Symmetry: Culture and Science

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Symmetry in a Kaleidoscope 2

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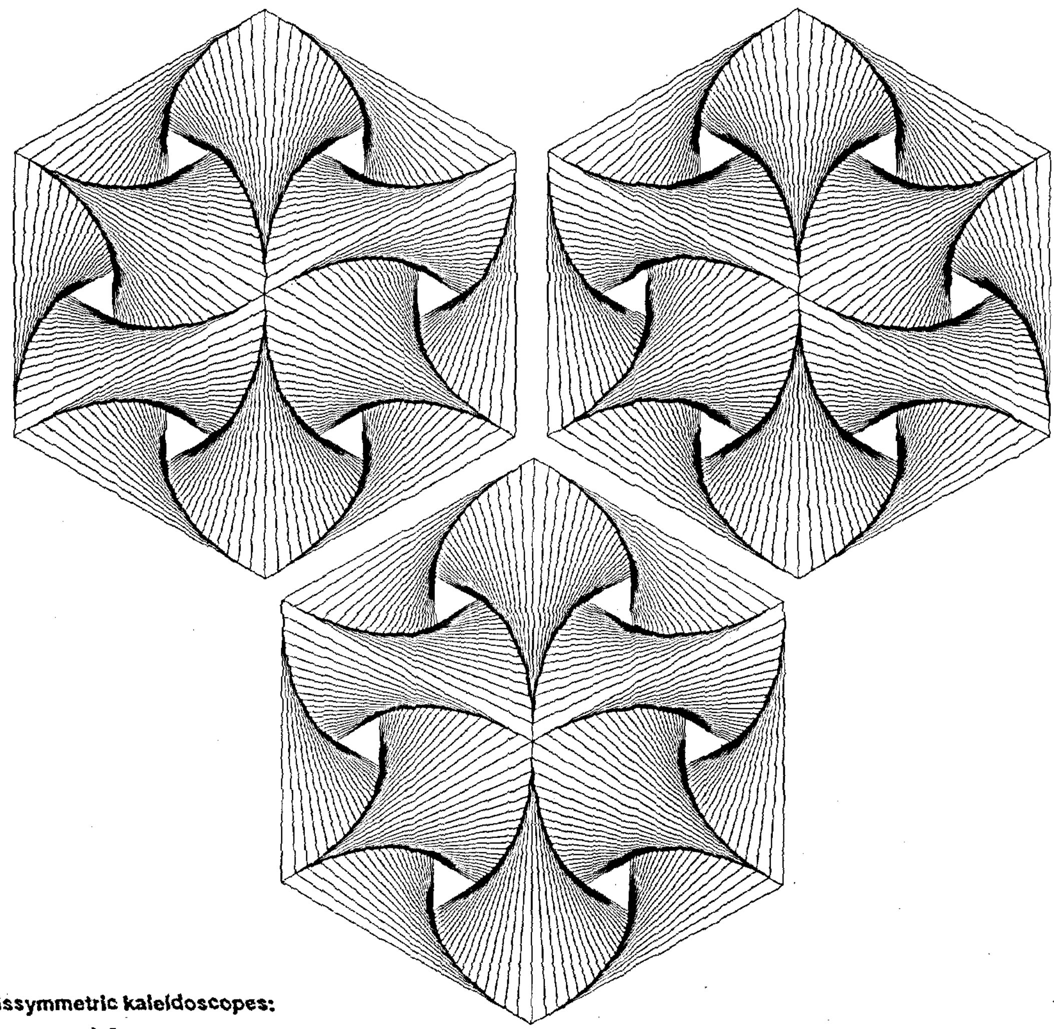

Dissymmetric kaleidoscopes:
Hommage à Pasteur

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SYMMETRY: CULTURE & SCIENCE

MATHEMATICS AND SYMMETRY
A PERSONAL REPORT

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QUESTION 1

what is symmetry?

Within mathematics the concept of symmetry is ubiquitous.
Perhaps more so in algebra and geometry than elsewhere, but in all branches of mathematics there have been practitioners who were particularly guided by the symmetry principle. In its simplest form, "twofold symmetry", this principle is expressed throughout mathematics by such words as "duality" or "complement", or even "if and only if". All mathematicians are familiar with the duality, and complementation, expressed in the laws of Boolean algebra which govern, for example, the unions and intersections of sets:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C),$$
$$(A \cap B)' = A' \cup B',$$
$$(A \cup B)' = A' \cap B'.$$

The appeal of these simple laws is certainly in large part due to their symmetry.

As has been pointed out by H. S. M. Coxeter (1948, pp. 162-163, 258-259) there can be little doubt that this instinctive search for, and response to, symmetry was inherited from George Boole by his daughter, Alicia Boole Stott. Coxeter tells us that as a young woman Alicia Boole was taught the rudiments of four-dimensional geometry by that Howard Hinton who later became known for his mystical books on higher space. In later years she determined, using only synthetic methods, the entire sequence of cross-sections of the regular four-dimensional polytopes. This led to a collaboration with P. H. Schoute, a skilled professional mathematician who had determined only the middle one of her sequence of sections by more orthodox analytic methods. At the age of 70 she was introduced to Coxeter with whom she collaborated on the study of a four-dimensional polytope he was investigating at the time. It is unimaginable that she accomplished all this work on regular (i.e. highly symmetric) polytopes, with no formal training in mathematics, except by the use of a powerful instinctive sense of symmetry.

