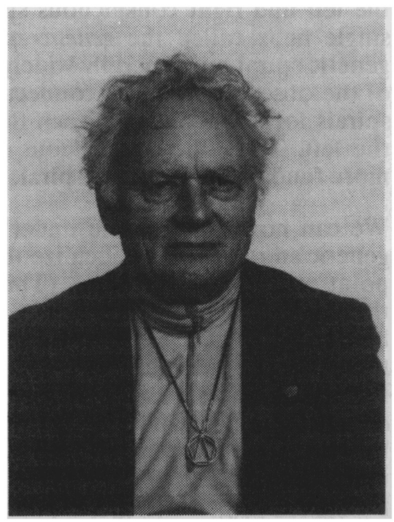
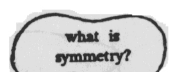
Publications: A model of contact pressure in phyllotaxis, *Journal of Theoretical Biology*, 45 (1974), 1-79; A model of space filling in phyllotaxis, *Journal of Theoretical Biology*, 53 (1975), 435-444; The consequences of contact pressure in phyllotaxis, *Journal of Theoretical Biology*, 65 (1977), 29-77; An application of the contact pressure model of phyllotaxis to the close packing of spheres around a cylinder in biological fine structure, *Journal of Theoretical Biology*, 67 (1977), 447-458; Plant spirals and Fibonacci numbers: A mathematical gold mine, *Mathematical Medley*, 12 (1984), 29-41.



QUESTION 1



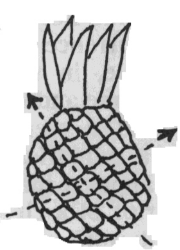


Figure 1: Conspicuous spirals on a pineapple

On a pineapple farm near Brisbane, Australia there is a one-story building intended to look like a giant pineapple. It shows clearly that a pineapple is like a living crystal: the closely packed florets on its surface are arranged in two intersecting sets of parallel spirals. One set goes up to the right. The other set goes up to the left. The architect who designed the building was aware of the fact that there is symmetry in the arrangement of the florets, since he had put 13 spirals into each of the sets, and both sets of spirals made the same angle with the ground. However, the symmetry he gave his pineapple building is wrong. In a real pineapple, the two sets of conspicuous spirals *do not* have equal numbers of spirals. In a typical pineapple, the two numbers are 8 and 13, not 13 and 13, and the angles they make with the base are not the same. The symmetry displayed by a pineapple is an *asymmetrical symmetry.*

To describe this symmetry we first idealize the picture by assuming the surface of the pineapple to be cylindrical, and we think of each of the conspicuous spirals as a helix drawn on the cylinder. A rotation of 360°/13 will bring the set of 13 spirals into coincidence with itself, and a rotation of 360°/8 will bring the set of 8 spirals into coincidence with itself. But neither of these rotations will bring both sets of spirals into coincidence with themselves. Moreover, while each of these rotations brings one of the sets of spirals into coincidence with itself, it does not necessarily bring the set of florets on them into coincidence with itself. To find a motion that will do this we have to uncover a more fundamental feature of the arrangement of the florets. The numbers 8 and 13 are relatively prime. Whenever the numbers of the left and right conspicuous spirals are relatively prime, the florets all lie on a single helix called the *genetic spiral* (Bravais, 1837). Consecutive florets on the genetic spiral are generally widely separated horizontally on the surface, so the eye of the observer does not connect them. Instead, the eye observes the conspicuous spirals formed by joining each floret to its nearest neighbors on the right and on the left. Thus, the conspicuous spirals are secondary spirals associated with the more fundamental genetic spiral.

We can now describe the underlying symmetry of the pineapple in terms of the genetic spiral. First we idealize the picture further by representing each floret by a point on the genetic spiral. (Think of it as the center of the floret.) The picture then becomes that of a point-lattice on a cylinder. The florets are arranged at equal distances on the genetic spiral. To bring a floret into coincidence with the next higher floret, it suffices to turn the surface around its axis through an angle, called the *divergence angle*, and then move the surface in a direction parallel to its axis through a distance called the *internode distance*. It is useful in the theory to normalize the surface by taking its girth as the unit of measure. In the normalized surface the divergence angle, expressed as a fraction of a turn, is numerically equal to the length of the horizontal component of the distance between consecutive florets on the genetic spiral, and is designated by d. The internode distance in the normalized surface is called the *rise* and is designated by r.

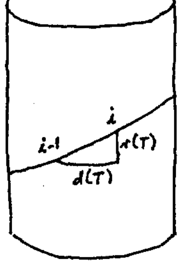


Figure 2: The genetic spiral on a normalized cylinder (girth =1).

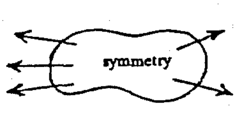
The phenomenon of units arranged in two sets of conspicuous spirals displayed by the florets of a pineapple is a common occurrence in plants. It appears also, for example, in the arrangement of the scales of a pine cone and in the arrangement of leaves around a stem. Moreover, it is not restricted to a cylindrical surface. The surface on which it appears may be approximately a disc (the interior of a circle), as on the head of a sunflower, or a parabolic surface, as at the growing tip of a stem where the embryo leaves emerge. By an appropriate conformal transformation each of these surfaces can be mapped onto a cylindrical surface, so, in what follows, we consider only the case of a cylindrical surface (Adler, 1977a, pp. 52-61). In some plants there is more than one genetic spiral. The parameters in such cases are easily related to the parameters of a single genetic spiral (Adler 1977a, p. 28), so we shall continue to restrict our attention to that case.

On a stem with a single genetic spiral, the units on it are formed one at a time at approximately equal intervals of time known as *plastochrones*. If we number the units consecutively starting with 0, and measure time in plastochrones, starting with the formation of unit 0, then unit T emerges at time T. The principal facts about phyllotaxis are these:
(1) There is a period in the early history of the stem when, as T increases, r decreases. (This simply means that during that period the girth is growing faster than the length.)
(2) In nearly all plants, the numbers of right and left conspicuous spirals are consecutive Fibonacci numbers, that is, consecutive terms of the sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., where each new term after the second is the sum of the preceding two. The ordered pair (m, n) of these numbers is called the *phyllotaxis* of the stem. As r decreases, the phyllotaxis increases, that is, it changes from $(1, 2)$ to $(3, 2)$ to $(3, 5)$, etc, where in each transition the smaller number is replaced by the sum of the two.
(3) As r decreases, the divergence angle d converges rapidly to the value g^{-2}, where g is the golden section ($g = (1 + \sqrt{5})/2$). Facts (2) and (3) are not independent. They are, in fact, equivalent.

The almost universal occurrence on plants of the Fibonacci numbers and the divergence angle g^{-2} has fascinated botanists and others and has inspired the search for an explanation. The fact that there are exceptions where the numbers m, n are not consecutive Fibonacci numbers has posed an interesting problem for model builders: find a model that can explain the almost universal occurrence of Fibonacci numbers in phyllotaxis, but will also explain the exceptions. I believe that my model has solved this problem (Adler, 1974; 1977a).

My model of phyllotaxis makes the following basic assumptions:
(1) The rise r decreases as T increases.
(2) At some time T_c the minimum distance between units is maximized and remains maximized for some time after that.
It is then proved that (*A*) as long as these two conditions hold, the transition to higher and higher phyllotaxis must follow the addition rule that if $m<n$, then (m, n) phyllotaxis must be replaced by $(m+n, n)$ phyllotaxis; and (*B*) if either $T_c<5$ or, at time T_c $r \geq \sqrt{3}/38$, then, if d at that time is between 1/3 and 1/2, the pairs (m, n), etc, must be consecutive Fibonacci numbers.

QUESTION 2



The study of the symmetry of phyllotaxis has had an impact on three other areas of study or practice: (a) elementary number theory, (b) the study of biological fine structure, and (c) the teaching of mathematics.

(a) Ever since the first systematic study of phyllotaxis was undertaken by Schimper (1830), Braun (1831, 1835) and the Bravais brothers (1837), the divergence angle has been represented by its expansion as a simple continued fraction. While investigating the connection between the divergence angle d and the numbers of conspicuous secondary spirals (m, n), I uncovered the geometrical meaning of a simple continued fraction: *it represents a mediant nest of intervals* (Adler, 1978). As is well known, for any given nest of intervals on the real line $i_1, i_2, ..., i_n, ...$, where each i_n is wholly contained in i_{n-1}, and where the length of i_n approaches zero as n approaches infinity, there is a unique real number that lies in all the intervals of the nest. A mediant nest is a special kind of nest constructed as follows with the help of the concept of the mediant between two rational numbers. If a/b and c/d are rational numbers in lowest terms, then the fraction $(a+c)/(b+d)$ is called their mediant. It is easily seen that it lies between them, and that it, too, is in lowest terms. Now begin with the positive half of the real line, and represent 0 by the fraction 0/1, and represent infinity by the fraction 1/0. The mediant between them is 1/1=1. This point divides the positive half of the real line into two intervals: the left interval is between 0 and 1; the right interval is between 1 and infinity. Choose one of these intervals as i_1, and represent your choice by either 0 or 1, according as you chose the left or right interval respectively. Then in the interval you chose, insert the mediant between its endpoints, dividing it into two intervals. Choose one of them as i_2 and represent your choice by 0 if you chose the left interval and by 1 if you chose the right interval. Continue in this way ad infinitum to obtain a nest of intervals $i_1, i_2, ..., i_n, ...$ represented by a sequence of zeros and ones. The zeros and ones in the sequence occur in clusters, with each cluster of ones ending where a cluster of zeros begins, and vice versa. Let a_1 be the number of ones in the first cluster. (It may be zero or a positive integer.) Let a_2 be the number of zeros in the next cluster, let a_3 be the number of ones in the next cluster, etc, so that a_n is a number of zeros if n is even, and is a number of ones if n is odd. Then the number defined by the mediant nest is the number represented by the simple continued fraction $a_1 + 1/a_2 + 1/a_3 + ... + 1/a_n + ...$ (in this notation we use the convention that everything that follows a fraction line / is understood to be under it). Thus every simple continued fraction can be interpreted as a set of instructions for constructing a mediant nest that contains the number that the simple continued fraction represents: begin with the positive half of the real line, and insert mediants as described above. Then construct the mediant nest by first choosing the right interval a_1 times, then choosing the left interval a_2 times, then choosing the right interval a_3 times, etc. The mediant nest associated with the simple continued fraction that represents the divergence angle d plays a significant role, as we shall see, in the connection between d and the phyllotaxis (m, n).

D'Arcy Thompson, in his famous book *On Growth and Form* (1942), after discussing the secondary spirals seen in phyllotaxis, said: "The determination of the precise angle of divergence of two consecutive leaves of the generating spiral does not enter into the above general investigation [...], and the very fact that it does not so enter shews it to be essentially unimportant." This conclusion is entirely wrong. Before the real connection between the divergence angle d and the phyllotaxis (m, n) could be established it was first necessary to make some precise distinctions among different kinds of secondary spirals. Botanists use the term *parastichies* for secondary spirals, so we shall use this term from now on. Although the terminology is botanical, the subject is really pure mathematics, since it concerns properties of a cylindrical point-lattice generated by a single genetic spiral.

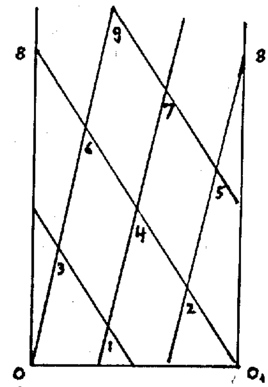


Figure 3: The opposed parastichy pair (2, 3).

We begin by slitting the cylindrical surface along the element through unit 0 and developing the surface in a plane (Fig. 3). The point 0 then appears twice in the plane development. To distinguish them, call its left-hand image 0 and its right-hand image 0_1. The entire cylindrical surface is contained in a narrow strip between 0 and 0_1. Repeat this strip over and over again to the left and to the right to form a plane point-lattice. For any given $n>0$ such that there is no lattice point between 0 and n on the line-segment that joins them, draw the line joining 0 and n. All the lattice points that are multiples of n lie on this line. For each of the integers i between 0 and n there is another line parallel to the line joining 0 and n that contains all the lattice points whose remainder is i when you divide by n. We then have n parallel lines going up to the right that contain all the lattice points. On the original cylinder they would be spirals. These n spirals are called a set of right parastichies. In a similar manner, by first joining 0_1 to any lattice point m in the original strip such that there is no lattice point between 0_1 and m on the line-segment that joins them, we determine a set of left parastichies. In this way, any two distinct positive integers (m, n) determine an *opposed parastichy pair* consisting of a set of n right parastichies and a set of m left parastichies.

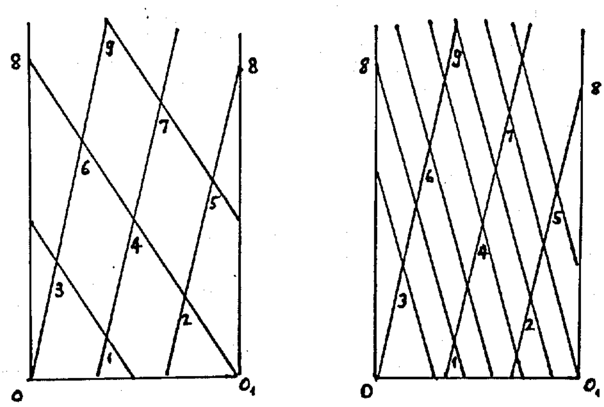


Figure 4: $d=3/8$.
(2, 3) is visible. (7, 3) is not visible.

In general, in an opposed parastichy pair, there need not be a lattice point at every intersection of a left parastichy and a right parastichy. In the special case where there is a lattice point at each intersection, we call the opposed parastichy pair *visible*. Figure 4 shows that when $d=3/8$, (2, 3) is visible, but (7, 3) is not. For any given divergence angle d, the identity of the lattice points to the right of 0 or to the left of 0_1 that are nearest to 0 depends on the value of the rise r. This is because the vertical component of the distance of a lattice point m from point 0 is mr, and, as r

decreases, this vertical component becomes negligible. Then a lattice point with a high lattice number can become the point nearest 0 if the horizontal component of its distance is small. If m and n are the lattice points nearest to 0 on the left and right respectively, then the opposed parastichy pair (m, n) is called *conspicuous*. It can be proved that a conspicuous opposed parastichy pair is necessarily visible (Adler, 1984, letter to Roger Jean). The significance of these distinctions is this: the divergence angle d determines which opposed parastichy pairs are visible, and vice versa, in a manner explained in the next paragraph, and, for a given value of d it is r that determines which visible opposed parastichy pair will be conspicuous.

To be able to explain the connection between the divergence angle d and the phyllotaxis (m, n) we must first introduce two more concepts. If (m, n) is a visible opposed parastichy pair and $m > n$, then $(m, n\text{-}m)$ is called its *contraction*. The contraction of a visible opposed parastichy pair is necessarily visible (Adler, 1974). It is important that *a contraction is not uniquely reversible*, contrary to what Tait assumed (Tait, 1872). If (m, n) is a visible opposed parastichy pair, then $(m+n, n)$ is called its left extension, and $(m, m+n)$ is called its right extension. In general, only one of these two extensions is visible, depending on what the value of the divergence angle d is: There is a certain maximal interval [a, b] in which d may be if (m, n) is visible. The mediant between a and b divides this interval into two segments. *The left extension of* (m, n) *is visible if and only if* d *is in the left segment, and the right extension is visible if and only if* d *is in the right segment*. The following propositions can be established : If the genetic spiral is a right spiral, every visible opposed parastichy pair (m, n) with $m, n > 1$ can be obtained as the end product of a sequence of extensions starting with a visible opposed parastichy pair of the form $(t, t+1)$, where t is a uniquely determined integer greater than 1. Moreover $(t, t+1)$ is visible if and only if d lies in the interval $[1/(t+1), 1/t]$. Now here is where the continued fraction for d becomes relevant. Since the terms of the continued fraction determine whether d is in the left segment or the right segment after each successive insertion of the mediant between the ends of the segment previously determined, they also determine whether the corresponding extension that is visible will be a left extension or a right extension. For example, (2, 3) is visible if and only if d is in the interval [1/3, 1/2]. The mediant between 1/3 and 1/2 is 2/5. (5, 3) is visible if and only if d is in the interval [1/3, 2/5]. (2, 5) is visible if and only if d is in the interval [2/5, 1/2]. Further visible extensions of a visible opposed parastichy pair go hand in hand with further restrictions of the range of d. The terms of the continued fraction choose in succession a left or right segment for the position of d as each of these segments is divided in turn by the mediant between its endpoints. To each segment there corresponds a visible opposed parastichy pair. The choice of left or right segment when the segment is divided by the mediant determines whether the left or right extension of the corresponding visible pair will also be visible. Since an opposed parastichy pair is conspicuous only if it is visible, the phyllotaxis of a stem is thus intrinsically tied to the value of d. (As already mentioned, it also depends on the value of r.)

