# Symmetry: Culture and Science

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Symmetry in a Kaleidoscope 1

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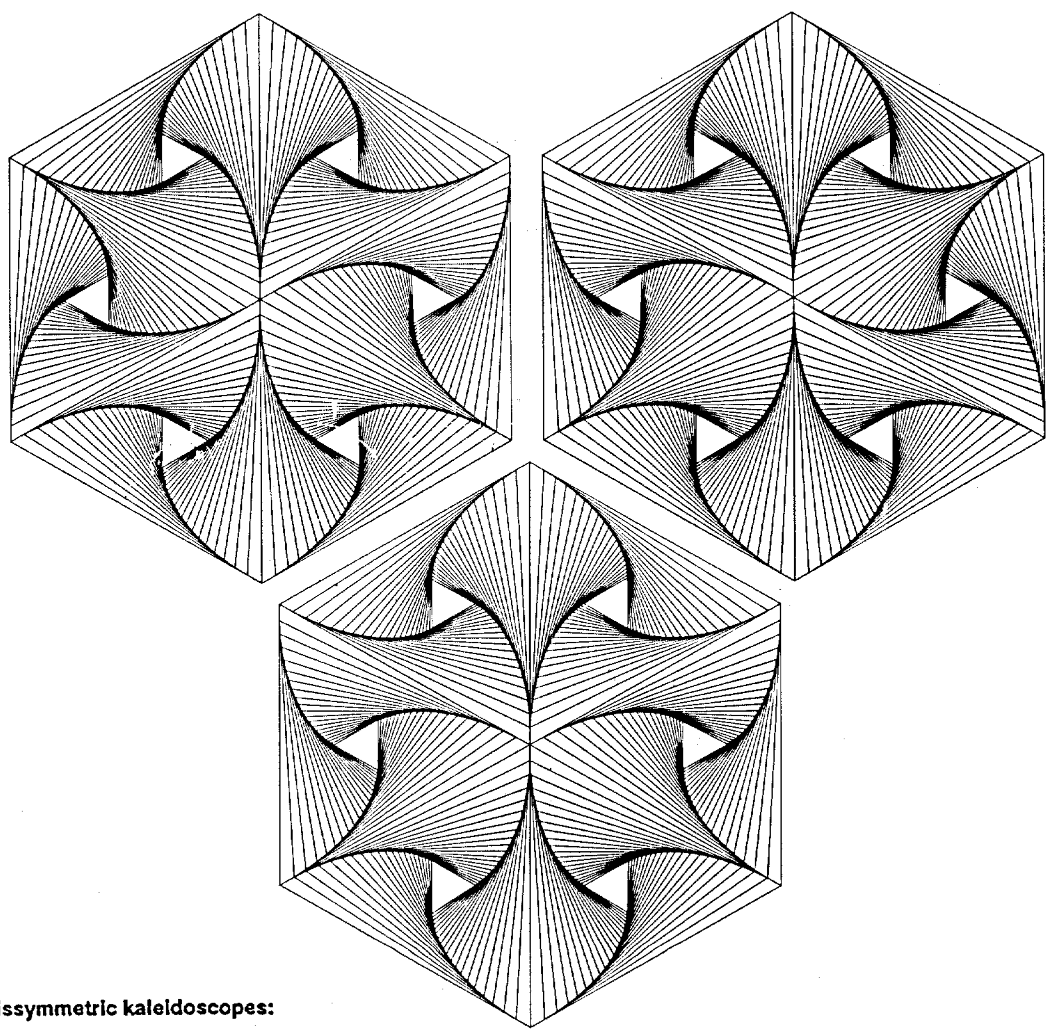

Dissymmetric kaleidoscopes:
Hommage à Pasteur

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# SYMMETRY IN GEOMETRY: A PERSONAL VIEW

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*This article is not a general survey of symmetry in geometry, but merely an account of some of the ways in which symmetry has played a part in my own work.*

## 1.

"Symmetry" is a noun used to describe a state or property possessed by certain patterns, objects and geometrical figures. If a plane figure is symmetrical about a *line of symmetry* or *mirror line,* as in Figure 1, then the transformation known as *reflection* in that line is called a *symmetry* of the figure. We thus have a different but related use of the word "symmetry", to describe a transformation that maps a figure to itself. There are other well known types of symmetry. The badge or symbol of the Isle of Man (Figure 2) is mapped to itself by a *rotation* through 360/3 degrees, so this rotation is a symmetry of the badge, a *three-fold rotational symmetry.* A frieze pattern (a pattern that repeats regularly in one direction) has a *translation* as its fundamental symmetry. These three types of symmetry all preserve lengths and angles: they map any figure to a congruent figure. Because the human eye regards congruent figures (the two wings of a butterfly; the three legs of the Isle of Man badge) as being in some sense "the same", these basic types of symmetry are easily recognized; we shall discuss more subtle forms of symmetry later.