In the foundations of geometry, the incomplete duality between the Euclidean axioms of plane geometry,

> Two points determine a unique line,

and

> Two lines determine a unique point, *except when they are parallel*,

leads to a formulation of the completely dual axioms for projective geometry. Projective geometry has been recognized since its invention as a particularly beautiful branch of mathematics, exactly because this duality, i.e. symmetry, between point and line (point and plane in space) does not need to be qualified by the exceptions which render it imperfect in Euclidean geometry.

Also, in the foundations of geometry, it has been suggested (Heath, 1956, vol. 1, p. 202) that the millenia-long search for a proof of Euclid's "parallel postulate" was motivated in part by the expectation of a symmetry between a theorem and its converse in geometry. The parallel postulate is equivalent to Euclid's Proposition I.29, "If two parallel lines are cut by a transversal, then the alternate interior angles are equal". Now, the converse to this proposition, "If two lines are cut by a transversal in such a way that the alternate interior angles are equal, then the two lines are parallel," was known to be true. (It is essentially the content of Proposition I.28, which immediately precedes I.29 in Euclid.) In Euclid's treatise, many facts occurred in *theorem — converse theorem* pairs. For example,

> I.5: If two sides of a triangle are equal then their opposite angles are equal,

is followed immediately by its converse,

> I.6: If two angles of a triangle are equal then their opposite sides are equal.

Such examples, combined with the complexity of the statement of the parallel postulate, led some commentators on Euclid's *Elements* to urge that it should be "struck out of the Postulates altogether; for it is a theorem involving many difficulties" (Heath, 1956). In this way the expected theorem–converse symmetry contributed to the persistent search for a proof of I.29, a search which culminated in the discovery of non-Euclidean geometries by Bolyai and Lobachevskii in the early nineteenth century. It is a commonplace now that this discovery is one of the most revolutionary in the history of thought because of its revelation of the independence of our mathematical model-making from any actual physical universe. This freedom led to the possibility of new models, such as relativity, to describe aspects of the universe unthought of by minds tied to the view that the human brain is constrained by its very construction to think only in the Euclidean way. It is interesting to speculate that this revolutionary development stems in part from the (misguided?) search for theorem—converse symmetry in Euclid's *Elements!*

Turning to more intricate manifestations of symmetry in geometry than the simple twofold symmetry of the preceding influential examples, the most familiar instance is probably that of the regular polyhedra and related highly symmetric structures. Recall that Euclid begins his *Elements* with the construction of the simplest of regular figures, the equilateral triangle (Proposition I.1), and concludes, nearly at the end of Book XIII, with the construction of the most complicated of the regular polyhedra, the icosahedron and dodecahedron (Propositions XIII.16,17). Apparently Proclus already suggested that this showed that the geometric purpose of the *Elements* was to provide a treatise on the construction of regular figures. In his own treatise on *Regular Polytopes*, Coxeter (1948, p. 13) reports that D'Arcy Thompson repeated this opinion to him.

Without believing that this is an entirely accurate characterization of Euclid's work, we still recognize the fundamental significance of these most symmetric figures throughout mathematics and science. As Marjorie Senechal recently put it: "Today we believe that it is not the classical form of the regular polyhedra that is significant: instead it is the high degree of order which they represent" (Senechal and Fleck, 1988). It is only necessary to mention Felix Klein's book *Lectures on the Icosahedron* to remind a mathematician of the unifying role of polyhedral symmetry, in the guise of group theory, in treating problems in analysis.

Group theory itself is, in one of its main aspects, the study of permutation groups or transformation groups, that is, the study of the symmetries of various mathematical or real-world objects. Even in "abstract" group theory, where the groups studied are not initially groups of symmetries of any particular mathematical objects, one of the main problems is often to find some such object on which the abstract group acts in a natural way as the group of all symmetries. In particular, now that the enumeration of the finite simple groups has been completed, much effort is being spent to understand them by creating more or less natural geometric objects whose symmetries are described by the new simple groups.

At a still deeper level, an interesting program was suggested, and partially carried out, by L. Fejes Tóth in his book *Regular Figures* (1964). He started from the observation that "extremum postulates often involve regularity". (Here by "regularity" he means "symmetry".) That is, "regular arrangements are generated from unarranged, chaotic sets by the ordering effect of an *economy principle*, in the widest sense of the word". Several such results are well-known. For example, among all polygons having a given perimeter and a given number of edges, the regular (that is, the most symmetric) polygon has the maximum area. Likewise, if we ask for the densest packing of congruent circles in the plane, the answer is the regular arrangement of circles at the centers of the cells of a hexagonal honeycomb.

In space such problems are much harder. If we ask for the convex polyhedron of maximum volume, having a given surface area and given number f of faces, the answer is known to be the regular polyhedron with f faces if $f = 4, 6,$ or 12. However, it has apparently not even yet been proved that the analogous question for polyhedra having 6 vertices is answered by the regular octahedron. An intriguing problem of this type, introduced by the biologist Tammes, is that of the optimal arrangement of orifices, or spines, on pollen grains. A proposed arrangement of 122 points on a sphere, derived from the regular icosahedron, was shown by Fejes Tóth (1964, pp. 232-233) not be optimal. Could it be that the (approximately) 122 spines on the pollen grain in Figure 1.1 constitute a solution to the optimization problem for 122 points? A brief discussion, with recent references, of these and similar problems is given by Fejes Tóth (1986).