The state of a system of phyllotaxis is determined by the two parameters d and r. It can therefore be represented by a point in the corresponding phase space, namely, the (d, r) plane. In my model of phyllotaxis it is assumed that r is a decreasing function of time. Then the changes the system undergoes with the passage of time are pictured as the path this point follows as r decreases. In my model I show that if the minimum distance between units is maximized, then if m and n are the units nearest to unit 0, then they must be equidistant from 0. The equation dist(0, m)=dist(0, n) is the equation of a circle. Consequently, as r decreases, the point (d, r) must descend along an arc of this circle. While it is on this arc, the phyllotaxis is (m, n) or (n, m) depending on whether m or n represents the number of left parastichies. The descent continues until another unit displaces either m (if $m<n$) or n (if $n<m$) in its role of nearest neighbor of unit 0. The condition that m and n are nearest neighbors of 0 requires that this point be $m+n$. If $m<n$, the point (d, r) then descends along an arc of the circle defined by the equation dist($m+n$, 0)=dist(n, 0), until the unit $m+2n$ displaces unit n as nearest neighbor of 0, etc. Consequently, the condition that the minimum distance between units is maximized requires the point (d, r) to descend along a zigzag path. If we let T increase without limit, the projection on the d axis of the separate arcs of this zigzag path is a nest of intervals, and d converges to the value of the unique point inside this nest. In the case where d is initially between 1/2 and 1/3, if maximization of the minimum distance begins early, that is, when either $T_c<5$ or $r\geq\sqrt{3}/38$, then, if the genetic spiral is a right spiral, the first value of the phyllotaxis along the zigzag path is either (1, 2) or (3, 2). Then, as the point (d, r) descends, the phyllotaxis increases according to the addition rule noted above with every transition from one arc of the path to the next. Thus the higher values of the phyllotaxis all consist of consecutive Fibonacci numbers. More generally, if the initial value of d is between $1/(t+1)$ and $1/t$, the first two possible values of the phyllotaxis for a right genetic spiral are $(1, t)$ and $(t+1, t)$. If maximization of the minimum distance begins when either of these two is the phyllotaxis at that moment, then the phyllotaxis, rising according to the addition rule, takes on these values in succession: $(t+1, 2t+1)$, $(3t+2, 2t+1)$, $(3t+2, 5t+3)$, etc., and d converges to the value $(t+g^{-1})^{-1}$. Since what we are talking about is the behavior of a cylindrical point-lattice, the propositions asserted above are really propositions of pure mathematics although they arose in the context of the study of phyllotaxis. In the special case where t=3, the possible values of the phyllotaxis (assuming early maximization of the minimum distance) are (1, 3),(4, 3),(4, 7), etc, and d converges to $(3+g^{-1})^{-1}$ =0.2764 approximately. Expressed in degrees it is 0.2764(360)°=99.5°. This fact is relevant to what will be discussed in paragraph b).

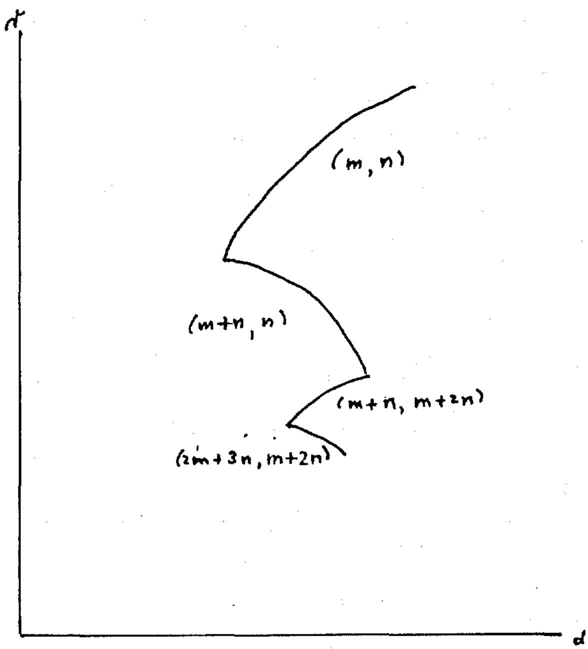


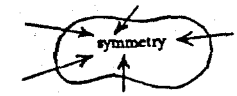
Figure 5: The path of (d, r) as r decreases while the minimum distance between units is maximized

(b) A brief note in *Nature* (Frey-Wissling, 1954) pointed out that divergence angles in a polypeptide chain are analogous to divergence angles in phyllotaxis, and can be derived from the sequence 1, 3, 4, 7, ... which is generated by the same recurrence relation as that which generates the Fibonacci sequence if you begin with 1, 3 instead of 1, 2. "The similarity", he said, "is certainly due to the same geometric cause, that is, as dense as possible an arrangement of identical objects along a helix". Erickson (1973) pursued this idea further and proposed the use of concepts and terminology of phyllotaxis to describe microscopic biological structures that are assembled from protein monomers in helical arrangements like those displayed by the close packing of equal spheres around a cylindrical surface. Using these concepts borrowed from phyllotaxis, he showed how the parameters of these structures can be calculated from observed data. He used two methods employing the notation and equations of Van Iterson (1907). The first method was based on using for the distance between the centers of two neighboring spheres the length of the straight line-segment that joins them. This method leads to transcendental equations that can be solved by an iterative procedure. In the second method, he used for the distance between the centers of two neighboring spheres the length of the helical arc that joins them on the cylindrical surface that contains them. This method leads to simple algebraic equations that are solved more directly than the transcendental equations of the first method and provides solutions that are close approximations to those obtained by the first method. The equations Erickson used in the second method, however, apply only to the triple-contact case (hexagonal packing). In 1977 I showed how this method could be made more general, so that it would be applicable to both the triple-contact case and the double-contact case (rhombic packing) (Adler, 1977b) by using the equations derived in my contact-pressure model of phyllotaxis (Adler 1974, 1977a).

(c) The study of phyllotaxis has had an important impact on the teaching of mathematics. Counting the spirals on a pineapple, a pine cone, or a sunflower , and discovering that the numbers obtained are consecutive terms of the Fibonacci sequence immediately arouses student interest in this sequence. As I have pointed out in talks to teachers in the United States, Australia, New Zealand, Hong Kong, Singapore, and Malaysia (Adler, 1984), introducing young people to the Fibonacci numbers opens the door to a host of mathematical concepts, with some suitable for every grade level. These numbers are joined by many threads to the rest of the fabric of modern mathematics. Here is a list of some of the subjects entered if the threads are followed: theory of limits, linear algebra (determinants and matrices, modules and vector spaces, spectral theory), theory of rings and fields, theory of numbers (Diophantine equations, congruences, continued fractions, theory of primes), differential equations, theory of linear recurrence relations, combinatorial analysis, and theory of functions of a complex variable.

The Fibonacci numbers also provide students with many opportunities to make their own discoveries. For example, even students in the elementary grades can discover for themselves the formula for the sum of the first n consecutive Fibonacci numbers, and the formula that, for any three consecutive Fibonacci numbers, connects the product of the first and third number to the square of the middle number.

QUESTION 3



Although the botanical literature usually credits Charles Bonnet (1720-1793) with being the first to notice and study the spiral arrangement of leaves, his observations and his theory were both anticipated by Leonardo da Vinci (1452-1519). So the study of phyllotaxis has a history of at least five hundred years. During this long period of time many different theories have been proposed to explain the occurrence of spirals and Fibonacci numbers on plants. Each of them shows the influence of some particular current of thought present at the time the theory was advanced. We illustrate this fact by describing briefly some of the theories and the cultural influences that they reflect.

Leonardo da Vinci observed that "nature has arranged the leaves of the latest branches of many plants so that the sixth is always above the first," while the first five branches "come forth in five different directions". This arrangement, he says, has two uses: it gives the lower branches access to water and to light coming from above (MacCurdy, 1955, p 30). Bonnet's theory is essentially the same (Bonnet, 1754). This explanation has two aspects. It is *functional* in that it proposes that the arrangement of leaves or branches is *useful* to the plant, and it is *teleological* in that, in addition, it assumes that the arrangement was designed by God or nature to serve that purpose.

Kepler observed the frequent occurrence of the number five in plants, as for example, in the five parts of the ovary of an apple. He explained its occurrence by the fact that 5 is a term of the Fibonacci sequence which is generated by a recurrence relation. He said, "I think that the seeding capacity of a tree is fashioned in a manner similar to the above sequence propagating itself" (Ludwig, 1896). Thus, he thought that the Fibonacci sequence is propagating itself, rather than being propagated by us through our choice of one of many possible recurrence relations. Then, because he attributed to the numbers the vital property of propagating more numbers, this property of the number five became his explanation of the ability of the plant to produce seeds. This is pure *Pythagoreanism*, similar to his attempt to explain the distances of the planets from the sun by a scheme of circumscribed regular solids whose number also happens to be five.

For Schimper and Braun the divergence angles in plants were always rational numbers from the sequence 1/2, 2/3, 3/5, 5/8, ... These occur, Braun said, because they are convergents of the simplest possible continued fraction, one whose terms are all equal to one (Braun, 1835, p. 158). This theory was correctly characterized by Sachs (1906, p. 162) as an example of *Platonism*: "Here too we have the idealistic conception of nature, which refuses to know anything of the causal nexus, because it takes organic forms for the ever-recurring copies of eternal ideas, and in accordance with this platonic sphere of thought confounds the abstractions of the mind with the objective existence of things."

Wright (1873) and Wiesner (1903), influenced by Darwin's theory of evolution, returned to the functional theory of Bonnet, but with them the usefulness of the spiral arrangement of leaves was presumably explained by *natural selection* instead of design. This explanation was rejected by Thompson (1942, vol 2, p. 933): "...if it be (so to speak) Nature's object to set them farther apart than they actually are, to give them freer exposure to the air or to the sunlight than they actually have, then it is surely manifest that the simple way to do so is to elongate the axis, and to set the leaves farther apart, lengthways on the stem. This has at once a far more potent effect than any nice manipulation of the 'angle of divergence'." Airy (1873), whose paper was presented to the Royal Society by Charles Darwin, also claimed that natural selection explained the spiral arrangement of leaves, but gave a different explanation of what it was selecting for. The usefulness of the Fibonacci pattern of phyllotaxis, he said, was found, not in the mature stem, where the leaves are widely separated, but in the bud, where the embryo leaves are closely packed: "In the *bud* we see at once what must be the use of leaf-order. It is for *economy of space*, whereby the bud is enabled to retire into itself and present the least surface to outward danger and vicissitudes of temperature."

Schwendener, influenced by the obvious successes of mechanics in physics, put forward a mechanical theory of leaf arrangement (1878) in which the convergence of the divergence angle to g^{-2} was explained by the contact pressure that leaf primordia exert on each other. Unfortunately, his mechanical transposition to botany of concepts derived from mechanics didn't succeed because his argument based on a force diagram was fallacious (see Adler, 1977a, p. 50).

Meanwhile a new branch of physics was coming to the fore, namely, thermodynamics. In this branch, attention was centered on the flow of energy, rather than on force diagrams. In the new intellectual climate created by thermodynamics, Church developed his theory that is based on assumed pulses of energy. He rejected the idea of a genetic spiral, and insisted instead that the parastichies are fundamental. Using the disc picture you get when you take a cross-section of the growing tip of a stem, he said that impulses of energy travel away from the center of the disc in spiral paths, and that new leaves emerge where the spirals intersect (Church, 1904, 1920). While his theory uses terminology borrowed from physics, it clearly reflects the influence of the *vitalist* school of thought in biology.

Meanwhile attention in biology was shifting from anatomy and histology to biochemistry. In this new climate of thought Schoute (1913) advanced the hypothesis that the initial position of a leaf primordium is determined by the action of an inhibitor secreted by the primordia already present. The inhibitor prevents another primordium from emerging too close, and the location of a new primordium is determined by the inhibition emanating from the two nearest primordia already growing. Whether such an inhibitor exists is still an open question.

My own thinking which led to the construction of a mathematical model for a contact-pressure theory of phyllotaxis was influenced by the philosophy of dialectical materialism. Because of my materialist outlook, I sought an explanation in terms of cause and effect. Because of my dialectical outlook, I looked for the internal contradiction in the dynamics of growth of a stem that might provide a clue to what happens. I found this contradiction in the fact that, as Schimper (1830, p. 25) had pointed out, while leaf primordia tend to separate as far as possible, they are also constrained to grow toward each other. Neighboring primordia grow toward each other until they make contact. After that, further growth compels their centers to move apart. But, since they are confined to a finite space, the distance between them ultimately becomes maximized. The essential content of my model is the rigorous determination of the consequences of the maximization of the minimum distance between primordia (Adler, 1977a).

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SYMMETRY AND ASYMMETRY IN PSYCHOLOGY

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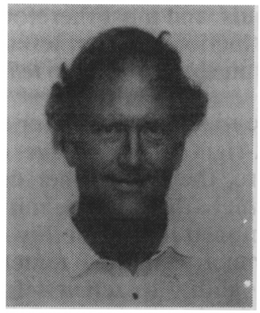


QUESTION 1

what is symmetry?

To a psychologist, symmetry generally means *bilateral* symmetry. Animals, including humans, are for the most part symmetrical about the sagittal plane, especially with respect to external bodily shape, and the structure of the brain and central nervous system. This must surely have consequences for their behavior and perhaps even their thought processes.

The simplest of these consequences follows from mechanical considerations: an animal that was *perfectly* bilaterally symmetrical could not tell left from right. In order to show this, it is important first to make clear what it means to be able to "tell left from right". There are two ways in which an animal might demonstrate such an ability. One is to give systematically different outputs to inputs that are left-right mirror-images of one another, with the proviso that the *outputs* are themselves *not* mirror-images of one another. Examples include the Pavlovian dog that salivates when touched on the left side but not when touched on the right side, or the Skinnerian pigeon that pecks a disk when it displays a 45-degree angled line but does not peck when it displays a 135-degree angled line, or the child learning to read who calls the lowercase letter *b* a "bee" and the lowercase letter *d* a "dee". This category of tests is known as *mirror-image discrimination* (Corballis and Beale, 1970).

The second way to demonstrate the ability to tell left from right is to systematically give a rightward output to one input, and a leftward output to a different input. Again, one must include a proviso, which is that the *inputs* are *not* left-right mirror-images of one another. Examples that meet the test include the dog that holds up the right forepaw when a bell rings and the left forepaw when a buzzer sounds, or the soldier who turns right on the command "Right turn!" and left on the command "Left turn!", or the child who correctly writes a *b* when told "bee" and a *d* when told "dee". This category of tests has been called *left-right response differentiation* (Corballis and Beale, 1970).

In mirror-image discrimination, the inputs are left-right mirror-images but the outputs are not, so that the animal must *decode* the left-right aspect, or parity, of the inputs in order to give the correct outputs. In left-right response differentiation, the left-right information in the outputs is not present in the inputs, and must therefore be supplied by the animal. This is an act of *encoding* the distinction between left and right. In both cases the animal is demonstrating, by its outputs, the ability to *tell* left from right.

The reader may wonder at the exclusion, from both definitions, of the case in which left-right mirror-image outputs are produced to left-right mirror-image inputs. Here, there is neither decoding nor encoding of the left-right information. The animal that gives mirror-image outputs to mirror-image inputs no more demonstrates the ability to tell left from right than does a billiard ball that responds symmetrically to symmetrical impacts. Similarly, an animal does not demonstrate any ability to tell left from right by following a winding track, or scratching an itching leg, or flicking away an annoying fly with its tail. By the same reasoning, the child who merely copies script cannot be said to demonstrate any ability to read or write.