It is not surprising that, in geometry, if we start with a symmetrical situation or figure, we can deduce further symmetrical properties of that figure. Once we have progressed beyond the stage of Euclid's "pons asinorum" (if a triangle has two congruent sides then it has two congruent angles), such deductions of symmetrical properties become almost second nature and are often made instinctively. More surprising are the unexpected occurrences of symmetry. As a simple example, the curves known as *conic sections* were first defined as sections or slices of a right

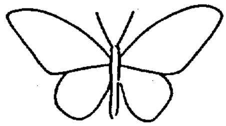


Figure 1

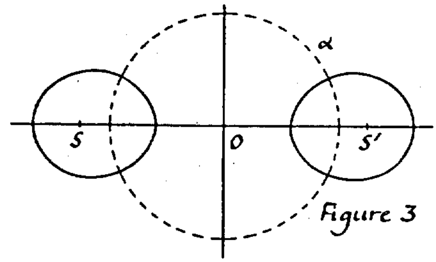



Figure 3

Figure 2

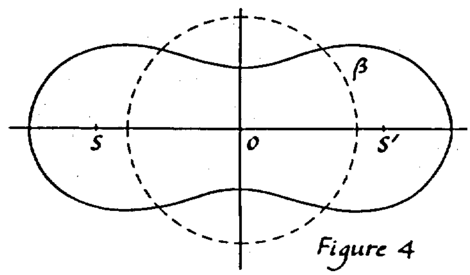



Figure 4

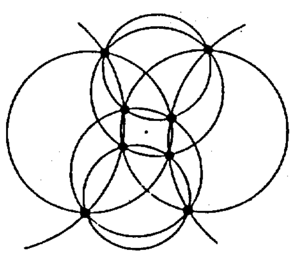


Figure 5

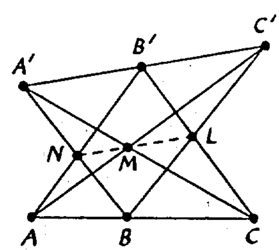



Figure 6

circular cone, as their name implies. When we obtain an ellipse by taking an oblique plane section of a right circular cone, it is not difficult to see that the complete three-dimensional figure has a plane of symmetry, and hence an ellipse has one line of symmetry; but we could be forgiven for guessing that an ellipse might be more pointed at one end like a bird's egg. The well known fact that an ellipse has two perpendicular lines of symmetry comes as a surprise.

Transformations that preserve length are called *isometries*. Another important type of transformation is *inversion in a circle*: if the circle $\alpha$ has centre O and radius r, if the points P and P' lie on the same radius vector from O, and if OP.OP' = $r^2$, we say that P and P' are *inverse* points in $\alpha$ and the transformation that maps each point to its inverse in $\alpha$ is called *inversion* in $\alpha$. Inversion alters the shapes as well as the sizes of figures, but every inversion maps circles to circles or lines. In the study of inversion, lines are regarded as circles of a special type, and reflection in a line as a special case of inversion in a circle; thus "inversion" is an extension of "reflection".

Let S and S' be two points in a plane, at a distance 2a apart. The locus of a point P such that SP.S'P = $b^2$, where b is a fixed length, is called a *Cassinian oval*, and it is clear from the definition that such a curve has two perpendicular mirror lines. When $b < a$, the curve is a bioval (Figure 3), and it can be shown that the curve is mapped to itself by inversion in the circle $\alpha$ with centre O (the midpoint of SS') and radius k, where $k^4 = a^4 - b^4$ (Rigby 1983; see also Kavanau 1982). This circle is called a *circle of symmetry* of the bioval, and here we have an example of another type of symmetry: *inversive symmetry*. In this case the inversive symmetry is unexpected and surprising. The symmetries of a figure always form a group; the isometric symmetries of the Cassinian bioval form a group of order 4 (consisting of the reflections in its mirror lines, the rotation about O through 180°, and the identity), but the inversive symmetry group of the bioval has order 8. When $b > a$, the curve is a monoval, as in Figure 4; this figure also shows the circle $\beta$ with centre O and radius h, where $h^4 = b^4 - a^4$. Inversion in $\beta$, followed by rotation about O through 90°, maps the monoval to itself (Rigby 1983), so the Cassinian monoval also has an inversive symmetry group of order 8; but this group is not isomorphic to the symmetry group of the bioval.