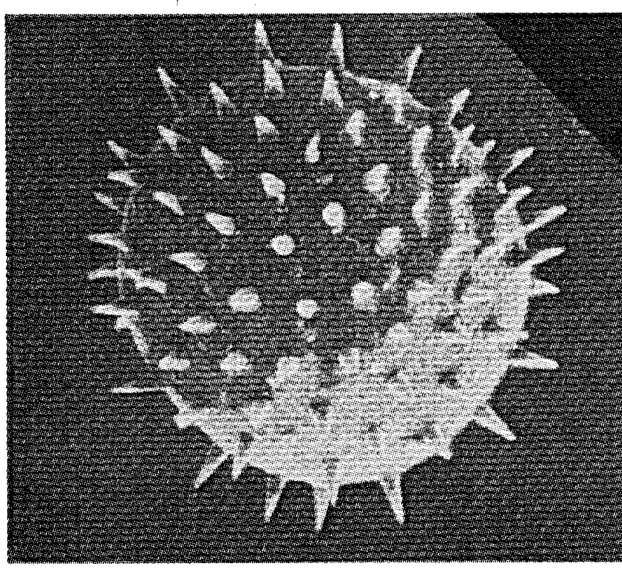


Figure 1.1: Pollen grain of *Hibiscus*.
(Scanning Electron Micrograph by Joan W. Nowicke, SmithsonianInstitution.)

To conclude this section I describe a famous example of a possible application of Fejes Tóth's program (not mentioned explicitly by him). This is the attempt, over more than a hundred years, to explain the regular arrangement of leaves around a stem in growing plants, or the apparently similar regular spiral arrangement of the florets in the heads of daisy-like flowers. Such an arrangement is most easily seen on a giant sunflower, where the numbers of spirals visible in two directions on the head are very often adjacent numbers of the Fibonacci sequence

$$\{f_i\} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

The photograph in Figure 1.2 shows a sunflower in which the numbers of spirals in the two directions are 55 and 89, a typical pair for large sunflowers. The purely mathematical properties of this sequence have been extensively studied since its introduction by Fibonacci in the 13th Century. The sequence is connected to aesthetic theory by the fact that the ratio of adjacent terms approaches the "golden ratio", $\varphi = (1 + \sqrt{5})/2 \approx 1.618\ldots$, which has been widely considered to be especially attractive. (Hence the prevalence of 3×5 and 5×8 notecards in America.)

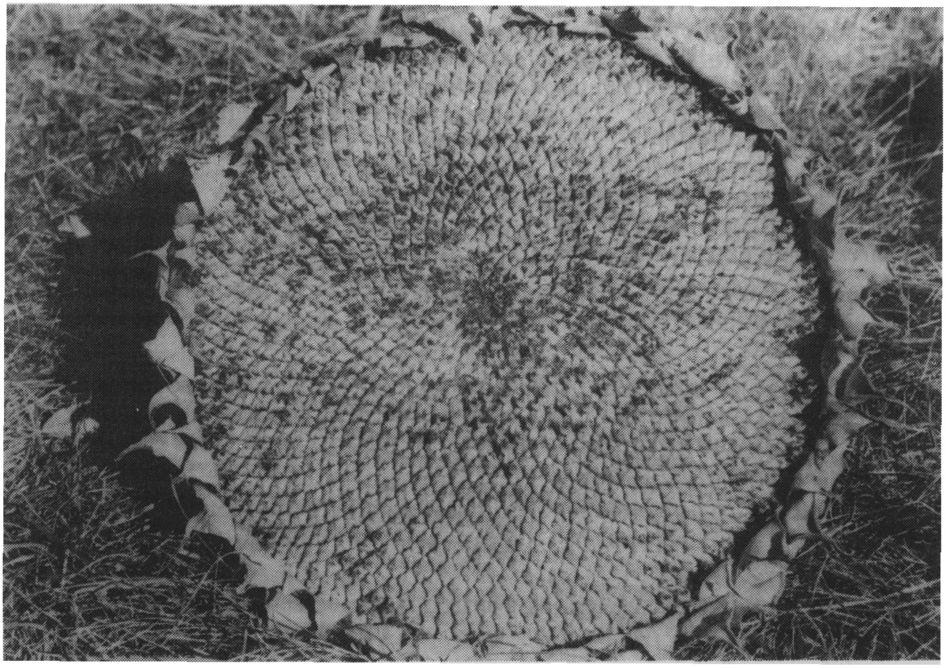


Figure 1.2: Sunflower. The number of spirals in one direction is 55; in the other it is 89. These constitute a Fibonacci pair.

Over the years many explanations have been proposed for the occurrence of such number pairs in plant growth, none of them completely satisfactory. The most recent ones imagine the plant is "trying" to apply some *economy principle*, such as to maximize the share of light received by each leaf, or to minimize the physical crowding of one floret by the next. Although other, perhaps chemical, mechanisms have been proposed as the means by which the plant carries out its "desires", there are aspects of the theory in which purely mathematical consequences of suitable economy principles lead to the observed symmetric spiral arrangements. A brief description, which owes much to the text and illustrations of Jean (1984), Marzec and Kappraff (1983), Dixon (1981), and Stevens (1974), of such a mathematical model which yields the observed shape and number of spirals with convincing accuracy follows. [See also the article "Symmetry in phyllotaxis" by Irving Adler in this issue — Eds.]