An animal that was perfectly bilaterally symmetrical could accomplish neither test of the ability to tell left from right. The best way to see that this is so is by means of what I like to call the *mirror test.* Suppose we have a perfectly symmetrical animal that *can* tell left from right; say it always lifts its right forepaw in response to a bell and its left forepaw in response to a buzzer. But left us now observe its behavior in a mirror. The animal is exactly the same, since it is perfectly symmetrical. But now we see that it is lifting its *left* paw in response to the bell and its *right* paw in response to the buzzer, which contradicts the original assertion that it always does the opposite. We must therefore conclude that the original behavior is impossible. A moment's reflection, so to speak, should convince the reader that this demonstration holds for any test of mirror-image discrimination or left-right response differentiation.

Readers may also like to amuse themselves by constructing perfectly symmetrical devices that *can* tell left from right by the criteria given above. I am willing to bet that they will be unable to do so, but the exercise of trying should help convince them of the logic of the argument.

Notice that the argument is a *mechanical* one, and depends on the assumption that an animal, even a person, is a purely mechanical entity. If Descartes was correct in asserting a nonmaterial basis for mind, at least in human beings, then perhaps it might be possible to transcend the mechanical restriction imposed by bilateral symmetry. But I do not think so. Perhaps this is a devious way of saying that I think that Descartes was mistaken.

Moreover, I do not think that the argument is merely specious. Most animals *do* have difficulty with tasks that meet the definition of telling left from right, and so do many humans (see Corballis and Beale, 1970, 1976, for reviews). It is a common observation that young children have difficulty telling left from right, and this is very often manifest in their left-right confusions in learning to read. It is therefore very tempting to conclude that difficulty in telling left from right is a consequence of the near bilateral symmetry of the body and nervous system. It is also of interest that it is very often difficult to *learn* to tell left from right. This suggests that the mechanisms of learning and memory may be such as to preserve bilateral symmetry in the face of asymmetrical experience. I have speculated elsewhere on how this might be accomplished (Achim and Corballis, 1976; Corballis and Beale, 1970, 1976).

In general, the bilateral symmetry of organisms is adaptive. It is achieved despite the fact that the *molecules* of living matter are *asymmetrical*, which prompted the biologist Jacques Monod (1969, pp. 16-17) to declare our outward bilateral symmetry to be "something of a fake." But I think it is not so much a fake as an adaptation to the fundamental indifference of the environment itself to left and right, an adaptation that overrides the asymmetry of the molecular building blocks. The French physician Bichat (1771-1802) long ago formulated what he called the "laws of symmetry", noting that the organs that serve "external relations" (including the sensory organs and limbs) tend to be organized in symmetrical pairs, while the organs of "organic life" (heart, stomach, etc.) tend to be asymmetrical. The reason for the symmetry of the organs of external relations was that the organism needs to be able to react to the environment with both sides of the body (Bichat, 1805).

Bilateral symmetry probably evolved first as a consequence of movement. Linear movement is most efficiently accomplished by limbs that are arranged in bilaterally symmetrical pairs. So it is that legs, wings, and fins and flippers are nearly always symmetrical, even where they have evolved independently. Given freedom to move, there is then in general no bias favoring one or other side of the body, so that sensory organs are also symmetrically located. As Martin Gardner (1967, p. 70) puts it: "The slightest loss of symmetry, such as the loss of a right eye, would have immediate negative value for the survival of any animal. An enemy could sneak up unobserved on the right!"

The reader might be tempted to think that the inability to tell left from right, a consequence of bilateral symmetry, might be a disadvantage. However this is probably true only in the esoteric and unnatural world of human beings. The natural world seldom if ever poses problems involving telling left from right. Quite to the contrary, it is generally much more important to treat left and right as *equivalent* rather than as different. Predator or prey are as likely to appear on one side as the other; if one is attacked from the left, it is as well to be equally alert to possible future attacks from the right. The face or body of another animal might be seen first in one profile, but it would be advantageous to recognize it if appears subsequently in the opposite profile.

There are possible exceptions. Animals that migrate might need to make absolute decisions about left and right in deciding which way to go, although it is not clear that this asymmetrical information need be coded structurally in the animals themselves. Directions may be specified environmentally by prevailing winds, configurations of stars, the direction of the sun, or specific landmarks. But the most obvious exceptions have to do with the human environment. The child learning to read must be able to tell left from right, at least in those cultures where script is written in a particular left-right direction. Human conventions such as driving a car or shaking hands require knowledge of which is left and which is right. The mirror test can be applied here: if you observe a group of animals other than humans in their natural habitat, there will generally be no way to determine whether the scene is viewed directly or through a mirror. But if you observe humans in human environments, it is usually easy to tell whether the view is veridical or mirror-reversed.

So far, I have dwelt on the implications of bilateral symmetry for the ability, or inability, to tell left and right. There may be another implication, to do with the salience of visual patterns that are themselves bilaterally symmetrical. In the seventeenth century, Blaise Pascal wrote that "Symmetry is what you see at a glance" (Stewart, 1950, p. 491). Ernst Mach (1893/1986, p. 94) later made it clear that this applies to left-right (or vertical) symmetry rather than to up-down (or horizontal) symmetry:

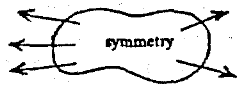
> The vertical symmetry of a Gothic Cathedral strikes us at once, whereas we can travel up and down the whole length of the Rhine or Hudson without becoming aware of the symmetry between objects and their reflections in the water. Vertical symmetry pleases us, whilst horizontal symmetry is indifferent, and is noticed only by the experienced eye.

Mach's observation was proven in more experimental fashion by Béla Julesz (1969), who showed that the symmetry of computer-generated patterns of dots that are random except for their symmetry is much more obvious if the symmetry is about a vertical axis than about a horizontal one.

I suspect, as Mach and Julesz did, that the perceptual salience of left-right symmetry is consequence of the bilateral symmetry of our own brains and bodies. No doubt it also has an adaptive significance. Many objects are bilaterally symmetrical or are nearly so, and detection of that symmetry may aid object recognition. In particular, the faces of other animals or people are bilaterally symmetrical when viewed frontally, and detection of the symmetry effectively halves the amount of detailed processing that must be accomplished. An axis of symmetry may also serve as a heuristic for discovering the top-down axis of an object (Marr, 1982).

In summary, the bilateral symmetry of animals, including ourselves, is an adaptation to the left-right indifference of the natural world in which we live and move. Notwithstanding the asymmetries observed at the molecular level, or in those fundamental nuclear forces thought to disobey the so-called law of conservation of parity, there *is* conservation of parity at the gross level at which the natural environment impinges on animals. Bilateral symmetry is a consequence of this, and in turn constrains the behavior of animals. Bilateral symmetry is adaptive because it allows animals to move linearly from one point to another, to be equally alert to events on either side of their bodies as they do so, and to be sensitive to the symmetry of other animals.

QUESTION 2



To some extent, the implications of bilateral symmetry for behavior and thought have been overshadowed in recent years by the remarkable obsession with lateral *asymmetry*, especially in humans. The most striking manifestations of that asymmetry are, first, that most humans are right-handed and, second, that an even greater majority have speech and language skills represented primarily in the left cerebral hemispheres. This left-hemispheric advantage is offset by right-hemispheric superiorities for certain nonverbal functions, such as those having to do with the encoding of spatial relations (see Corballis, 1983, for a review). But this obsession with asymmetry is no doubt dependent in part on the very symmetry that, in effect, gives it meaning.

Immanuel Kant (1783/1953, p. 42) wrote: "What can more resemble my hand or my ear, and be in all points more alike, than its image in a looking glass? And yet I cannot put such a hand as I see in the glass in the place of its original [...]" This illustrates what might be called the paradox of mirror images: structurally, they are the same, in that every point on one is matched by an equivalent point on the other. Yet they are also in a sense opposites. I recall an uncle of mine who possessed a mirror-image corkscrew and used to watch in amusement as unsuspecting guests tried to use it to extract a cork from a bottle. It *looked* like a normal corkscrew, but in order to make it work one had to turn it in the *opposite* direction. This paradox of sameness and opposition may partly explain why left and right have exerted such a fascination, and served as a potent source of myth and superstition.

The asymmetry boom in psychology began in the early 1960s with the studies of the so-called split-brain patients, who had had the fibers connecting the two sides of the brain cut in order to alleviate intractable epilepsy. With skilful testing it was possible to test the functional capacities of each side of the brains of these patients independently of the other, and so reveal something of the extent of the differences between them — an enterprise for which Roger W. Sperry (1974) was awarded the Nobel Prize. Many commentators began to insist that the two sides of the brain were not merely functionally different, they were complementary-functional *opposites*. Thus the left hemisphere was seen as rational, analytical, and logical, the right as emotional, holistic, and intuitive (e.g., Ornstein, 1972). This duality spread well beyond the confines of neuropsychology, and influenced thinking in such diverse fields as education, anthropology, and the creative arts. The terms "left-brained" and "right-brained" are now commonly used in everyday language.

This notion of hemispheric duality may owe more to the paradox of mirror images than to the evidence, which in fact implies a good deal of functional similarity between hemispheres. The two hemispheres are taken to represent the two opposite "sides", as it were, of human nature. The metaphor even gained a geographic dimension, with the left brain associated with materialistic Western culture and the right with the more spiritual East.

The symbolic potency of the left and right hands, or of the left and right sides of the brain, may be due to more than just the paradox of mirror images, however. At least, the French social anthropologist Robert Hertz (1909) seemed to have rather more in mind when he exclaimed: "What resemblance more perfect than that between the two hands, and yet what a striking difference there is!" (cited in translation by Needham, 1973, p. 5). It was not just that the hands were mirror images, it was also that they were *functionally* so different. Try asking a person to write, or to throw a ball, with each hand in turn, and the difference in function will surely be apparent. This functional difference belies the close *structural* similarity of the hands.

And so it is with the sides of the brain. They *look* like mirror-images of one another, and yet in the great majority of people one side can produce speech and the other cannot. The paradox is again not just that of mirror images, it is also that of function versus structure. Part of the fascination of this paradox lies in the suggestion of a nonmaterial basis for functional asymmetry that transcends the material symmetry — a surreptitious (and probably mostly subconscious) appeal to Cartesian metaphysics (Corballis, 1980).

The appeal to Descartes goes further. Descartes thought that the influence of a nonmaterial soul was unique to humans, and that other animals were mere automata. Human uniqueness was manifested in the extraordinary flexibility of human language, and in the exercise of free will. Functional laterality also seems uniquely human, for although other animals do exhibit functional asymmetries they do not seem so marked or distinctive as those in humans. Moreover, the functions for which laterality is most apparent, namely, manual skill and language representation, are precisely those that seem to distinguish humans from other animals. Only humans, it is said, possess true language (e.g., Chomsky, 1966), and our extraordinary mechanical inventiveness may reasonably be traced to our ability to fashion things with our hands — or to fashion things that fashion things with our hands. (Like language, manufacture also has a recursive quality.)

It is therefore very tempting to see in lateral asymmetry a basis for the Cartesian idea that humans uniquely possess a nonmaterial soul that transcends physical structure. But where Descartes saw the soul as operating through the pineal body, it now seems more appropriate to attribute a uniquely human consciousness to the left hemisphere of the brain. Aspects of this idea are more or less explicit in the writings of Jaynes (1976), Eccles (1965), and Popper and Eccles (1977). But I believe that it has been a powerful *implicit* source of fascination, and may have added to the popularity of ideas about left-brain/right-brain in everyday culture as well as in neuroscience.

However if laterality holds the key to human uniqueness, I do not think it does so through any nonmaterial intervention. Rather, I think that humans may have discovered a principle of *generativity* that is in most (but not all) people mediated by the left cerebral hemisphere. This is indeed the main feature that distinguishes human language from other forms of communication between animals; humans can use the rules of language to produce and understand an essentially unlimited number of sentences. Probably every sentence in this essay will be new to the reader, yet I hope my meanings are generally clear. The manufacture of objects has a similar generative property; humans have an extraordinary capacity to make *new* things, and to comprehend what new things made by others are for.

There are reasons to believe that this generativity may carry over even to visual perception. Biederman (1987) has suggested that recognition of visual objects is accomplished by segmenting them into parts, which he calls *geons*. He argues that about 36 geons are sufficient to generate, in idealized form, virtually all the objects we know. There is an obvious parallel here to the human ability to generate meaningful words and sentences from the small number of elementary speech sounds known as *phonemes*. Only about 44 phonemes are necessary to create the corpus of utterances in American English.

There is some evidence that people's ability to form visual mental images is largely, if not exclusively, under the control of the left cerebral hemisphere (Farah, 1984; Kosslyn, 1987). More recently, Kosslyn (1988) has qualified this conclusion by suggesting that the right hemisphere *can* generate visual mental images, but only in a holistic manner; the left hemisphere, by contrast, does so in piecemeal fashion, by arranging component parts into a whole. This again suggests a left-hemispheric basis for *generation*.

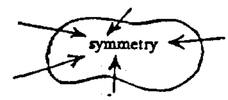
A generative mode of representation may have come about initially simply because the early hominids gradually evolved the ability to manufacture a diversity of objects. Generativity is a powerful heuristic, since it allows the description, representation, or construction of an enormous variety of composites, given only a limited vocabulary of elements. For example, Biederman (1987) calculates that from a total pool of only 36 geons, and choosing only three at a time, one can construct about 154 million possible three-geon objects! Similarly, there is virtually no limit to the number of new words that can be coined from the limited number of phonemes at our disposal, and many of these words are of course required to name new objects.

A generative mode of representation may have limitations, however, especially when it comes to capturing subtle properties. For instance, one may describe a human face in words, or one may represent it visually in terms of Biederman's standardized geons. But such representations are more appropriate for describing faces in general rather than a particular person's face. In capturing the subtleties of a particular face, a picture is worth a thousand words. It is beyond the scope of representation by geons.

I therefore think it plausible to suppose that the *right* hemisphere may have retained a holistic, iconic mode of representation that may be more appropriate for naturalistic stimuli, but relatively inflexible for the representation of manufactured objects. So there may be some truth after all to the idea of the right hemisphere as predominantly holistic, with the left as predominantly analytic and generative. However I have no wish to suggest that the differences between the hemispheres owe anything to a nonmaterial component. I suspect, moreover, that the differences are not absolute. A generative mode of representation may have been superimposed on a left hemisphere that, in our primate forebears, may have sustained a mode of representation basically the same as that of the right.

In summary, it is asymmetry rather than symmetry that has had the stronger centrifugal influence, coloring many aspects of human life and culture. But the symbolic potency of human asymmetry owes much to the symmetry from which it derives. The functional asymmetry of hands and brain belie their structural symmetry, and stand in contrast to the lack of comparable asymmetries in other animals. Of course these comparisons should not be exaggerated; the human brain does exhibit consistent structural asymmetries that seem to have some correlation with functional symmetries (e.g., Geschwind and Levitsky, 1968; Galaburda *et al.*, 1978), and nonhuman animals show some functional asymmetries that bear at least some resemblance to those in humans (e.g., Denenberg, 1981; Hamilton and Vermeire, 1988). Even so, human functional asymmetry is sufficiently singular to suggest that it may well hold the key to our humanity.

QUESTION 3



Although concepts derived from the study of cerebral and manual asymmetry in humans have influenced our culture in diverse ways, the influence has not been all unidirectional. At least some of the dichotomies that have been linked to the two sides of the brain are old, and predate the discovery of cerebral asymmetries. The sides of the brain have served as a modern vehicle for pre-existing and often ancient polarities.

Before the discovery of cerebral asymmetry, it was the two *hands* that attracted opposite poles of various dichotomies. In the Pythagorean Table of Opposites, recorded by Aristotle, the right was associated with the limited, the odd, the one, the male, the state of rest, the straight, the light, the good, and the square, while the left was associated with the unlimited, the even, the many, the female, the moving, the curved, the dark, the evil, and the oblong (Lloyd, 1962). Very similar associations are found among contemporary tribes of Africa (Needham, 1967). In most cases, the attributes associated with the right were of superior status to those associated with the left, a consequence no doubt of the universality of right-handedness among human populations.