Pappus' Theorem provides an example of a different type of symmetry. Suppose that A, B, C and also A', B', C' are two sets of three collinear points. Denote the points $BC \cap B'C$, $CA' \cap C'A$, $AB' \cap A'B$ by L, M, N as in the Figure 6; then L, M, N are collinear. In the figure we now have nine points and nine lines; each line contains three of the points and each point lies on three of the lines, and the figure is called a $9_3$ *Pappus configuration*. Any one-to-one mapping of the sets of nine points and nine lines to themselves that preserves incidence is called an automorphism or *combinatorial symmetry* of the figure; alternatively we can say that a combinatorial symmetry maps collinear points to collinear points. The combinatorial symmetry group of a Pappus configuration has order 108. These symmetries are of a type different from the previous ones: in general a combinatorial symmetry of a figure is not induced by a symmetry of the whole plane such as a reflection, an inversion or a collineation (a type of symmetry that we shall not need to define here).

Another famous configuration is the $8_4$ *Clifford configuration* of eight points and eight circles, where each circle contains four of the points and each point lies on four of the circles (Figure 5). The combinatorial symmetry group of this configuration has order 192. A unique Clifford configuration can be built up starting with any four circles in general position through a common point; a surprising result is that, given any Clifford configuration, by applying a suitable inversion we can transform the

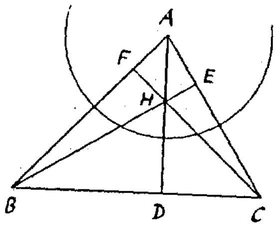



Figure 7

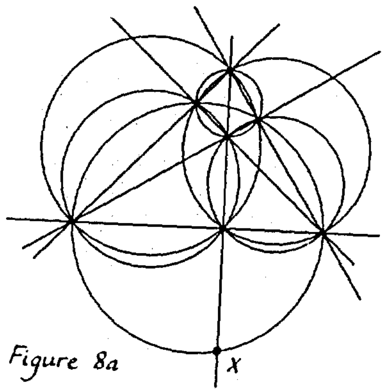



Figure 8a

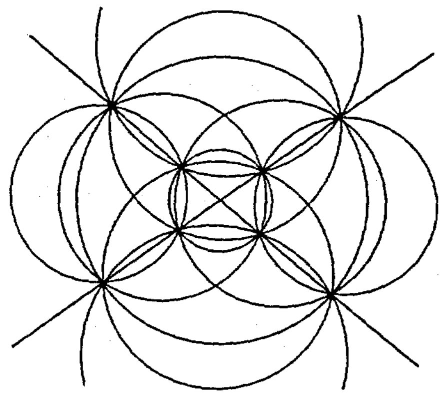


Figure 8b

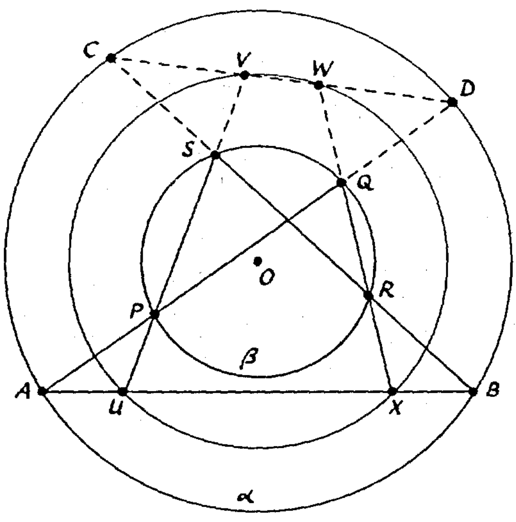



Figure 9

configuration to another Clifford configuration that has twofold rotational symmetry like the one in Figure 5; thus every Clifford configuration has an inversive symmetry group of order 2 (Rigby 1977).