It has been well documented that the florets (which later become the seeds) are laid down successively along a logarithmic "growth spiral" whose equation in polar coordinates (r,θ) is $r = e^{k\theta}$ (Marzec and Kappraff, 1983, p. 205-207). This is also called an "equiangular spiral" because it is the locus of points (r,θ) such that at any point the radius vector makes a constant angle with the tangent vector at that point. The constant angle is A, where $\cot A = k$. Thus if $A = 90°$ we have $k = 0$ and the equation becomes $r = 1$, which is a circle of radius 1. If $k > 0$ is very near to 0 then A is very near to 90°; in this case the spiral is tightly wound around the origin — almost a circle.

It also seems that the florets are laid down at equal intervals along such a spiral, i.e. at points (r,θ) where θ successively takes on the values $\alpha, 2\alpha, 3\alpha, \ldots,$ for some fixed angle α. We want to apply a suitable *economy principle* to determine α. Since the angles $\alpha, 2\alpha, 3\alpha, \ldots$ can be represented as points $(1,\alpha)$, $(1,2\alpha)$, $(1,3\alpha), \ldots,$ on the unit circle, such an economy principle is one which will ensure that the first n of these points are "equally distributed" around this circle. (This is a mathematical version of the biological requirement that all florets should have equal access to the sunlight, or that later ones should not physically crowd any of those already laid down.) At first glance, a reasonable such principle seems to be:

First Attempt: For each n, the number of different lengths among the n arcs into which the n points $(1,\alpha)$, $(1,2\alpha)$, $(1,3\alpha)$, $\ldots$ $(1,n\alpha)$ divide the circle is minimized.

For if, among the n arcs, there are many different lengths, then some points are bunched together and others are not. However, it is a remarkable, though simple, fact (the truth of which the reader can readily verify experimentally by moving a paper angle around a circle) that:

For each choice of angle α the number of different lengths among the n arcs into which the points $(1,\alpha)$, $(1,2\alpha)$, $(1,3\alpha), \ldots,$ $(1,n\alpha)$ divide their circle is *never greater than three.*

Thus this first attempt is useless as a criterion for selecting one α instead of another. We discard it.

A slight modification is much more successful. Note first that if $n-1$ of the points have already been placed around the circle then the nth point divides one of the $n-1$ arcs into two parts. (Of course we are assuming that α is an irrational multiple of 360°; otherwise two points will eventually coincide.) It would seem desirable that it should divide this arc into nearly equal parts. We say that the point $(1,n\alpha)$ causes a "bad break" if one of the two arcs it creates is more than twice the length of the other. The appropriate economy principle is that *no bad break occurs*, that is, NBB Economy Principle: For each n, the point $(1,n\alpha)$ does not cause a "bad break" in the arc in which it appears.

An astonishing combination of beauty and function is expressed by the unexpected mathematical fact,

Theorem (Knuth, 1973, p. 543): The only value of α for which NBB is satisfied is $360°/\varphi^2 \approx 137.5°$ (or its complement, $360° - 360°/\varphi^2$, which is just $360°/\varphi$), where φ is the golden ratio, $(1+\sqrt{5})/2$.

Thus, for purely mathematical reasons, a plant might "choose" the angle 137.5° for laying down successive florets, or, in the case of leaf growth, for laying down successive leaves around a stem. There is considerable documentary evidence for the actual occurrence of just this angle.

Now, assuming that points are placed on a tight logarithmic spiral at equal intervals of $360°/\varphi^2$, what would be seen by the human eye? The eye does not see the original spiral, because it is so tightly wound, but sees certain "secondary" spirals. These are the ones seen in the sunflower photo in Figure 1.2. A simple computer drawing shows vividly how this can happen. In the example of Figure 1.3 some 750 points have been plotted at intervals of $360°/\varphi^2$ along the curve $r = e^{k\theta}$, with $k = 1/800$. It is a simple matter to count the spirals and see that there are 21 in one direction and 34 in the other direction. To obtain the numbers 55 and 89 seen in Figure 1.2 it is only necessary to choose a still smaller value of k, i.e. a still more tightly wound logarithmic spiral.

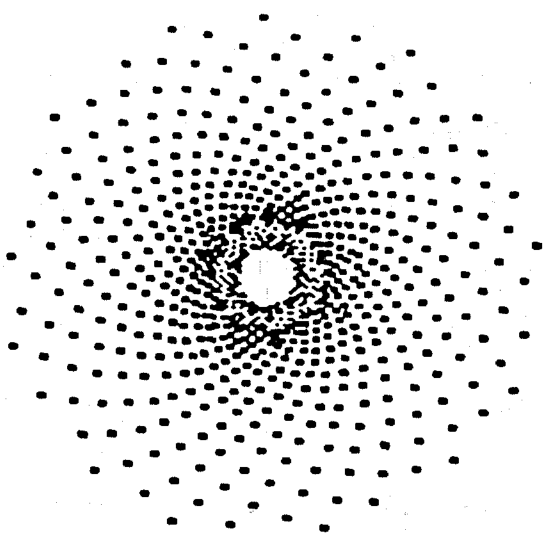
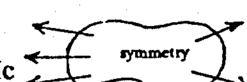


Figure 1.3: Points equally spaced along the logarithmic spiral $r = e^{k\theta}$ ($k = 1/800$) at intervals of $360°/\varphi^2$ appear to form 21 secondary spirals in one direction and 34 in the other.