The particular associations of right with the male and left with the female seems to have been virtually universal, notwithstanding the evidence that women are if anything more likely to be right-handed than are men (e.g., Oldfield, 1971). The Maori expression *tama tane*, meaning literally "male side", referred to the right side, and *tame wahine*, "female side" referred to the left. Empedocles, in the 5th century B.C., thought that males were hotter than females and the right hotter than the left, so that the sex of a child was determined by the leftward or rightward location in the womb. (Empedocles is also said to have destroyed himself by leaping into the crater of Mt. Etna.) The association even persists in modern biology. Ursula Mittwoch (1977) notes that in hermaphrodites with mixed sex organs testes are found more often on the right and ovaries on the left, and suggests that the same lateralizing tendencies may be present in normal males and females. A more subtle twist is provided by Wilhelm Fliess (1923), a one-time friend and colleague of Sigmund Freud, who argued that left-handedness brings out the tendencies of the opposite sex, so that left-handed men tend to be feminine and left-handed women masculine. I do not know of any evidence that this is in fact the case.

With the discovery of cerebral asymmetry, these dichotomizing influences went to our heads. For instance, a recent theory by Geschwind and Behan (1982) might also owe something to the age-old association between sex and the left and right sides. They suggest that testosterone slows the development of the left side of the brain, so that males tend to be more "right-hemispheric" than females. This explains, they say, why there is a higher proportion of left-handedness among males than among females, and why males are more likely to suffer from disabilities in reading or speech (which are thought to depend primarily on the left side of the brain). Geschwind and Behan also suggest that a high level of intra-uterine testosterone increases susceptibility to disorders of the immune system, so that these, too, tend to be associated with left-handedness. It is too early to tell whether these speculations are true, or whether they are yet another manifestation of left-right mythology.

Joseph E. Bogen (1969), one of the surgeons who pioneered the split-brain operation for the relief of epilepsy, enthusiastically interpreted the evidence on functional differences between the two sides of the brain in terms of complementary modes of thought, referring to the left brain as "propositional" and the right brain as "appositional". But he was aware that the distinction he sought to characterize was an old one, predating the neuropsychological evidence. He referred, for instance, to the pre-Confucian Chinese concepts of yin and yang, the Hindu distinction between Buddhi and *manas*, Lévi-Strauss's dichotomy between the positive and the mythic. One might add C.P. Snow's (1959) distinction between the sciences and the arts.

Harrington (1985) has reminded us that there was a epidemic of speculation about differences between the two sides of the brain in the latter part of the 19th century, following the realization in the 1860s that Bichat's "laws of symmetry" did not strictly hold. Some of the dichotomies that were proposed resemble those of the modern era of speculation (but were strangely forgotten until Harrington reminded us of them), while others seem to reflect obsessions peculiar to the 19th century. Not surprisingly, it was suggested that the left hemisphere was male, the right hemisphere female (Delaunay, 1874). In view of the 19th-century concern over Darwinian theory, it is perhaps also not surprising to find the left hemisphere associated with humanness, the right with animality (Broca, 1869). There was also much speculation about the organic basis of racial differences, so we find the left hemisphere associated with white superiority and the left with black inferiority (Delaunay, 1874). The association of the left hemisphere with intelligence and the right with emotion (Luys, 1881b) was extrapolated so that the left hemisphere stood for reason and the right for madness (Luys, 1881a).

As Harrington points out, similar associations were revived in the 1970s. However the dichotomies of the earlier era reveal rather more of 19th-century values, and were colored by notions of the superiority of whites over blacks, males over females, and reason over emotion. The dichotomies that emerged in 1960s and 1970s, by contrast, probably owe at least something to black power, women's liberation, the protest against the Vietnam War and the military-industrial establishment, and the rise in popularity of Eastern religions. So the right hemisphere is accorded a more romantic status, the symbol of the flower children. In the 1960s slogan "Make love not war", it was surely the right hemisphere that stood for love, and the left hemisphere for war.

Yet even this characterization can be found in earlier writings, as in the following remarkable passage written in 1914 by Maurice Maeterlinck, the Belgian man of letters, distinguishing between what he called the Western and Eastern lobes of the human brain:

> The one produces here reason, science, and consciousness; the other secretes yonder intuition, religion, and subconsciousness. The one reflects only the infinite and the unknowable; the other is interested only in what it can limit, what it can hope to understand. They present in an image that may be illusory, the struggle between the material and moral ideals of humanity. They have more than once tried to penetrate each other, to work in harmony; but the Western lobe, at least over the most active part of our globe, has up to the present paralyzed and almost destroyed the efforts of the other. We owe to it not only our extraordinary progress in all the material sciences, but also catastrophes such as we are experiencing today, which, unless we take care, will not be the last nor the worst. It is time to rouse the paralyzed Oriental lobe! [quoted by Massis, 1926, p. 487].

A remarkably similar theme was pursued, apparently independently, by Hertz (1909), who maintained that thinking in dichotomies was "inherent in primitive thought", and that handedness was actually fabricated in human culture to resolve the conflict between oppositions:

> For centuries the systematic paralyzation of the left arm has, like other mutilations, expressed the will animating man to make the sacred predominate over the profane, to sacrifice the desires and interests of the individual to the demands felt by the collective unconscious, and to spiritualize the body itself by marking upon it the opposition of values and the violent contrasts of the world of morality [translated by Needham, 1973, p. 21].

To counter the repressive influence of handedness, Hertz advocated ambidexterity, "to develop the energies dormant in our left side and in our right cerebral hemisphere" [Needham, 1973, p. 22].

These cries for the "release" of the left hand—right hemisphere have their modern counterpart in pleas for education of the right hemisphere. Garrett (1976, p. 244), for example, bemoans "the tragic lack of effort to develop our children's right brain strengths. That potential — a source of [...] creative, artistic, and intellectual capacity — is largely unawakened in our schools."

So is lateral asymmetry *merely* a cultural phenomenon, imposed for political or moral ends? In its extreme form, as proposed by Hertz for example, there is surely a touch of paranoia in that view. I have argued elsewhere that the evidence overwhelmingly favors a biological basis for human right-handedness and hemispheric asymmetry (Corballis, 1983). Given the ancient and universal practice of assigning opposing values to the two sides, however, we must surely remain alert to the mythological component in the interpretation of left-right asymmetries. But as Harrington points out, the fact that similar ideas about hemispheric and manual asymmetries have been repeated at different stages in history does not mean that they are wrong. The very persistence of the ideas, and the fact that they tend to be rediscovered in ignorance of earlier formulations, suggests at least some measure of truth.

And I do not think we should be *too* concerned at the mythical element. I hope I may be forgiven the luxury of concluding by quoting myself:

Another dangerous dichotomy lurks, however: that between science and myth. The two surely lie at the extremes of a continuum; no healthy science is without a dose of myth, just as all myths convey a measure of truth. I have no doubt that conceptions of human laterality will continue to evolve both as a result of careful scientific evaluation and in response to broader human concerns [Corballis, 1985, p. 637].

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SYMMETRY IN EDUCATION

CREATIVE CEREBRAL ASYMMETRY

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QUESTION 1
THE ORIGIN OF CEREBRAL ASYMMETRY

what is symmetry?

My cultural circle is science, namely, the human effort to define order in the kaleidoscope of phenomena. My specific field of study within the scientific system is the investigation of the cognitive function of the human brain, in particular the investigation of the cerebral mechanisms related to the cognition of mathematics and of ontology.

One of the major discoveries in this field of study is the discovery of the functional asymmetry of the human cerebral hemispheres. The left hemisphere is somewhat specialized in the analysis of details, in perceiving temporally ordered phenomena, in lingual functions, and in supervising the order of performing the movements of the hands, in particular of the right hand (each hemisphere controls the contralateral part of the body). On the other hand the right hemisphere is somewhat specialized in perceiving spatial and simultaneous phenomena, like forms. This specialization of the hemispheres is a specific human quality (it appears in a smaller degree also in humanoid apes).

Luria (1966, p. 577) suggested another, though rather similar, functional dichotomy of the human brain. Luria suggested that the central part of the left hemisphere is related to serial synthesis, while the posterior part of the left cerebral hemisphere is related to simultaneous synthesis. Luria's dichotomy can be applied in an attempt to explain the development of the hemispheric asymmetry in human beings.

It is well known that the central part of both hemispheres of the mammalian brain differentiates between temporally close auditory signals. This differentiation is so delicate that a mammal can determine the direction in space of a sound source by the minor temporal difference between the arrival times of the sound to the left and right ears. The details of a series are perceived by us one after another temporally, therefore it is natural that the central part of the brain has a role, at least a partial role, in serial synthesis. On the other hand the posterior part of both hemispheres of the mammalian brain receives visual information which is presented spatially; space is the synthesis of many simultaneously presented details. Therefore Luria's observation may be related to the rather temporal mode in which we perceive auditory data versus the rather spatial (and therefore rather simultaneous) mode of perceiving visual data.

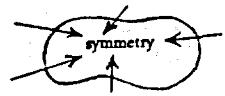
It is possible that this functional difference between the central and posterior parts of the brain developed evolutionarily into the left-right asymmetry in the following process. When the pre-human ancestors of humanity left the trees and began to make their living by hunting, they applied their hands, which originally evolved for climbing trees, to produce and use weapons. Simultaneously they developed speech which enabled them to communicate and organize hunting parties. Both these tasks involve fine temporal analysis of muscular movements, moreover, the producing and understanding of speech involve temporal perception. Therefore they are executed by the central part of the brain.

These new neural tasks apparently require a relatively large quantity of energy. The human body is not entirely symmetric, the heart is in its left side. Therefore the blood supply to the brain in not symmetric and the left hemisphere receives a larger supply of blood than the right one, see Harris (1985, pp. 242-243). Therefore these new serial tasks developed more in the left hemisphere than in the right one, i.e., there was an evolutionary advantage to individuals who executed these tasks in the left hemisphere. The cerebral hemispheres supervise the contralateral side of the body, which accounts for the preference of the right hand for motoric tasks.

The right hemisphere specialized in the less energy consuming tasks of spatial and simultaneous perception which are related to visual perception. However, some simultaneous tasks remained in the posterior part of the left hemisphere as Luria observed, see Kinsburne and Warrington (1962).

Thus the human brain developed with two cognitive functions located asymmetrically. However, the term "cognitive duality" seems to be more appropriate than the term "cognitive asymmetry" for characterizing the two cerebral functions cognitively.

QUESTION 3
CHARACTERIZATION OF THE CEREBRAL ASYMMETRY WITHIN THE CULTURAL KALEIDOSCOPE



Before answering the second question, we shall answer the third one: how the cultural surroundings influenced the meaning given to the left-right duality in human cognition by brain researchers.

There are several dichotomical characterizations of human culture. Some brain researchers tried to characterize the duality in human cognition related to hemispheric asymmetry. Bogen (1969) prepared a list of such cognitive and cultural dichotomies, which their authors related to the hemispheric *asymmetry*. He also prepared another list of dichotomies which he related to the hemispheric asymmetry, though their authors did not. Orenstein (1977) suggested a far-reaching theory that western culture is related to the left hemisphere while far-eastern culture is related to the right hemisphere. We shall see now how such dichotomies can be obtained from a single dichotomy, that is, the multi-axis asymmetry of the cultural and cognitive kaleidoscope can be reduced into a one axis asymmetry.

Bogen and Gazzaniga (1965) suggested that the functions of the left and right hemispheres are related, respectively, to the dichotomy of *verbal* versus *visuospatial* aspects of cognition and culture. This dichotomy is obtained from the observation that the left hemisphere is related to the performance of verbal and lingual functions, while the right hemisphere is related to the visual perception of forms and to the orientation in space. May be that this dichotomy was noticed since it presents two modes of transferring information, by words and by pictures; two modes of art, literature and plastic arts; and two modes of thinking, verbal versus image-creation.

Carmon and Nachshon (1971) suggested another dichotomy of the left and right hemispheres: temporal versus spatial perception, respectively. This cognitive dichotomy too is related to culture, for example temporal history versus spatial geography.

The verbal versus visuospatial dichotomy can be obtained from the temporal versus spatial dichotomy as follows: each of the lingual functions is performed one step after another temporally; speaking and hearing are performed one syllable after another, writing and reading are executed one letter after another and one word after another. Thus we see that the lingual functions are performed applying temporal analysis of one item after another. On the other hand forms are perceived visually in space.

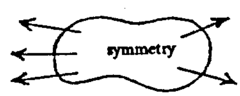
Levy-Agresti and Sperry (1968) suggested a dichotomy based on information theory, namely, analytic processing of single items at one time by the left hemisphere versus synthesis of a new whole from several data by the right one. For example, right-brain damaged patients draw pictures with accurate details, but they cannot organize these details into forms. On the other hand left-brain damaged patients can draw general forms, but cannot draw details. Thus this dichotomy can explain the existence of two styles of drawing. The first is detail-oriented like the pictures of Breughel, while the second emphasizes the form and structure of the entire picture, like the Italian Renaissance artists.

Ben-Dov and Carmon (1976) obtained the temporal versus spatial dichotomy (and through it also the verbal versus visuospatial dichotomy) from the analytic versus synthetic dichotomy of Levy-Agresti and Sperry. They argued that the analysis of details is necessarily performed one item after another temporally. On the other hand spatial presentation is the synthesis of many details presented simultaneously. Ben-Dov and Carmon argued further that when data from the external world arrive at the brain, the left hemispheric data processing mechanism presents them to consciousness organized one after another temporally, while the right hemispheric mechanism presents the same data to consciousness simultaneously, and therefore they are perceived spatially. Ben-Dov and Carmon concluded that space and time are no more than the subjective modes in which the hemispheric mechanisms present the just arrived data to consciousness. Thus they obtained from neurology Kant's doctrine about the subjectivity of space and time.

Ben-Dov and Carmon (1976) argued also that since the left hemisphere does not perceive any datum between two consecutively perceived data, time cannot be continuous and it must be quantizied. This idea originates from the quantization of various entities in modern physics.

Thus Ben-Dov and Carmon (1976) showed that the whole of cognition can be explained by the analytic versus synthetic data processing dichotomy of Levy-Agresti and Sperry.

QUESTION 2
CREATION OF AN EPISTEMOLOGICAL KALEIDOSCOPE BY THE ASYMMETRIC BRAIN



Now we shall answer the second question: how the duality in human cognition, which was discovered by brain research, influenced, or can influence, the scientific and philosophical cultural spheres.

We are surrounded by an enormous number of data, which emerge into our consciousness from the physical world through our senses. The brain processes this kaleidoscope and orders it. According to the model of Ben-Dov and Carmon (1976) the construction of all our cognition is performed by the analytical and synthetical data processing mechanisms. The modes by which we perceive this information are defined by the metaphysical realistic ontologies. There are two such ontologies, namely, nominalism and Platonism. According to nominalism the phenomena are perceived as concrete and discrete details, while according to Platonism phenomena are properties which can be distinguished in the kaleidoscope of phenomena.

It was suggested by Fidelman (1985; 1987a; 1988a; 1989a) that the two ontological modes of perception exist as a result of the existence of the two hemispheric data processing mechanisms. This concerns perceiving phenomena of experience as well as perceiving mathematical objects and ideas.

The left hemispheric mechanism extracts details out of the external information and presents them to consciousness in the nominalistic mode of perceiving phenomena. On the other hand the right hemispheric mechanism perceives wholes which are synthesized from the details. Sets, the elements of which are the details, are such wholes. According to Frege the concept of a set and the concept of a property characterizing the elements of this set are virtually equivalent, therefore we may relate the cognition of the Platonistic concept "property" to the right hemispheric mechanism.

We may conclude that while the left hemispheric mechanism perceives phenomena as discrete details, the right hemispheric mechanism perceives the same phenomena as physical properties. Therefore we may consider nominalism and Platonism to be our two only possible subjective modes of ontological perception of phenomena. Thus we extend Kant's view that while the things in themselves exist, space and time, the two extensional modes of perceiving experience, are subjective. Thus we also extended the idea of Ben-Dov and Carmon (1976) that Kant's extensional modes of perception are, indeed, subjective and related to the hemispheric mechanisms, to the ontological modes of perception.

An experiment in which this idea was tested is described by Fidelman (1989a). Subjects with a dominant left hemisphere preferred a nominalistic ontological approach to the physical world, while subjects with a dominant right hemisphere preferred a Platonistic ontology.