QUESTION 2



Many examples of the interdisciplinary impact of geometric symmetry are comparatively well-known. Of these, one of the most striking is the story of the artist M. C. Escher who found, in an illustration from a paper by the geometer H. S. M. Coxeter, the solution to an artistic problem. As he explained in a 1958 letter to Coxeter (Coxeter, 1979), he had for a long time been interested in patterns containing motifs which kept getting smaller and smaller. He had solved the problem of creating such patterns when the small motifs approached a single point at the center of the pattern, as in his 1939 print *Development II* and the 1956 print *Smaller and Smaller* (Ernst, 1976, pp. 102-103). He had even made such patterns where the motifs became smaller and smaller toward a line limit. However, he had never been able to make a pattern whose motifs grew smaller towards an outside circle which would form a natural artistic boundary for his print. But in the copy of Coxeter's paper for a Canadian Symmetry Symposium (Coxeter, 1957), sent to him by Coxeter, he saw an illustration of the symmetry group [4,6] in the non-Euclidean hyperbolic plane which was exactly the inspiration he needed. He enclosed with the letter a copy of his *Circle Limit I*, based on this [4,6], and later made the famous *Angels and Devils* (*Circle Limit IV*) based on the same illustration. When he learned that there were infinitely many other symmetry groups which should serve the same purpose he used one of them, [3,8], as the basis for the two other *Circle Limit* prints, *II* and *III*.

An even better-known example of such an interplay between geometric symmetry and the rest of the world is the famous Rubik's Cube. Here was a toy, originally devised by Ernő Rubik to illustrate symmetry principles in a particularly concrete fashion, whose popularity swept the world in a way not seen since the "15 Puzzle", also based on symmetries (odd and even permutations) many decades earlier.

Of the hundreds of other, less well-known, examples, let me choose two which are quite different from each other. The first is a result of the researches of the archaeologist, Dorothy Washburn. Early in her career she was confronted with the problem of analyzing a collection of pottery in the Peabody museum of Harvard University. This pottery had been collected from well-documented sites in the Upper Gila River in what is now New Mexico. As she reported it to me, she began by using one of the conventional tools of pottery analysis, "typology", which had been particularly carefully developed for this heavily studied area of the Southwest USA. However, each morning when she returned to work on the classifications she had produced in the preceding day she found that she could no longer remember why pot x had been assigned type y. Not only would it be difficult for subsequent investigators to duplicate her "scientific" conclusions, but she herself could not confirm her own previous day's work after a 12 hour time lapse!

In this context she asked herself whether some truly objective attribute could be found for pottery so that today's classification would still stand up to tomorrow's scrutiny. *Symmetry* turned out to be just such an attribute. Most of the pottery in question was decorated, either with finite designs (often with rotational symmetry), one-dimensional designs ("bands"), or two-dimensional designs having a variety of symmetries. She extended the earlier ideas of Brainerd (1942) and Shepard (1948) by incorporating the two-color symmetry classification of the textile scientist H. J. Woods (1935-36). In this way she developed a suitable tool for the study not only of patterned pottery, but of patterned material of any type (Washburn, 1977).

Without knowing it at the time, Washburn had thereby repeated some of the work of crystallographers such as Belov and Tarkhova (1964), but in a context more directly suitable for use by archaeologists, anthropologists, and art historians. A systematic treatment of this method of symmetry analysis, incorporating the relatively standardized crystallographic notation, presented specifically for the study of patterned material of any kind is found in a more recent monograph (Washburn and Crowe, 1988).

To see how this symmetry tool could be applied, we look at an example suggested by Washburn herself. In Chaco Canyon, in what is now New Mexico, USA, there are dozens of impressive ruins, one of which is shown in Figure 2.1. At first glance these ruins, each containing many separate rooms, appear to have been inhabited by a large number of people. However, Chaco Canyon is extremely hot and dry, and evidence suggests that it was not much different when it was still inhabited, some 700-800 years ago. When archaeologists first became aware of these ruins, about 100 years ago, the apartment houses and "skyscrapers" of New York were the talk of the world. Thus it is not surprising that archaeological opinion of that time, and for many years afterward, estimated a huge population for Chaco Canyon, in spite of the obvious difficulties of living in such inhospitable country.