It was argued by Fidelman (1987a; 1988a) that the duality in physics also originates from the two cerebral mechanisms. The left hemisphere can perceive only concrete and discrete objects, and it interprets the physical world as comprising particles. Moreover, according to Ben-Dov and Carmon (1976), time is necessarily quantizied. Since the left hemisphere can perceive phenomena only one after another in time, this quantization of time is necessarily extended to every physical dimension. On the other hand the right hemisphere cannot perceive discrete details, therefore it necessarily perceives phenomena as continuous and spatial. The dual perception of physical phenomena as quanta or elementary particles on one hand and as continuous waves on the other hand, are two possible interpretations of the things in themselves by the hemispheric mechanisms. May be that these two mechanisms evolved evolutionarily so that they suit the thing in themselves, but we cannot know for sure.

Interactions between the hemispheres construct complex cognitive structures. According to the model of Ben-Dov and Carmon (1976), the output of each hemispheric mechanism is available to the other one as an input. For example, children who learn to read perceive each letter as a form by the right hemisphere, which integrates it from its details. Then the left hemisphere perceives the letters one after the other temporally. At the next stage the right hemisphere integrates a word from several letters. Then the left hemisphere reads the words one after another temporally, and so on.

The letters, initially perceived by children through the right hemisphere as forms comprising details, are perceived by experienced readers through the left hemisphere as single details. The same is true regarding the reading of whole words. Shanon (1982) found that reading words in a foreign language is related more to the right hemisphere than to the left one, while reading words in a native language is related more to the left hemisphere than to the right one.

Another example is the cognition of music. While notes are perceived one after another temporally by the left hemisphere, the synthesis of chords and melodies is performed, according to Kimura (1964) and Gordon (1970), by the right hemisphere of ordinary persons. This example manifests that the ordinal synthesis related by Luria to the central part of the left hemisphere is, indeed, a combination of ordinal (in time) processing in the central part of the left hemisphere and synthesis of the series by the right hemispheric mechanism. However, according to Bever and Chiarello (1974), professional musicians perceive chords and melodies as individual items through the left hemisphere.

We may formulate a universal principle: familiarity with a whole, perceived by the right hemisphere, causes it to be perceived as a single item by the left hemisphere.

In a paper by Fidelman (1988a) this principle was applied to explain how the cosmos, i.e., the whole of phenomena, is perceived by the human brain. The holistic concept "cosmos" (which according to Einstein is finite) is integrated by the right hemisphere from details which are phenomena. However, we may expect that professional cosmologists, who deal with this concept frequently, will eventually treat it as a new single item through the left hemisphere. When this happens, the consciousness of the trained cosmologist needs a stage, i.e., extensional modes of perception, in which the new item will exist. Therefore this finite cosmos is perceived within some spatial continuum. This continuum may be inhabited by additional such items, namely additional cosmoses. It was suggested by Fidelman (1988a) that the cosmological "bubbles theory" of Linde (1983a; 1983b) may emerge from such a neuropsychological process. It should be noticed that Linde's extended cosmos emerged outside the domain of experience. Therefore the neuropsychological model for the creation of such a theory indicates the possibility that, though the dual models in physics may be alternative presentations of the things in themselves to our consciousness, the extension of experience may exist only mentally, having an ontological status similar to that of some mathematical structures.

We observe that the cerebral functional asymmetry explains epistemologically scientific theories. This explanation may have an impact on our appreciation of scientific theories and on the future evolution of science.

The hemispheric asymmetry concerns also another domain of human cognition: mathematics. There is a duality in mathematics: ordinal numbers versus cardinal numbers, series versus sets, and potential infinity versus actual infinity. A natural assumption is that the first item of each pair is related to the left hemisphere while the second item is related to the right one. This assumption was confirmed experimentally in Fidelman (1984; 1987c; 1989b).

A duality exists also in the philosophical approaches by which mathematics is founded. Kant (1964) obtained mathematics from the two modes of perceiving experience, space and time. According to Kant ordinal arithmetic is obtained from the intuition of one after another in time, while geometry is obtained from the intuition of space. Two dual schools emerged from Kant's approach. The first is Brouwer's intuitionism which establishes all of mathematics on the intuition of one after another in time, the other is Frege's late approach of establishing the whole of mathematics on the intuition of space alone. It is explained in Fidelman (1987a; 1988a) how the temporal approach to mathematics is related to the left hemisphere, while the spatial approach is related to the right one. Therefore Kant's approach is related to both hemispheres.

There are also two extreme metaphysical realistic approaches to mathematics. The first is Frege's logicism which is Platonistic and demands the existence of sets, but not of atomic elements. The second approach is Hilbert's formalism which is nominalistic and establishes mathematics on the intuition of concrete and discrete individual objects. It is explained by Fidelman (1987a; 1988a) how Frege's logicism is related to the right hemispheric mechanism, while Hilbert's formalism is related to the left one. There is also a dualistic approach, namely, Russell's theory of types. It is a logistic approach too, but unlike Frege's approach, Russell demanded the existence of both atomic elements and sets. Therefore this approach is related to both hemispheres.

Frege's logicism collapsed because Russell's paradox was formulated within it. This paradox can be formulated as follows:

Let s be a set having the property P that all its elements are sets which do not include themselves as elements. Then s itself has this property P, otherwise one of its elements, namely, s itself, does not have the property P. Let R be the set of all sets having the property P. Since R itself has this property, it must be an element of itself. But this implies that R has not the property P, which leads to a contradiction.

It was shown by Fidelman (1987b; 1988a; 1988b) that the cognition of this and other foundational paradoxes in mathematics can be explained as follows:

The set R is integrated by the right hemispheric mechanism. Then this product of the right hemisphere is treated by the left hemisphere as a new individual element, which is not one of the elements integrated by the right hemisphere into the set R. But this element too has the property P, contrary to the definition of R. That is, the left hemisphere continues to extract additional elements having the property P after the final construction of the set of all the elements having this property by the right hemisphere. Thus the left hemisphere causes the disintegration of the set R which was integrated by the right hemisphere. This leads into a cognitive conflict. This cognitive model was confirmed in experiments described in Fidelman (1987b; 1988b).

We conclude that the foundational paradoxes are the product of a lack of coordination between the hemispheric mechanisms regarding infinite sets. In this case the hemispheric interactions have a destructive role. However, a similar hemispheric-interaction related cognitive conflict has a constructive role in proving the existence of certain infinite mathematical structures by negating their non-existence. Cantor's diagonal process is an example for such proofs.

We found that the dual hemispheric mechanisms are the source from which mathematical knowledge originates. Moreover, they construct more and more complicated mathematical structures by hemispheric interaction, i.e., integration of sets by the right hemisphere from elements presented by the left hemisphere. According to Fidelman (1987c; 1988b), there is some evidence that when the right hemisphere integrates a series presented by the left hemisphere in a potentially infinite process which does not terminate, the integration of the series by the right hemisphere involves a termination of the process. This lack of coordination between the hemispheres involves a cognitive conflict which was described by Zeno in his paradoxes of *Achilles and the tortoise* as well as *the runner*. In this conflict the integration of the infinite set is accomplished when the right hemisphere "overcomes" the left one, while in Cantor's diagonal process, as well as in set theoretical paradoxes, the left hemisphere "overcomes" the right one.

In conclusion, mathematics originates from the hemispheric mechanisms, hemispheric interactions are responsible for the construction of mathematical structures, and they are responsible for the collapse of too comprehensive mathematical systems, as happened to Frege's logicism. This observation may have an impact on our understanding of what mathematics is.

The experimental results regarding the relation of mathematics to the hemispheric mechanisms can influence culture considerably through its possible application in mathematical education. In an experiment described by Fidelman (1984), grown up students studied Peano's ordinal arithmetic first, and then Frege's cardinal arithmetic. The students were examined on both arithmetics and participated in tests for the performance of the cerebral hemispheres. The results were that students with a dominant left hemisphere succeeded more in learning ordinal arithmetic, while students with a dominant right hemisphere succeeded more in learning cardinal arithmetic.

In a similar experiment described by Fidelman (1987c), students studied calculus simultaneously in the ordinary "standard" approach of potential infinity, and in the "non-standard" approach of actual infinity according to Keisler. The results were that students with a dominant left hemisphere succeeded more in the "standard" approach, while students with a dominant right hemisphere succeeded more in the "non-standard" approach.

These results can be applied to the classification of children before entering school in order to teach them arithmetic in a mode which suits their brain. It is proposed that children whose left hemisphere is more efficient than the right one will learn reading letter by letter and arithmetic based on ordinal numbers. On the other hand children whose right hemisphere is more efficient than their left one will learn reading in the global method, namely, a whole word as one form, and arithmetic based on cardinal numbers. A similar classification can be tried with university students who study calculus.

The existing educational system imposes a uniform method of learning on all the students without considering individual differences. Thus students whose brain function does not suit this method have a smaller chance of success. The contribution of the discovery of the brain's functional asymmetry to education may be the establishment of a dual educational approach which adapts the method of teaching to the student.

This educational approach will increase the knowledge level of the entire population through more efficient teaching, but it may be most important to students with an extreme difference between the efficiencies of their hemispheric mechanisms. In many cases they can hardly succeed with a method of learning which does not suit their aptitude, and they may be considered, sometimes, by their teachers as lacking learning capability. Nevertheless, they may succeed in an educational method which suits their brain. After graduating, these students may apply their unique and extreme modes of thinking and contribute unusual achievements to culture.

May be that Einstein's extreme geometrical conception of the physical world is a result of an extremely larger efficiency of his right hemisphere relatively to his left one (indeed, he was not considered by his teachers to be a brilliant student). Similarly, may be that Frege's early Platonistic approach and his later geometrical approach to mathematics are a result of a relatively more efficient right hemisphere, while Brouwer's extreme temporal and ordinal approach to mathematics is a result of a relatively more efficient left hemisphere.

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Vol. 1, No. 2, 1990, 205-208

SFS: SYMMETRIC FORUM OF THE SOCIETY (BULLETIN BOARD)

All correspondence should be sent to the editors. For the list of publications of members (and non-members) refer to the "Symmetric Reviews" in Symmetro-graphy.

FIRST CIRCULAR

CALL FOR PAPERS, WORKSHOP TOPICS, AND EXHIBITION ITEMS

SYMMETRY OF PATTERNS

Second Interdisciplinary Symmetry Symposium and Exhibition
August 17-23, 1992, Hiroshima, Japan

Organized by the
International Society for the Interdisciplinary Study of Symmetry (ISIS-Symmetry)
together with the Synergetics Institute of Japan, Hiroshima

Fields of interest

The regularity (and the beauty) of *patterns* — as well as their formation and chaotic dissolution — has become an important focus in various scholarly fields, including non-linear mathematics (catastrophe, chaos, fractal), computer graphics and design, geometric crystallography (including the Penrose patterns and related quasicrystal-models), physics of dissipative systems, theory of periodic reactions, biomorphology, geomorphology, experimental psychology, ethology, as well as linguistics, literature, and aesthetics, etc. The same topics have obvious connections with artistic and technical fields, from ornamental art to structural design and architecture.

The Symposium and the Exhibition will provide a forum for interdisciplinary and intercultural approach to the concept and phenomenon of symmetry (asymmetry, dissymmetry, antisymmetry) and to related concepts (proportion, rhythm, invariance, etc.), where representatives of different disciplines, arts, and skills — from East and West, North and South — have an opportunity to exchange and enrich their experience in a holistic form. The actual title of our second symposium *Symmetry of Patterns* emphasizes the interest in those aspects which can be visualized and discussed without too much technical details.

The papers submitted should either (1) describe concrete inter-disciplinary "bridges" between different fields of art, science and technology using the concept of symmetry; or (2) survey the importance of symmetry in a concrete field with an emphasis on possible "bridges" to other fields. The meetings of ISIS-Symmetry are informal ones and do not substitute for the disciplinary conferences, only supplement them with a broader perspective.

Working language, program

The working language of the symposium is English (no interpreters will be available).

The main program will consist of three parts:
(1) *Mornings*: interdisciplinary plenary lectures aiming at the interest of an audience with different backgrounds;
(2) *Afternoons*: workshops discussing either some topics informally ("symmetric [round] tables"), or focusing on further details of some topic for more specialized groups of people (minisymposia);
(3) *Evenings*: informal meetings and workshops of recreational type with art and crafts, as well as various performances.

Note that the philosophy of the symposium tries to avoid both extremities: the too specialized disciplinary approaches and the too general indirect surveys. Discussions at the Exhibition Hall — located within the conference center — will form an organic part of the event, providing forum for dialogue between artists and scientists.

Proposals, formats, deadlines

A lecture proposal should include an *extended abstract* with a length between 1½ and 2 pages (neither shorter nor longer, single spaced!) in *camera ready* version.

In connection with the interdisciplinary goals, please, try to help the readers outside of your main discipline, e.g., by explaining some special concepts, using intuitive approaches, or giving comprehensive tables and illustrations. Regarding further questions about selecting your topic or preparing the text, consult the detailed articles "Aims and scope", or "Instructions for contributors" in the quarterly *Symmetry: Culture and Science*.

You are also encouraged to submit proposals for

workshop topics (approximate title, short description, other possible contributors; papers emphasizing interactions, mediated by symmetry, between different fields of science, science and art, cultural origins, and relying upon the interest of participants with different background, are preferred),

items for the exhibitions (short descriptions and photographs of art works, models, etc. identifying the dimensions), and

evening activities or performances, including music, dance, video, laser, etc. (send two-page extended abstracts, similar to lectures, adding by pencil at the top of the front page "evening").

Items for the exhibition should be carried with you (we cannot organize extra shipment, insurance, etc.).

The program will also include a General Meeting of ISIS-Symmetry.

The deadline for proposals is *February 1, 1992. Mailing address*: György Darvas, ISIS-Symmetry, Budapest, P.O. Box 4, H-1361 Hungary. (In the case of the accepted proposals we will not request further versions of the extended abstracts, but we will use them in the submitted camera ready forms. There will be, however, a short period for modifications either requested by the organizers or suggested by the contributors.)

The Scientific Advisory Committee of the Symposium is identical with the Board of ISIS-Symmetry (see inside cover).

Location of the Symposium

The Symposium will be hosted by the *Synergetics Institute* in Hiroshima, Japan. This Institution was founded in 1988 as the Japanese Division of the Buckminster Fuller Institute in the United States. Address: Yasushi Kajikawa, Director, Synergetics Institute, 5-4 Nakajima-cho, Naka-ku, Hiroshima 730, Japan.

Travel: Although Hiroshima has no international airport, it can be easily reached by domestic flights, by various trains, including the "Shinkansen" super express, from Tokyo, Nagoya, Osaka and Fukuoka, or by ferries from Osaka and other ports. The Synergetics Institute is located very close, about five-minute walk, to the Peace Memorial Park in downtown Hiroshima.

Accommodation: there are several hotels near to the Synergetics Institute, including both budget and luxury hotels (the city also has a youth hostel). A list of recommended hotels and reservation details will be announced in the next circular.

Climate: the mean temperature in August is 26.8 °C (80.2 °F), the average rain-fall in the same month is 124 mm (4.9").
Registration fee (in US$)

	Until February 1, 1992	February 2 — July 1, 1992	After
Members:	250.00	300.00	350.00
Non-Members:	325.00	375.00	425.00
Accompanying persons:	100.00	135.00	150.00
Students:	100.00	100.00	100.00
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Student Members: Free, with a letter of support from a Regional or Project Chairperson; but there will be a small charge for the Abstract Volume.

The registration fee includes a copy of the proceedings, refreshments, free pass to the exhibition and the symposium programs.

Members, in the case of special circumstances, may apply for a waiver.

Please send checks to the address of the Executive Secretary of ISIS-Symmetry (György Darvas, ISIS-Symmetry, Budapest, P.O. Box 4, H-1361 Hungary), or transfer funds to the account number of ISIS-Symmetry, 401-6585-844-99, Hungarian Foreign Trade Bank, Budapest, Szt. István tér 11, H-1821 Hungary (Telex: Hungary 22-6941 extr h; Cable address: exterbank) and a parallel letter to the Executive Secretary, with statement of the transfer.