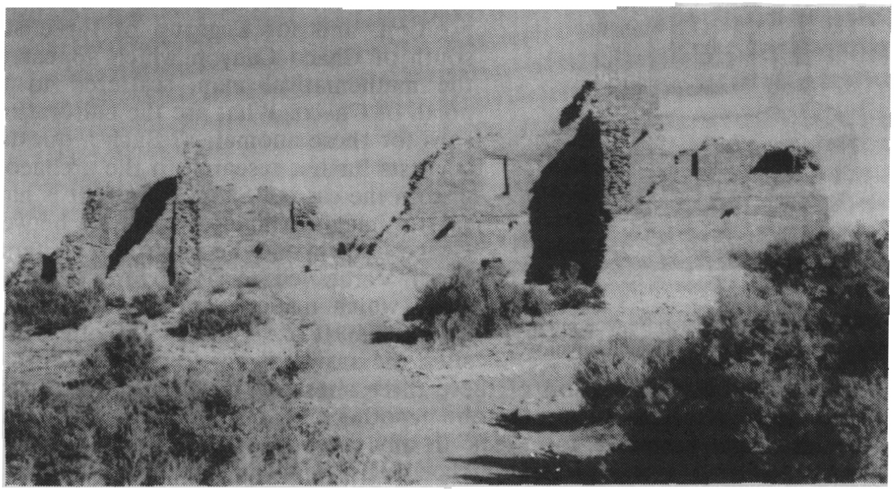


Figure 2.1: A portion of the ruins of Chaco Canyon as they appear today.

As archaeology became aware of the fact that there are many other, smaller, sites in the general vicinity of Chaco Canyon (and as the marvels of American culture came to be the "shopping malls" instead of skyscrapers) a new possibility presented itself. Perhaps the buildings of Chaco Canyon were not really densely inhabited, but were mainly the store rooms of a vast "shopping center", which was the center of a broad trade area. The fact that the pottery found in the Chaco outliers was much like the pottery of Chaco itself (see Figure 2.2) was consistent with this idea.

In collaboration with a statistician (Washburn and Mattson, 1985) Washburn compiled information about the relative frequency of occurrence of symmetry types (which she had already developed for her earlier study) of patterns found on pottery at the various sites in and around Chaco Canyon. If indeed these sites were in constant communication and trade with each other, it is a reasonable hypothesis that two sites with similar distributions of pattern types were close together, whereas those which have dissimilar distributions of pattern types were farther apart geographically. Using this hypothesis, she used "multidimensional scaling" to make a map of the various sites so that their distances from each other best agreed with the percentage correspondences between pattern types. And, indeed, this

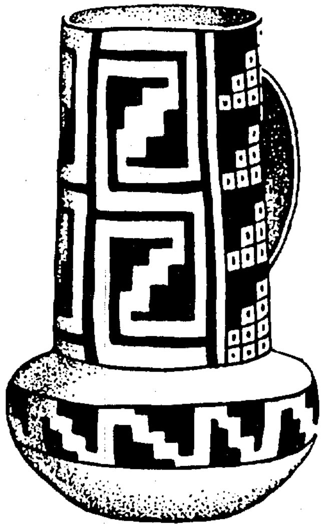


Figure 2.2: Typical Chaco Canyon pitcher (from Pueblo Bonito, Chaco Canyon).

mathematical map agreed in a general way with the actual known location of the sites round Chaco Canyon.

However, it is the *disagreements* which are the most instructive, since they suggest directions for further investigation. The two most notable disagreements are for Salmon Ruin, which appears on the mathematical map far to the east of its true geographical position; and for a group of three sites south of Chaco Canyon which appear, on the mathematical map, scattered to the north of Chaco. What are the cultural reasons for these anomalies? Such a question suggests further research to the archaeologist. In the case of Salmon Ruin, it is natural to imagine that its location between Chaco Canyon and the important centers at Mesa Verde led to a mixture of design styles which made it appear farther from Chaco than it really is. For the other three sites, the answer is not so obvious. Perhaps the fact that the actual location of these three sites was quite separate (farther south) from the other sites meant that the hypothesis of constant cultural trade and communication is not valid for them. In any case, the study of symmetry has suggested specific sites to which the archaeologist might give further attention to determine what particular aspects of culture contributed to these discrepancies.

My second example is very different. It represents the combined work of a mathematician, W. F. Orr, and a professor of French, C. W. Carroll. It began with the observation that one of the oldest of French verse forms, the Provençal sestina, was based not on *rhyme*, but on a symmetrical rearrangement of the final words of each line. More specifically, the sestina form consists of six stanzas of six lines each, followed by three final lines. The six final words of the six lines of stanza one are permuted to reappear as the final words of the six lines of stanza two. Applying the same permutation yields the final words of stanza three, and so on, in such a way that the same permutation takes the final words of stanza six back to the original order of stanza one. (Moreover these same six words occur in the three culminating lines of the sestina, in the same order as in the first stanza. In the present description, however, these three lines will be ignored.)

The oldest known sestina, and the one whose analysis led to the formulation of the general rule of construction, is the late 12th Century "Lo ferm voler", by Arnaut Daniel. It is reprinted in Carroll and Orr (1975). Two modern sestinas, "Paysage moralisé" and "Have a good time", were composed by the English poet W. H. Auden. ("Paysage moralisé" appears in the collection by Williams, 1951, pp. 750-751.)

The actual permutation used in Arnaut Daniel's sestina is $P = (163542)$. The literary name for it is *retrogradatio cruciata* ("crossed-reverse"); it leads to the spiral pattern in Figure 2.3 when $P(1), P(2), \ldots, P(6)$ are connected in order. As a generalization of the sestina, the French author Raymond Queneau, known for his applications of mathematics to literature, suggested the problem: "For which n can an n-ina exist?", and apparently answered that question for $n < 100$.