PROJECT

Proposal for a taxonomy project

I believe it appropriate for the new society to host a taxonomy project as one of its initial activities. The primary goal of the project is to induce interdisciplinary dialogue through examining how each worker or group relates its work within the larger context of symmetry. As a secondary goal, a useful taxonomy may result, in whole or in part, as a useful tool in cross-indexing papers, etc.

Traditional taxonometric approaches involve two devices which are proposed to be avoided here. The first is the common indentured list, or hierarchical tree. Certainly the society recognizes that the relationships among areas of interest are multi- (or non-) hierarchical and multidimensional. Who better to devise a more robust conceptual system than the society; and what better focus to apply it than the component work of the society? It should be expected that fundamental issues of what constitutes generic, or underlying "deep" structure will arise. Other interesting controversies may include concepts of what constitutes the study of symmetry anyway.

The second reflex which is proposed to be avoided concerns the tendency to categorize. What is more important are the relationships among areas, not what those areas are. The philosophy of the proposed project will stress commonality not differentiation per se. Exactly what is meant here is itself expected to be a strange attractor of dialogue!

To get the project rolling, I volunteer to contribute to and edit the discussion in the appropriate forum, as determined by the society's editors.

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SYMMETRO-GRAPHY

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BIBLIOGRAPHY

Kaleidoscopes in art, science, and education

The kaleidoscope has never lost its place in popular scientific and hobby literature, although we may observe cyclic changes in the amount of interest. Usually the invention of new kinds of kaleidoscopes revitalized the whole topic, including the classical devices. Indeed, there are very many papers which do not go far beyond the description of the usual kaleidoscopes and some practical advice on how to build them. This bibliographic survey does not focus on these works, but rather on the history of the topic in art and science, including some questions of education. The aspect of symmetry has a special emphasis; we list here all the works cited in the introductory essay "The kaleidoscope and symmetry" (Parts 1-2). This bibliography also includes items which do not refer directly to the kaleidoscope, but investigate related arrangements of mirrors. We do not list, however, works which are connected with the kaleidoscope only metaphorically. The bibliographic data in brackets are those that are not directly available from the publications; they are added by us.

Anderson, D.H. (1975) [Projection kaleidoscope], *Scientific American*, 233, No. 1, 124-125.

Brief description of the kaleidoscope with an illustration; it is inside the section called *The amateur scientist*.

Arn, W. (1990) *Phänomene zwischen Natur und Technik: Erleben, Experimentieren, Werken, Forschen*, [Phenomena between Nature and Technique: Experiencing, Experimenting, Creating, Investigating, in German], Zürich: Orell Füssli, 1990, 79 pp.

See chap. "Spiegeln" [Mirrors], 71-79. Note a mistake (p. 73): the angles of 72° (= 360°/5) and 51.4° (= 360°/7) are not suitable for a dihedral kaleidoscope; the allowed angles are 180°/n, where n = 2, 3, 4, ..., not 360°/n (Brewster, 1819; cf., our remarks on the Brewster-Kircher controversy in Part 1 of the article "The kaleidoscope and symmetry", this quarterly, Vol. 1, No. 1, pp. 29-30). There is a well-illustrated discussion on Caspar Schwabe's foldable kaleidoscope (p. 79) and the presentation of the image of a star-polyhedron (cf., this issue, p. 122, Fig. 4), but unfortunately the layout is not accurate.

Asakura, N. (1990) Experiments with mirror reflections, *Leonardo*, 23, 71-74.

Baker, C. (1985) *Through the Kaleidoscope ... and Beyond*, Annapolis, Md.: Beechcliff Book, 200 pp.

We did not have access to this book, but we hope to review it later.

Baltrusaitis, J. (1986) *Der Spiegel: Entdeckungen, Täuschungen, Phantasien*, [The Mirror: Discoveries, Delusions, Fantasies, in German], Giessen: Anabas.

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A complete geometrical discussion of those kaleidoscopes which we referred to as dihedral and cylindrical; the later article by S. Tolansky in the 14th edition also mentions these types, but the 15th edition in 1974 and its more recent reprints omit this part.

Brewster, D. (1817) *New Optical Instrument Called the Kaleidoscope for Exhibiting and Treating Beautiful Forms and Patterns of Great Use in All the Ornamental Arts*, Patent No. 4136, London: United Kingdom Patent Office.

Brewster, D. (1818) Description du kaleidoscope, [Description of the kaleidoscope, in French], *Bibliothèque Universelle des Sciences, Belles-Lettres et Arts: Partie des Sciences*, 8, 155-160.

Brewster, D. (1819) *A Treatise on the Kaleidoscope*, Edinburgh: Constable, vii + 166 pp.

This monograph gives a comprehensive survey of the classical kaleidoscopes, including all the types of two-mirror (dihedral) and polycentral (cylindrical) ones. Brewster also discusses some earlier devices.

Brewster, D. (1858) *The Kaleidoscope, its History, Theory, and Construction with its Application to the Fine and Useful Arts*, 2nd enlarged ed. [of Brewster, 1819], London: Murray, vii + 189 pp.; 3rd ed., London: Hotten, 1870, vii + 189 pp.

Chai, A.-T. (1971) The number of images of an object between two plane mirrors, *American Journal of Physics*, 39, 1390-1391.

Costello, E.D. (1988) *Kaleidoscope Patterns: Art Education in an Elementary Classroom*, [M.A. Thesis], Vancouver: University of British Columbia; Microfiche, Ottawa: National Library of Canada, 1989.

[Coxeter, H.S.M.] (1939) Polyhedra, In: Ball, W.W.R., *Mathematical Recreations and Essays*, 11th revised ed. by Coxeter, H.S.M., London: Macmillan, Chapter 5, 129-160; Reprints and new editions; Russian trans.

See "The Kaleidoscope", 154-160. Coxeter gives credit for this topic to Hess (1889). Note that the earlier editions do not have the cited chapter on polyhedra. The authorship of this chapter is revealed in the editor's preface (p. vii).

Coxeter, H.S.M. (1948) *Regular Polytopes*, London: Methuen; American ed., New York: Pitman, 1948; 2nd ed., New York: Macmillan, 1963; 3rd ed., New York: Dover, 1973, xiii + 321 pp.

See chap. 5 "The kaleidoscope", 75-92; chap. 11 "The generalized kaleidoscope", 187-212. Note that these chapters deal rather with reflection groups in the 3- and *n*-dimensional spaces, respectively, than with optical devices.

Coxeter, H.S.M. (1961) *Introduction to Geometry*, New York: Wiley, xiv + 443 pp.; 2nd ed., ibid., 1969, xvi + 469 pp.; Reprint: 1989; German, Hungarian, Polish, Russian trans.

See chap. 2.7 "The kaleidoscope", 34-36; chap. 15.7 "The polyhedral kaleidoscope", 279-282 (these page numbers are the same in both editions). Coxeter's term "polyhedral kaleidoscope" refers to devices with trihedral arrangement of mirrors that are able to present polyhedral images.

Coxeter, H.S.M. (1965) *Dihedral Kaleidoscopes*, [13-minute color film], Minneapolis, Minn.: College Geometry Project, University of Minnesota.

Coxeter, H.S.M. (1974) *Regular Complex Polytopes*, London: Cambridge University Press, x + 185 pp.; 2nd ed., [forthcoming].

See chap. 3 "Polyhedral kaleidoscopes" (i.e., trihedral devices for presenting polyhedral images).

Courant, R. and Robbins, H. (1941) *What is Mathematics? An Elementary Approach to Ideas and Methods*, London: Oxford University Press; Many new editions and translations.

See chap. 3.6.3 "Repeated reflections".

Fedorov, E.S. (1883) *Gonoedricheskie demonstrativnye pribory po kristallografii*, [Gonohedral Demonstrative Devices for Crystallography, in Russian], [Lecture], Sankt-Peterburg: Imperatorskoe Sankt-Peterburgskoe Mineralogicheskoe Obshchestvo, February 15, 1883; Summary, *Zapiski Imperatorskago Sankt-Peterburgskago Mineralogicheskago Obshchestva*, 2-ya seriya [2nd series], 19 (1884), 181.

Fedorov (1885, part 3, appendix 1) refers to a lecture as the presentation of his trihedral kaleidoscopes – giving credit to Möbius for the basic idea, but also describing the further development made by himself – during a session of the Imperial Mineralogical Society of St. Petersburg in early 1883; see also Fedorov (1890, p. 234). We have found the data about this lecture in other sources (Fedorov's list of publication, item no. 7; see in: Fedorov, E.S., *Simmetriya kristallov*, Moskva: Nauka, 1949, p. 599; Shafranovskii, I.I., *Evgraf Stepanovich Fedorov*, Moskva: Akademiya Nauk SSSR, 1963, p 242, cf., p. 90). Note that the term "gonohedron" was introduced by Fedorov to replace the term "polyhedral angle" (cf., Fedorov, E.S., *Symmetry of Crystals*, New York: American Crystallographic Association, 1971, p. 23, note 2).

Fedorov, E.S. (1885) Nachala ucheniya o figurakh, [Elements of the study of figures, in Russian], *Zapiski Imperatorskago Sankt-Peterburgskago Mineralogicheskago Obshchestva*, 2-ya seriya [2nd series], 21, i-viii, 1-278; German summary, Elemente der Gestaltenlehre, *Zeitschrift für Krystallographie und Mineralogie*, 21 (1893), 679-694; 2nd annotated ed. of the Russian monograph by Ansheles, O.M., Shafranovskii, I.I., and Frank-Kamenetskii, V.A., Moskva: Akademiya Nauk SSSR, 1953, 409 pp.

Fedorov describes the importance of kaleidoscopes for crystallography in part 3 "Uchenie o simmetrii" [Study of symmetry], appendix 1 (in the 1953 edition pp. 224-225); see also part 5, appendix (p. 374). Interestingly, he does not use the Russian word "kaleidoskop", only the analogous German expression when quoting Möbius. He speaks about "gonohedral mirrors", or specifically about "trihedron" (*trekhgrannik*).

Fedorow, E. [Fedorov, E.S.] (1890) Gonoëdrische demonstrative Apparate in Anwendung auf die Krystallographie, [Gonohedral demonstrative apparatuses for the use in crystallography, in German], *Neues Jahrbuch für Mineralogie, Geologie und Palaeontologie*, 1890, Band 1, 234-247 and table 2.

Fedorov emphasizes the priority of his apparatuses (Fedorov, 1883) over the kaleidoscopes of Hess (1889), without realizing that Hess published about this subject and demonstrated his devices earlier (Hess, 1879; 1882); cf., Fedorov's introductory remarks on p. 234 and the comment of the editor. In connection with the idea of using liquid as object inside the surface of the device, see pp. 244-245. Note that kaleidoscopes are useful to demonstrate only those crystal systems (point groups) that can be generated just by reflections. Therefore Fedorov developed another apparatus, based on a wooden ball and appropriate drawings on its surface, to demonstrate all the possible 32 crystal systems (pp. 245-247 and table 2).

Fedorov, E.S. (1891) *O polnoi kollektsii gonoedricheskikh priborov po kristallografii*, [A full collection of gonohedral devices for crystallography, in Russian], *Zapiski Imperatorskago Sankt-Peterburgskago Mineralogicheskago Obshchestva*, 2-ya seriya [2nd series], 28, 515.

Fejes Tóth, L. (1964) *Regular Figures*, New York: Macmillan, xi + 339 pp.; German trans., *Reguläre Figuren*, Leipzig: Teubner, Budapest: Akadémia, 1965, 316 pp.

Chap. 1 "Plane ornaments" – all the possible types of 2-dimensional symmetric patterns (two infinite series of rosette groups, seven frieze groups, and 17 wallpaper groups) are illustrated with patterns; those ones which can be generated by kaleidoscopes, i.e., just by reflections, are listed on p. 41 (or, in the German ed., p. 45).

Fialowsky, L. (1889) *Kristályalakok magyarázata tükrökkel*, [Explanation of crystal-forms with mirrors, in Hungarian], [Lecture], Budapest: Királyi Magyar Természettudományi Társulat, May 22, 1889; Summary, *Természettudományi Közlöny*, 21 (1889), 282; A later demonstration, *A kristályok fölépítését szemléltető tükrös eszköz*, [A device with mirrors for demonstrating crystal-structures, in Hungarian], [Demonstration], Pozsony: A magyar orvosok és természetvizsgálók 34. vándorgyülése, August 25-29, 1907; Cf., *Természettudományi Közlöny*, 39 (1907), 655.

Finkel, N.Y. and Finkel, L.G. (1980) *Kaleidoscopic Designs and How to Create Them*, New York: Dover.

Fuller, R.B. (1979) *Synergetics 2: Explorations in the Geometry of Thinking*, New York: Macmillan, xxiv + 592 pp.

The surface of the Coxeter kaleidoscope, which can be folded from a square, is identical with Buckminster Fuller's *T quanta module*, see the layout, pp. 257-258, and its folding, p. 283. Although "Bucky" does not refer here to kaleidoscopes, but later (p. 320) – in a slightly fuzzy statement on the kaleidoscopically nonrepetitive local patterning events between the phases of icosahedron, octahedron, and tetrahedron – he uses the expression in a sense which may be associated with the partial kaleidoscope (cf., this issue, p. 121, Fig. 3).

Gombrich, E.H. (1979) *The Sense of Order: A Study in the Psychology of Decorative Art*, Oxford: Phaidon Press; 2nd ed., ibid., 1984, xii + 411 pp.; German trans., *Ornament und Kunst: Schmucktrieb und Ordnungssinn in der Psychologie des dekorativen Schaffens*, Stuttgart: Klett-Cotta, 1982, 420 pp.

Chap. 6.1 "The kaleidoscope".

Gray, N. (1989) *Laughter in the Ruins: The Kaleidoscope as a Problematic-Theoretical and Visual Excess in Cultural Theory, Science and Language Theory*. [B.A. Thesis], Sydney: Department of Fine Arts, University of Sydney; Abstract, *Leonardo*, [forthcoming].

Greenslade, T.B., Jr. (1982) Multiple images in plane mirrors, *Physics Teacher*, 20, 29-33.

Hahn, W. (1989) *Symmetrie als Entwicklungsprinzip in Natur und Kunst*, [Symmetry as a Developmental Principle in Nature and Art, in German with English summary], Königstein: Langewiesche, 320 pp.

See chap. 5.3 about symmetrizations in space, including photographs showing the application of dihedral kaleidoscopes; chap. 11.8.6 about the kaleidoscopic aspects of *ars evolutoria* and the remarks about Gombrich's view; and color table no. 10 with examples of *ars evolutoria* composed by the combination of photographs and paintings.

Henderson, L.D. (1983) *The Fourth Dimension and Non-Euclidean Geometry in Modern Art*, Princeton: Princeton University Press, xxiii + 453 pp., 116 figs. on plates.

See especially in chap. 3 "Marcel Duchamp and the new geometries" the subchap. "The final solution: The mirror of the fourth dimension", 150-157. The "3-sided mirror" is mentioned on p. 154. Note that this book is a source of very many rarely cited documents and references. There was a series of articles in connection with this book in the journal *Leonardo*: see Henderson's conclusions of her book, 17 (1984), 205-210; A.L. Loeb's criticism against the premise that mathematical theory was transformed into works of art, 18 (1985), 193-196; Henderson's response and Loeb's reply, with comments by S. Edgerton and A.C. Danto, 19 (1986), 153-158.

Hess, E. (1879) Ueber ein Problem der Katoptrik, [About a problem of catoptrics, in German], *Sitzungsbereichte der Gesellschaft zur Beförderung der gesammten Naturwissenschaften zu Marburg*, 1879, 7-20; Review, *Beiblätter zu den Annalen der Physik und Chemie*, 6 (1882), 742.

The term *catoptrics* in the title refers to that part of optics which treats reflections. Hess discusses here the number of images of an illuminated point that is placed inside the surface of a trihedral or multihedral corner shaped with mirrors. He connects this topic with his own research on the theory of polyhedra. Hess emphasizes in both the beginning and the end of the paper (pp. 7 and 20) that the discussed problem is the spatial generalization of the topic considered by Brewster (1819) in connection with his kaleidoscopes.

Hess, E. (1882) Ueber Polyeder-Kaleidoscope, [About polyhedral kaleidoscopes, in German], *Sitzungsbereichte der Gesellschaft zur Beförderung der gesammten Naturwissenschaften zu Marburg*, 1882, 9-12; Review, *Beiblätter zu den Annalen der Physik und Chemie*, 6 (1882), 742-743.

Using our terminology, this article deals with trihedral kaleidoscopes. Hess, independently of Möbius, rediscovered these devices and demonstrated them on February 15, 1882, during the session of a scientific society (Gesellschaft zur Beförderung der gesammten Naturwissenschaften zu Marburg). The actual devices were constructed by the Hamburg optician Krüss.

Hess, E. (1883) *Einleitung in die Lehre von der Kugelteilung mit besonderer Berücksichtigung ihrer Anwendung auf die Theorie der gleichflächigen und gleicheckigen Polyeder*, [Introduction to the Study of Spherical Tiling with Special Regard to its Application to the Theory of Isohedral and Isogonal Polyhedra, in German], Leipzig: Teubner, x + 475 pp., 16 tables.

See chap. 57 "Anwendung auf die Theorie der räumlichen Winkelspiegel (Polyederkaleidoskope)" [An application for the theory of spatial corner-mirrors (polyhedral kaleidoscopes)], pp. 262-265.

Hess, E. (1889) Ueber Polyëderkaleidoskope und deren Anwendung auf die Krystallographie, [About polyhedral kaleidoscopes and their application for crystallography, in German], *Neues Jahrbuch für Mineralogie, Geologie und Palaeontologie*, 1889, Band 1, 54-65.

This paper deals with trihedral kaleidoscopes (not with polyhedral ones, according to our terminology) and gives detailed data about the representation of various crystal-forms (polyhedra). Hess remarks that earlier he was not aware of the related works of Möbius (p. 55; cf., Hess, 1882, p. 10). Note that this paper inspired Fedorov (1890) to publish a paper in German about his similar devices and claiming his priority over Hess by referring to a demonstration in 1883. It is interesting to compare Hess's activity with Fedorov's. Hess, after a theoretical paper (1879), demonstrated his trihedral kaleidoscopes on February 15, 1882 during the session of a scientific society (Hess, 1882; see also Hess, 1883, p. 265; 1889, p. 55), i.e., exactly one year before the similar demonstration by Fedorov on February 15, 1883 (Fedorov, 1883). Both of them referred to kaleidoscopes very briefly in their comprehensive monographs on polyhedra (Hess, 1883; Fedorov, 1885). Note that there was a scholarly debate between Fedorov and Hess in connection with some mathematical aspects of Fedorov's monograph of 1885. We should add that other scholars, besides Hess and Fedorov, also discovered the applicability of kaleidoscopes or mirror systems in the teaching of crystallography, see, e.g., Werner (1882), Fialowsky (1889). Hess became aware of Werner's work only later (cf., Hess, 1889, p. 55).

Kennedy, J. and Thomas, D. (1982) *Kaleidoscope Maths*, Palo Alto, Calif.: Creative Publications, 160 pp.

Suggested for Grades 4-10; mirrors are enclosed.

Kükelhaus, H. (1975) *Fassen, fühlen, bilden: Organerfahrungen im Umgang mit Phänomenen*, [Perceiving, Feeling, Creating: Sensory Experiences in Association with Phenomena, in German], Köln: Gaia, 141 pp., 5 tables.

See chap. 20 "Symmetrie, Kaleidoskop", 78-83. Although only dihedral kaleidoscopes are discussed, the methodology and educational philosophy of Kükelhaus make this book interesting.

Mackay, A.L. (1967) A polyhedral kaleidoscope for demonstrating several point symmetry groups, *Physics Education*, 2, 266-267.

The use of "corner kaleidoscopes" (i.e., trihedral devices) for the demonstration of various polyhedra, sphere packings, tilings, and polyhedral viruses. The paper refers to the discussion by Coxeter (1961).

Marshall, L.A. and Marshall, E.B. (1983) Reflection in a polished tube, *Physics Teacher*, 21, 105.

Möbius, A.F. (1849) Ueber das Gesetz der Symmetrie der Krystalle und die Anwendung dieses Gesetzes auf die Eintheilung der Krystalle in Systeme, [On the symmetry law of crystals and the application of this law to the distribution of crystals into systems, in German], *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig: Mathematisch-Physische Klasse*, 1849, 65-75; Reprint, *Journal für die reine und angewandte Mathematik*, [Crelle's Journal], 43 (1852), 365-374; Reprint, Möbius, A.F., *Gesammelte Werke*, Vol. 2, ed. by Klein, F., Leipzig: Hirzel, 1886, 349-360; 2nd ed., Wisbaden, 1967.

See the concluding remark (last sentence) about crystals as kaleidoscopic figures and about the correspondence between crystal systems and mirror-corners of kaleidoscopes (*Spiegelwinkel des Kaleidoskopes*). The paper is a pioneering work on mathematical crystallography, written by one of the leading mathematician of the 19th century.

Möbius, A.F. (1851) Ueber symmetrische Figuren, [On symmetrical figures, in German], *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig: Mathematisch-Physische Klasse*, 1851, 19-28; Reprint, *Journal für die reine und angewandte Mathematik*, [Crelle's Journal], 44 (1852), 335-343; Reprint, Möbius, A.F., *Gesammelte Werke*, Vol. 2, ed. by Klein, F., Leipzig: Hirzel, 1886, 361-372; 2nd ed., Wisbaden, 1967.

This paper is a continuation of the earlier one (Möbius, 1849), including a detailed discussion of Bravais's parallel work of 1849. Although Möbius does not return to the topic of kaleidoscopes, he does discuss the related crystallographic symmetries. It is interesting to note that, when preparing Möbius's collected works in the 1880s, the corresponding papers were edited by Felix Klein, a pioneer of group-theoretic thinking in mathematics and physics.

Robertson, J.M., (1986) Geometric constructions using hinged mirrors, *Mathematics Teacher*, 79, 380-386.

Routledge, R. (1891) *Discoveries and Inventions of the Nineteenth Century*, 8th extended ed., London: Routledge, xiv + 681 pp.; Reprint, London: Bracken, 1989.

See the descriptions of the kaleidoscope, the polemoscope, and various apparatuses for "ghost illusion", pp. 303-309.

Series, C. (1990a) Non-Euclidean kaleidoscopes, *Interdisciplinary Science Review*, 15, No. 3, [forthcoming].

The described kaleidoscopes are made with circular mirrors. Note that Stewart (1986) uses the term "non-Euclidean kaleidoscopes" in a different sense.

Series, C. (1990b) Fractals, reflections and distortions, *New Scientist*, [forthcoming].

[Schwabe, C. and N.] (1986) Spiegel-Welt, [Mirror-world, in German], In: Mazzola, G., ed., *Symmetrie in Kunst, Natur und Wissenschaft*, Vol. 3, [Exhibition Catalog: Mathildenhöhe Darmstadt, June 1-August 24, 1986], Darmstadt: Roether, 23-39.

The exhibition presented large models of various trihedral kaleidoscopes, including a 120-fold device for presenting icosahedral and other related structures (a kaleidoscope is called *n*-fold if the number of images, including the original object, is *n*). The cited part (chap. A2) of the catalog includes photographs of the exhibited kaleidoscopes with brief comments by Caspar or Nikolaus Schwabe.

Shubnikov, A.V. (1939) Prostranstvennye kaleidoskopy, [Spatial kaleidoscopes, in Russian], *Trudy Laboratorii Kristallografii AN SSSR*, 1, 21-24.

Shubnikov, A.V. (1940) *Simmetriya: Zakony simmetrii i ikh primenenie v nauke, tekhnike i prikladnom iskusstve*, [Symmetry: The Laws of Symmetry and their Application in Science, Technology, and Applied Art, in Russian], Moskva: Akademiya Nauk SSSR, 175 pp.

See, pp. 27, 54-55, 74-76, 126-128, 150-153; cf., Shubnikov and Koptsik (1972) which is a considerably enlarged version of this book.

Shubnikov, A.V. and Koptsik, V.A. (1972) *Simmetriya v nauke i iskusstve*, Moskva: Nauka, 339 pp.; English trans., *Symmetry in Science and Art*, New York: Plenum Press, 1974, xviii + 420 pp.

See, in the English edition, the remark on simple kaleidoscopes, pp. 32-33; "Fedorov kaleidoscopes for producing figures with a singular point", 68-69; "Kaleidoscopes for forming one-sided bands", 92-93; "Kaleidoscopes for network patterns", 171-172; "Kaleidoscopes for three-dimensional periodic discontinua of the highest symmetry", 200-203.

Stewart, I. (1986) Non-Euclidean kaleidoscopes, *Nature*, 323, 114

A brief review of the result of Vinberg (1981), based on his more detailed paper of 1985, in a popular form. Note that Series (1990a) constructs her non-Euclidean kaleidoscopes using circular mirrors, while Stewart explains in his paper the impossibility of higher dimensional (hyper)plane mirror kaleidoscopes in non-Euclidean hyperbolic (or Bolyai-Lobachevskii) spaces.

Vinberg, E.B. (1981) Otsutstvie kristallografichekikh grupp otrazhenii v prostranstvakh Lobachevskogo bol'shoi razmernosti, *Funktsional'nyi analiz i ego prilozheniya*, 15 (1981), No. 2, 67-68; English trans., Absence of crystallographic groups of reflections in Lobachevskii spaces of large dimension, *Functional Analysis and its Applications*, 15 (1981), 128-130; More detailed paper with the same Russian title, *Trudy Moskovskogo Matematicheskogo Obshchestva*, 47 (1984), 68-102 and 246; English trans., The non-existence of crystallographic groups of reflections in Lobachevskii spaces of large dimension, *Transactions of the Moscow Mathematical Society*, 47 (1985), 75-112.

There exists no crystallographic reflection group (generated by reflections in hyperplanes) with a bounded fundamental polytope in n-dimensional hyperbolic (Bolyai-Lobachevskii) spaces where $n \geq 30$. Although the cited papers

are very technical, we may refer to a popular scientific review of the topic, and its connection with generalized kaleidoscopes, written by Ian Stewart (1986).

Walker, J. (1978) The amateur scientist: Moiré effects, the kaleidoscope and other Victorian diversions, *Scientific American*, 239, No. 6, 138-147.

Walker, J. (1985) The amateur scientist: The kaleidoscope now comes equipped with flashing diodes and focusing lenses, *Scientific American*, 253, No. 6, 124-130; Further remarks, 254 (1986), No. 1, 102-103; No. 2, 117.

Walker, J. (1986) The amateur scientist: Mirrors make a maze so bewildering that explorer must rely on a map, *Scientific American*, 254, No. 6, 98-104.

Walter, M. (1985) *The Mirror Puzzle Book*, New York: Parkwest Publications, 32 pp.; Reprint, Stradbroke, England: Tarquin.

Werner, G. (1882) Ein kristallographisches Anschauungsmittel, [A crystallographic demonstrative device, in German], *Program des königlichen Realgymnasium in Stuttgart*, Stuttgart: Realgymnasium.

See also the remarks about Werner's device by Hess (1889, p. 55).

Wright, V. (1989) [Kaleidoscopes for presenting Platonic solids], [Demonstrations for elementary school students], Cambridge, Mass.; Brief summary in: Kappraff, J. *Connections: The Geometric Bridge Between Art and Science*, New York: McGraw-Hill, [forthcoming].

Yoder, W.D. (1988) *Kaleidoscopes: The Art of Mirrored Magic*, Albuquerque, N.M.: Yoder [private publication], xvi + 317 pp.

We did not have access to this book, but we hope to review it later.

The address of the *Gallery and Shop AHA*, co-founded by Nikolaus and Caspar Schwabe, is: Spiegelgasse 14, CH-8001 Zürich, Switzerland. Note that "Spiegelgasse" means in English "Mirror Street"; it is clearly a good place for a Gallery which is interested in kaleidoscopes.

The cited papers by Möbius (1849), Hess (1879; 1882; 1883), and Fedorov (1883; 1885) coincide with those early cycles of the interdisciplinary study of symmetry which were discussed in the "Manifesto on (dis)symmetry": 1847 ± 3 and 1882 ± 3 (cf., this quarterly, Vol. 1, No. 1, p. 12).

We do not give further references to those Greek contexts where *kalos* and *symmetria* are used together (Plato's *Philebus*, 64e, and Aristotle's *Metaphysica*, 1078a), because these indexes, used frequently in classical philology, make possible to locate the quoted fragments in any critical edition or translation (usually the indexes are printed parallel with the text on the margin). Note that the Greek word *symmetria* here means "proportion". The transition of meaning from proportion to mirror reflection came in the scholarly literature only in early 19th century. There were, however, many earlier stages of this interesting process which we will discuss later in another paper. In connection with Legendre's terminology see J.J. Burckhardt's remarks in his monograph *Die Symmetrie der Kristalle* (Basel: Birkhäuser, 1988, pp. 14-15).

The monographic presentation of the foundations of geometry by reflection is due to the following work (note that the Russian title refers to symmetry):

Bachmann, F. (1959) *Aufbau der Geometrie aus dem Spiegelungsbegriff*, [Construction of Geometry with the Reflection Concept, in German], Berlin: Springer; 2nd enlarged ed., 1973, xvi + 374 pp.; Russian trans. of the 1st ed., *Postroenie geometrii na osnove ponyatiya simmetrii*, Moskva: Nauka, 1969, 379 pp.

Anamorphic art has a surprisingly wide-ranging literature. Here we list only a survey article and two books; the work of Leeman and his coauthors also discusses the history of this specific art from Leonardo to the present, while the booklet by Moscovich gives practical advice on how to draw the "distorted" pictures:

Gardner, M. (1975) Mathematical games: The curious magic of anamorphic art, *Scientific American*, 232, No. 1, 110-116.

Leeman, F., Elffers, J., and Schuyt, M. (1976) *Hidden Images: Games of Perception, Anamorphic Art, Illusion*, New York: Abrams, 166 pp.

This book grew out of an exhibition at the Rijksmuseum in Amsterdam.

Moscovich, I. (1988) *The Magic Cylinder Book*, Stradbroke, England: Tarquin, 34 pp.

Finally, we list the works where the expressions "kaleidocycle", "kaleidometrics", and "kaleidophone" were coined. These publications are associated with our topics at least in a semi-metaphorical form. Specifically, none of them are connected with the optical device, but all of them deal with patterns similar to those seen in kaleidoscopes. The first title cited below is a joint effort by a mathematician and a sculptor; we may build the cardboard models of the kaleidocycles using the enclosed layouts. The kaleidocycles are kinetic polyhedra, made of chains of tetrahedra, which can be turned to obtain new patterns based on the ornamentation of the surface (the authors cover them with Escher's periodic drawings). The second book on "kaleidometrics" can be recommended for younger children, although it is much more than just a coloring book. The young pupils can learn intuitively many ideas about symmetry and other concepts of elementary geometry. The third publication is a classical paper by Charles Wheatstone about a specific visualization of sound waves. Although the kaleidophone is a somewhat forgotten device, it would be interesting to revise this idea using the advantages of modern technology.

Schattschneider, D. and Walker, W. (1977) *M.C. Escher Kaleidocycles*, New York: Ballantine Books, iv + 43 pp., 16 plates; 2nd ed., ibid., 1987; Reprint, Stradbroke, England: Tarquin Publications; Revised ed., Petaluma, Calif.: Pomegranate Artbooks.

Shaw, S. (1982) *Kaleidometrics: The Art of Making Beautiful Patterns from Circles*, Washington, D.C.: Paul Hamilton, 40 pp.; Reprint, New York: Parkwest Publications, 1986.

Wheatstone, C. (1827) Description of the kaleidophone, or phonic kaleidoscope: A new philosophical toy, for the illustration of several interesting and amusing acoustical and optical phenomena, *Quarterly Journal of Science, Literature, and the Arts*, New Series, 1, 344-351; Reprint, Wheatstone, C., *Scientific Papers*, London: Physical Society of London, 1879; German trans., *Annalen der Physik und Chemie*, [Poggendorff's Annalen], 10 (1827), 470-480.

Of course this list about the artistic, scientific, and educational aspects of the kaleidoscope and related questions is far being complete, while the very rich hobby literature is deliberately excluded. Our readers are kindly encouraged to inform us about any notable kaleidoscopes and further publications, in any language, because the same topic may feature at the *Second Interdisciplinary Symmetry Symposium and Exhibition* in 1992 (cf., "Call for kaleidoscopes", Vol. 1, No. 1, p. 93).