Orr and Carroll made this more precise by defining a spiral permutation P on the n symbols $1, 2, \ldots, n$ by

$$P(2r) = r, \qquad 1 \le 2r \le n,$$
$$P(2r+1) = n - r, \qquad 1 \le 2r+1 \le n.$$

An n-ina exists, by definition, if this permutation is *cyclic*. Their complete result, which contains Queneau's previously obtained results, is the Theorem (Carroll and Orr, 1975): An n-ina exists if and only if

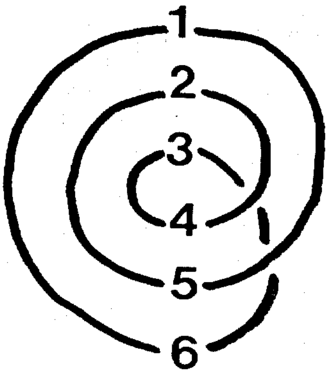


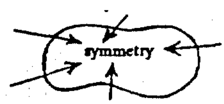
Figure 2.3: The spiral symmetry of the sestina.

(i) $2n + 1$ is a prime, p, and
(ii) either $+2$ or -2 is a generator of the multiplicative group of the finite field of p elements.

For example, in the range $3 \le n \le 20$, the values $3, 5, 6, 8, 9, 11, 14, 15, 18, 20$ satisfy condition (i). Of these, $8, 15$, and 20 do not satisfy condition (ii). Thus for $3 \le n \le 20$, an n-ina can exist exactly when $n = 3, 5, 6, 9, 11, 14$, or 18.

Surely this is a remarkable instance of an occurrence of symmetry in the world of literature which inspired a purely mathematical investigation and theorem!

QUESTION 3



What is the origin, in my own life and cultural background, of a continuing interest in symmetry and pattern? Are there any particular events or experiences which formed this interest?

Kh. S. Mamedov (1986, pp. 512-514) has related what in retrospect seems a dramatic contrast that aroused his curiosity about the prevalence of symmetry. He observed that in his early nomadic life the decorative objects of that culture were geometric (i.e. symmetric); on the other hand the nomads' physical surroundings were "a wonderful kingdom of various curved lines and forms". But when he moved to town to go to school he found the opposite to be true. The townspeople's physical environment was predominantly straight-line geometry; in contrast the decorative objects they chose were less symmetric, more ornate and curvilinear.

I can claim no such picturesque background to my later personal perceptions of symmetry and duality. Perhaps the austerity of a Nebraska childhood in the Great Depression, and the dryness of the climate in the Dust Bowl of the 1930's had an influence, but the connection is not quite apparent to me now, and certainly was not apparent to me then. I didn't know until later that I was living through a

Depression, and thought the hot dry winds were just a constant feature of Nebraska life.

Four particular memories from my 1930's childhood suggest the beginnings of a career in geometry and symmetry. The first was a sort of game which someone bought me, consisting of a 10 inch square board with some 400 round indentations (the integer points of a Cartesian coordinate system?), and several hundred small clay "marbles" in a variety of solid colors. With these marbles placed in the indentations a colored design could be produced. Today this seems like an exact precursor of a color computer monitor screen with its pixels which can be lit up to make colored designs on the screen, but of course this was long before the advent of electronic computers, or even ordinary television screens. In the 1960's I worked for a time on finite geometries, and at that time I always felt a direct connection between the kinds of pictures one draws to illustrate finite affine geometries and the pictures I plotted, as a child, on this marble picture board.

I also had a small loom on which I made beadwork "watchfobs", belt decorations, and the like in the style of some American Indian beadwork. In retrospect these beads were also the points of a finite geometry. Other visual images of this same type came from the game boards for the game of "Chinese checkers", played with marbles on an indented board much like my "pixel board", but placed in a hexagonal, rather than square, array whose boundary is a star hexagon. Of course the universal game of tic-tac-toe was also played in Nebraska. Its nine cells naturally correspond to the nine points of the finite affine geometry having three points per line, and the much desired three in a row is one of the lines of that geometry.

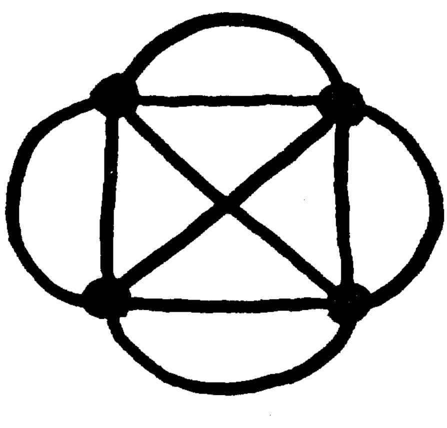


Figure 3.1: A problem from childhood: Draw this figure without lifting the pencil from the paper, or redrawing any line.

My second example is a simpler one. No elaborate commercial aids were required to while away the time in elementary school classrooms by trying to trace out an Euler circuit on the network shown in Figure 3.1! (An *Euler path* is a path containing each edge of the network exactly once. It is an *Euler circuit* if it ends at its starting point.) Most of us discovered fairly soon that if one of the diagonals was left out, or one of the outside loops, then we could find an Euler path. But so far as I know none of us ever realized just why. We were a long way from discovering Euler's result that such a path could be traversed if and only if there were exactly two (or no) vertices of odd degree. Indeed, the whole idea of "generalization" was foreign to us. We were only interested in this particular network, and never considered the possibility of devising other similar puzzles which might lead us to a general solution. Certainly we never imagined such beautiful and elaborate networks as the sand drawings of the Tchokwe people of Angola (Gerdes, 1988, 1990; Ascher, 1988b), or the *nitus* from the island of Malekula in Vanuatu (Ascher, 1988a). Figures 3.2 and 3.3 show turtles drawn by the Tchokwe and on Malekula, respectively.