Dénes Nagy

SYMMETRIC REVIEWS 1.2 (BRIEF NOTES ABOUT NEW PUBLICATIONS)

The "Symmetric Reviews" (SR) combines two separate sub-sections of the first issue: the list of new publications and the brief reviews (cf., Vol. 1, No. 1, pp. 98-99 and 102-105). The brief notes about books and papers, usually not longer than 20 lines, are not conventional reviews; their main goal is to emphasize the connections with symmetry and, in same cases, the required background (sometimes a simple title may hide a paper with very many technical details; in other cases a complicated title does not definitely exclude the broader audience). Unfortunately the original abstracts of papers are not always satisfactory for our purposes; those are often addressed to experts on specific fields. Currently, all notes are written, or adapted from the indicated sources, by the section editor. We hope, however, to form a group of reviewers in the future.

The SR is open to everyone who volunteers to send us books, reprints, and photocopies of his or her symmetry-related publications. All items will also be included in the computerized data bank of ISIS-Symmetry (cf., "The Bibliographic Project of ISIS-Symmetry", Vol. 1, No. 1, pp. 100-101). For the purpose of easy citation, each item has an SR-number, referring to the volume and issue numbers of this journal, as well as to the serial number of the item based on the alphabetical order of the authors or editors. It is followed, in parentheses, by subject classification and other keywords (currently it is just an ad hoc *system which should be revised and corrected later). If the full name of an author is not available on the title page, but we could find it in other sources, this one is added in brackets. Hopefully these reviews will provide a good experimental basis for the "Taxonomy project" proposed by Ted Goranson (see in this issue, p. 208).*

Correspondence should preferably be sent to both the section editor (for reviewing) and the Budapest Office (for the data bank).

SR 1.2 - 1 (Ornamental art: Hungarian; Ethnomathematics: geometric design; History of technology)
Bérczi, Szaniszló, *Szimmetria és techné a magyar, avar és hanti díszítőművészetben*, [Symmetry and *technê* in the Hungarian, Avar, and Hanti Ornamental Art, in Hungarian], Budapest: Eötvös Loránd Tudományegyetem, Általános Technika Tanszék; Leuven: Leuveni Katolikus Egyetem [Catholic University of Louvain], Collegium Hungaricum, 1986-1987, 59 pp.

The brochure starts with a very useful introduction to the theory of plane symmetry groups, emphasizing the technological aspects of ornamental arts as well. There is an interesting illustration on how the seven frieze groups appear in the nature. True, the group that is generated by a translation and a 2-fold rotation, without any mirror reflection, is illustrated by a rope in profile, which resembles an artificial object rather than a winding plant (note that ropes can be analyzed by spatial rod groups). Some of the wallpaper groups are derived by weaving, which is a very important addition to the topic. The author illustrates the 17 wallpaper groups by his own patterns built with an Old-Hungarian motif. It is nice to see these beautiful illustrations, because very many authors do not take the trouble to create similar patterns (the literature is full of patterns reprinted from Shubnikov and Koptsik's book *Symmetry in Science and Art*, New York: Plenum Press, 1974). The study of Old-Hungarian ornamental art led the author to the introduction of the concept of double friezes; cf. also his paper "Escherian and non-Escherian developments of new frieze types in Hanti and Old Hungarian communal art", In: Coxeter, H.S.M. et al., eds., *M.C. Escher: Art and Science*, Amsterdam: North-Holland, 1986, 349-358. Another novelty of the reviewed publication is the table of wallpaper groups "weaved" by double friezes. This brochure is a masterpiece of the anthropological application and generalization of the crystallographic symmetry groups. Illustrations: very many unnumbered figures, including photographs, diagrams, maps. References: 0. Address: Department of General Technology, Eötvös Loránd University, Budapest, Rákóczi u. 5, H-1088 Hungary.

SR 1.2 - 2 (Structural biology: phyllotaxis; Quasicrystal)
Bursill, L[es], Peng, Ju Lin, and Fan, Xudong, Spiral lattice concepts, *Modern Physics Letters*, B1 (1987), 195-206.

Note in advance that the discussed topic is very important in various fields, e.g., in the biomathematical study of visible spiral structures on plants (including the leaf arrangement, or phyllotaxis, cf., Jean, SR 1.2 -3); in the theory of the quasicrystalline structures (a fact emphasized by Bursill and his coauthors); as well as in musicology (cf., Tusa, SR 1.1-11, in Vol. 1, No. 1, p. 99). The paper briefly surveys some results in generalized crystallography, including the Penrose tilings and Mackay's approach. The authors emphasize a new step in the departure from classical crystallographic symmetries "[...] whereby both translational and orientational [rotational] symmetries are discarded, yet the structure is nevertheless described by a simple mathematical algorithm, and hence must be regarded as perfectly ordered". After describing some common 2-dimensional spirals in polar coordinates (reciprocal, parabolic, golden, Archimedian, logarithmic, and exponential), a method of quantization is introduced for the coordinates by which lattice points can be generated. The parabolic spiral lattices with irrational divergence angles are discussed in a separate chapter because of their importance for phyllotaxis, as well as the Penrose tilings (the visible spirals, or parastichies, have rotational symmetry around the center, but the lattice points lack this symmetry. Analogous lattices can be introduced in 3-dimensional space using cylindrical coordinates. Illustrations: 7. References: 10. Address: School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia.

SR 1.2 - 3 (Biomathematics: phyllotaxis)
Jean, R[oger] V., Phyllotactic pattern generation: A conceptual model, *Annals of Botany*, 61 (1988), 293-303.

This paper is written by one of the leading experts on phyllotaxis who significantly contributed to the newest Renaissance of this topic, by authoring several publications and organizing a conference (Berkeley, Calif., 1985). Author's abstract: "A conceptual model is proposed here that shows how all types of whorled and peculiar patterns in phyllotaxis derive straightforwardly from normal and anomalous spiral patterns. This is a complete model of phyllotaxis, integrating the author's interpretative model for generating spiral patterns. The paper underlines that a better understanding of the variety of phyllotactic patterns, and of the transition between them, involves a phylogenetic perspective. It stresses the working hypothesis that spiral patterns are primitive and that all other patterns, such as whorled systems, are by-products of evolution from spirality. An important epistemological consequence on mathematical modelling is drawn out of this hypothesis, namely that models of knowledge or interpretative models, able to take care of the spiral patterns, must be formulated and then followed by simulation, mechanistic or conceptual models that are able to reproduce the transition to all other types of patterns." Illustrations: 1. References: 26. Address: Refer to the Board of ISIS-Symmetry.

SR 1.2 - 4 (Geometry: 4-dimensional)
Miyazaki, Koji, Visualization of a die and other objects in four dimensional space. *Leonardo*, 21 (1988), 56-60 and color plate (between pp. 70 and 71).

Author's abstract: "This paper proposes a geometric method for imagining and constructing the 4-dimensional analogue of a 3-dimensional object, in this case a cubical die. First, a 4-dimensional analogue of a rolling 3-dimensional die is imagined in 4-space according to geometric properties. Second, it is projected orthogonally into 3-space as a solid that is shown by orthogonal projection onto a plane. Lastly, this procedure is applied to other images to illustrate additional 4-dimensional analogues of 3-dimensional objects." An interesting feature of the paper is a 4-dimensional Einstein portrait, having three eyes and three ears. Illustrations: 8, references: 10. Address: Refer to the Board of ISIS-Symmetry.

SR 1.2 - 5 (Interdisciplinary collection)
Tyukhtin, V.S. and Urmantsev, Yu[nir] A[bdullovich], eds., *Sistema, simmetriya, garmoniya*, [System, Symmetry, Harmony, in Russian], Moskva: Mysl', 1988, 315 pp.

This collection of essays includes two parts: (1) "Sistema i khaos. Polimorfizm i izomorfizm" [System and chaos. Polymorphism and isomorphism], papers by V.S. Tyukhtin, Yu.A. Urmantsev, Yu.S. Larin, V.Ya. Dalin, I.P. Shapanov, and A.V. Malikov; (2) "Simmetriya i dissimmetriya. Garmoniya i disgarmoniya" [Symmetry and dissymmetry. Harmony and disharmony], papers by Yu.A. Urmantsev, V.A. Koptsik, A.M. Zamorzaev, Yu.K. Didyk, S.V. Petukhov, and Yu.I. Artem'ev. The collection covers a broad set of topics, from system theory to crystal-physics, from biology to fine arts. This interdisciplinarity is also emphasized by the cover design of the book: the three keywords of the title are associated there with chemical elements of the periodic system, dancing girls forming a frieze pattern, and musical notes. References (collected at the end of the book): pp. 299-315. Address: Yu. A.

Urmantsev, Institut fiziologii rastenii AN SSSR (Institute of Plant Physiology), SU-127276 Moskva, ul. Botanicheskaya 35, USSR.

SR 1.2 - 6 (Crystallography: space groups; History of science)
Wilson, A.J.C., Space groups rare for organic structures: 1. Triclinic, monoclinic and orthorhombic crystal classes, *Acta Crystallographica*, A44 (1988), 715-724.

Although the total number of the theoretically possible 3-dimensional space groups (repetition-types) is 230, about half of the organic compounds (as recorded in the Cambridge Structural Database through 1979) typically crystallize in the space groups that permit close packing of ellipsoids (especially in two groups, $P1$ and $P2_1/c$), and there are some space groups for which there are no entries. The author reconsiders the distribution among the space groups, because the earlier data may have been infected by "sociological aspects", that is, the subjective choice of structures to be determined (e.g., crystallographers disliked the determination of non-centrosymmetric structures). The more recent computer programs exclude the above subjectivity. The author reduced the sample to those cases where no problems (ambiguities, disorders, etc.) were reported in connection with the determination of the space groups. Finally, his reduced sample included 34,730 substances. The number of space groups with no examples is about 75 in this restricted set, a much higher number than for the earlier data. The paper includes useful distribution tables and other statistical data for specialists. Illustrations: 0 (there are 7 tables). References: 14. Address: Crystallographic Data Centre, University Chemical Laboratory, Lensfield Road, Cambridge CB2 1EW, England.

Dénes Nagy

There are many disciplinary periodicals and symposia in various fields of art, science, and technology, but broad interdisciplinary forums for the connections between distant fields are very rare. Consequently, the interdisciplinary papers are dispersed in very different journals and proceedings. This fact makes the cooperation of the authors difficult, and even affects the ability to locate their papers.

In our "split culture", there is an obvious need for interdisciplinary journals that have the basic goal of building bridges ("symmetries") between various fields of arts and science. Because of the variety of topics available, the concrete, but general, concept of symmetry was selected as the focus of the journal, since it has roots in both science and art.

SYMMETRY: CULTURE AND SCIENCE is the quarterly of the *INTERNATIONAL SOCIETY FOR THE INTERDISCIPLINARY STUDY OF SYMMETRY* (abbreviation: *ISIS-Symmetry*, shorter name: *Symmetry Society*). ISIS-Symmetry was founded during the symposium *Symmetry of Structure (First Interdisciplinary Symmetry Symposium and Exhibition)*, Budapest, August 13-19, 1989. The focus of ISIS-Symmetry is not only on the concept of symmetry, but also its associates (asymmetry, dissymmetry, antisymmetry, etc.) and related concepts (proportion, rhythm, invariance, etc.) in an interdisciplinary and intercultural context. We may refer to this broad approach to the concept as *symmetrology*. The suffix *-logy* can be associated not only with knowledge of concrete fields (cf., biology, geology, philology, psychology, sociology, etc.) and discourse or treatise (cf., methodology, chronology, etc.), but also with the Greek terminology of proportion (cf., *logos*, *analogia*, and their Latin translations *ratio*, *proportio*).

The basic goals of the *Society* are
(1) to bring together artists and scientists, educators and students devoted to, or interested in, the research and understanding of the concept and application of symmetry (asymmetry, dissymmetry);
(2) to provide regular information to the general public about events in symmetrology;
(3) to ensure a regular forum (including the organization of symposia, and the publication of a periodical) for all those interested in symmetrology.

The Society organizes the triennial *Interdisciplinary Symmetry Symposium and Exhibition* (starting with the symposium of 1989) and other workshops, meetings, and exhibitions. The forums of the Society are *informal* ones, which do not substitute for the disciplinary conferences, only supplement them with a broader perspective.

The Quarterly — a non-commercial scholarly journal, as well as the forum of ISIS-Symmetry — publishes original papers on symmetry and related questions which present new results or new connections between known results. The papers are addressed to a broad non-specialist public, without becoming too general, and have an interdisciplinary character in one of the following senses:
(1) they describe concrete interdisciplinary "bridges" between different fields of art, science, and technology using the concept of symmetry;
(2) they survey the importance of symmetry in a concrete field with an emphasis on possible "bridges" to other fields.

The Quarterly also has a special interest in historic and educational questions, as well as in symmetry-related recreations, games, and computer programs.

The regular sections of the Quarterly:
- **Symmetry: Culture & Science** (papers classified as humanities, but also connected with scientific questions)
- **Symmetry: Science & Culture** (papers classified as science, but also connected with the humanities)
- **Symmetry in Education** (articles on the theory and practice of education, reports on interdisciplinary projects)
- **Mosaic of Symmetry** (short papers within a discipline, but appealing to broader interest)
- **SFS: Symmetric Forum of the Society** (calendar of events, announcements of ISIS-Symmetry, news from members, announcements of projects and publications)
- **Symmetro-graphy** (biblio/disco/software/ludo/historio-graphies, reviews of books and papers, notes on anniversaries)
- **Reflections: Letters to the Editors** (comments on papers, letters of general interest)

Additional non-regular sections:
- **Symmetrospective: A Historic View** (survey articles, recollections, reprints or English translations of basic papers)
- **Symmetry: A Special Focus on ...** (round table discussions or survey articles with comments on topics of special interest)
- **Symmetry: An Interview with ...** (discussions with scholars and artists, also introducing the Honorary Members of ISIS-Symmetry)
- **Symmetry: The Interface of Art & Science** (works of both artistic and scientific interest)
- **Recreational Symmetry** (problems, puzzles, games, computer programs, descriptions of scientific toys; for example, tilings, polyhedra, and origami)

Both the lack of seasonal references and the centrosymmetric spine design emphasize the international character of the Society; to accept one or another convention would be a "symmetry violation". In the first part of the abbreviation *ISIS-Symmetry* all the letters are capitalized, while the centrosymmetric image iSIS! on the spine is flanked by "Symmetry" from both directions. This convention emphasizes that ISIS-Symmetry and its quarterly have no direct connection with other organizations or journals which also use the word *Isis* or *ISIS*. There are more than twenty identical acronyms and more than ten such periodicals, many of which have already ceased to exist, representing various fields, including the history of science, mythology, natural philosophy, and oriental studies. ISIS-Symmetry has, however, some interest in the symmetry-related questions of many of these fields.

continued from inside front cover

ISIS-Symmetry
Budapest, P.O. Box 4
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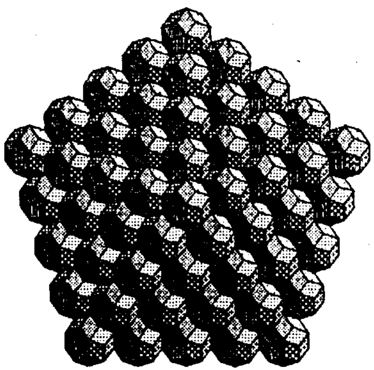
Symmetry of STRUCTURE

an interdisciplinary Symposium

Abstracts

II.



Edited by Gy. Darvas and D. Nagy

Buda Pest

August 13-19, 1989

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Symmetry

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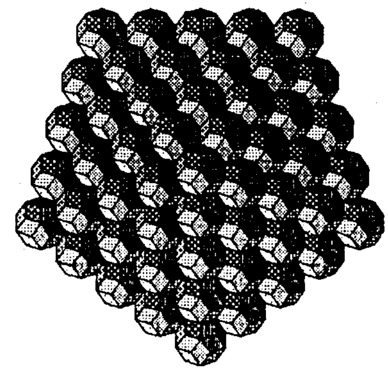

I.

Abstracts

an interdisciplinary Symposium

Symmetry of STRUCTURE

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Budapest, P.O. Box 4
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