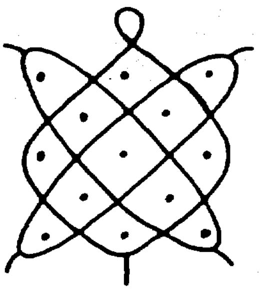


Figure 3.2: Tchokwe sand drawing of a turtle (Gerdes, 1988, p. 8)

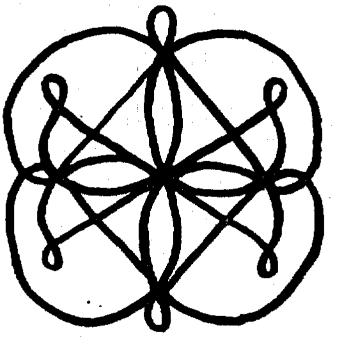


Figure 3.3: Turtle, as drawn on the island of Malekula, Vanuatu (Ascher, 1988a, p. 216)

There was, however, something about the symmetry of Figure 3.1 which appealed to us. Although the deletion of one arc to make it traversable might have made it still more appealing because of the subsequent availability of an Euler path, the resulting decrease in symmetry made it entirely unacceptable. Incidentally, I don't recall that we ever distinguished between an Euler path and an Euler circuit. Since it was only a pencil and paper problem it was a matter of indifference to us whether the path returned "home" to complete a circuit or not.

The third of these early symmetry influences was the image of long rows of upright cornstalks in cornfields on the flat Nebraska steppes. These were a common sight as we passed by in an automobile following the Platte River to visit relatives in Colorado. Of course, the cornfields had been *planted* in straight rows by tractors, these rows often being perpendicular to the direction of the highway. But I always noticed that "rows" were visible in many directions, not just at right angles to the highway, but at 45° angles, and others as well, with the angles varying depending on my line of sight through the field. In fact, I had a certain reputation for "squinting" when I thought no one was looking; I was only sighting along those unplanned rows of corn to confirm that the cornstalks really lined up.

The effect is the same as that of the regularly placed marbles on their board (in my first example), except that now the scene is viewed not *vertically* from above the board (cornfield), but *horizontally*, from the same level as the board. This experience, thus, was not a precursor of a study of finite geometries, but of the problem known as "Sylvester's Problem". This problem was originally formulated by Sylvester (1893) in terms of orchards (a comparative rarity in Nebraska at that time), not cornfields. In the interest of symmetry, he asked whether it is possible to plant the trees in an orchard so that they are all in rows, that is, so that any straight line ("row") containing two trees also contains at least one other tree. Sylvester'sproblem fascinated me from the first time I heard of it, and the mental picture I used to describe it was always in terms of cornfields. After Gallai proved that the answer to the original question in "no", one revision of the problem was to determine the minimum number of "short rows" (containing only two trees) in an orchard of n trees. My small contribution to that problem was made in a paper (Crowe and McKee, 1968).

When I was about 13, I found instructions in a boys' *How To Do It* book for making a "Tower of Hanoi" puzzle. This, my fourth example, is known to everyone nowadays, because it is a favorite recursion exercise in beginning computer programming courses. It consists of three pegs, on one of which is placed a "tower" formed of n disks of decreasing size. The puzzle is to move the disks from one peg to another, one at a time, and thus transfer the tower to another peg, never placing a larger disk on a smaller in the process.

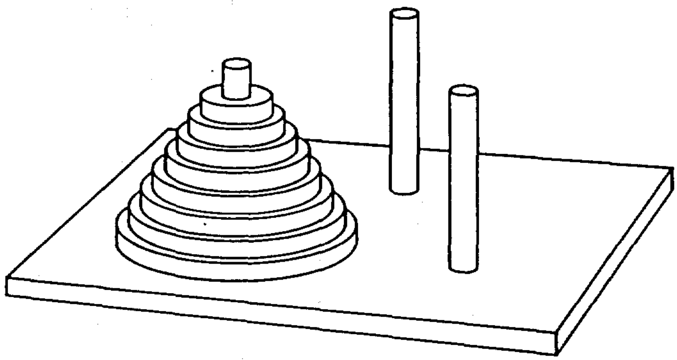


Figure 3.4 The Tower of Hanoi puzzle.

I don't think I realized then that the minimum number of moves required for a solution was $2^n - 1$, but I certainly realized that the kinetic symmetry and hypnotic monotony of the *process* of solution was very soothing and relaxing. This feature of the Tower still makes it more appealing to me than the "Chinese Rings" puzzle, which is almost equivalent from the purely mathematical point of view.

The fact that there was a direct connection between a simplest solution to the Tower and a particularly symmetric Hamilton circuit (that is, a circuit which contains each vertex exactly once) on the n-dimensional cube only became obvious to me when I was a beginning graduate student. That discovery became my first published paper (Crowe, 1956), and the wooden Tower I built at age 13 is used for lecture demonstrations to this day.

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