The three internal bisectors of the angles of a triangle ABC meet at the incentre of the triangle. Even a simple figure like this has a combinatorial symmetry group of order 6 (because we can permute A, B, and C amongst themselves in six ways), but we do not usually bother to mention such a simple fact. However, if we consider the three altitudes of ABC, meeting at the orthocentre H, the situation is more interesting, because A is the orthocentre of HBC, B is the orthocentre of AHC, etc., and the combinatorial symmetry group of Figure 7 (ignoring the circle) has order 24. The symmetries of this figure are more than just combinatorial symmetries; they map collinear points to collinear points, but they also map perpendicular lines to perpendicular lines: they *preserve the structure* of the figure. In Figure 7, the points D, E and F clearly play a different rôle from A, B, C and H, but by adding some circles to the figure we can make it more symmetrical, as we shall see in the next paragraph.

In inversive geometry, it is convenient to adjoin a single *point at infinity* to the plane, and to postulate that all lines pass through this point at infinity. Thus two lines, unless they are parallel, now meet at two points, one of them at infinity. This is not unreasonable, since we regard lines as circles of a special type, and if two circles meet once then in general they meet again. Because of the right angles in Figure 7, if we draw the circles whose diameters are BC, CA, AB, AH, BH and CH, we obtain Figure 8a. This figure contains eight points, including the point at infinity $\Omega$, and twelve circles (six of which are lines). Each point lies on six circles, and each circle contains four of the points. We have an $8_6 12_4$ configuration of points and circles; all eight points, and all twelve circles, play identical rôles in the configuration, and its symmetry group has order 48. Eight of the symmetries (including of course the identity) are induced by inversions or products of inversions; for instance, inversion in the circle shown in Figure 7 interchanges B and F, C and E, H and D, and A and $\Omega$ (inversion in any circle maps the centre of the circle to the point at infinity, and conversely). If we invert Figure 8a using a circle centre X as the circle of inversion, we obtain Figure 8b which shows the symmetry more clearly.

Suppose that two opposite points of a sphere are designated as the north and south poles, and suppose that a plane touches the sphere at the south pole. If a point P of the plane is joined by a line to the north pole, this line will meet the sphere again at a point P', called the *stereographic projection* of P on the sphere; the stereographic projection of $\Omega$ is the north pole itself; circles and lines in the plane are projected to circles on the sphere. It can be shown that, by choosing a suitable sphere, Figure 8a can be stereographically projected to the vertices of a rectangular box, together with twelve circles on the sphere each containing four vertices. Thus the familiar figure of a triangle and its orthocentre can be transformed into a figure with even more "visual symmetry" than Figure 8b. We should note also that Figure 7 shows the internal and external bisectors of the angles of triangle DEF, so it illustrates not just one but two familiar figures: a triangle and its altitudes, and a triangle and its angle-bisectors.

I recently found a similar example, of less basic importance than the previous one, but showing how an extension of the original figure can increase the overall symmetry. The following theorem is an extension of a result whose proof was asked for as a problem in the *Mathematical Gazette* (Rigby 1987). Let $\alpha$ and $\beta$ in Figure 9 be circles with centre O; let P, Q, R, S lie on $\beta$, and let PQ, RS meet $\alpha$ at A, B; then U and X (as shown in the figure) are equidistant from O. A second

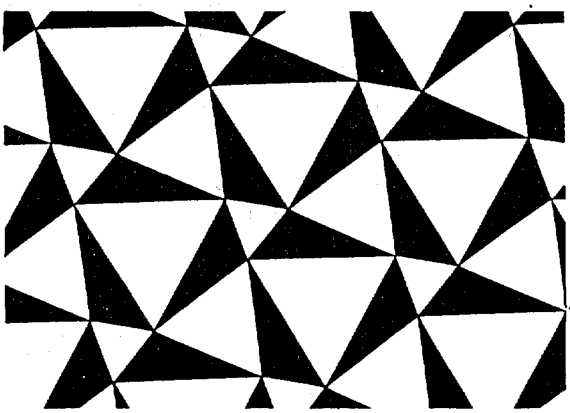


Figure 10

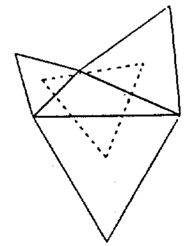


Figure 11

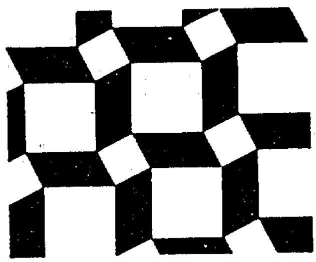


Figure 12

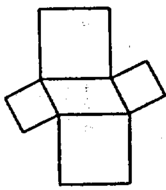


Figure 13

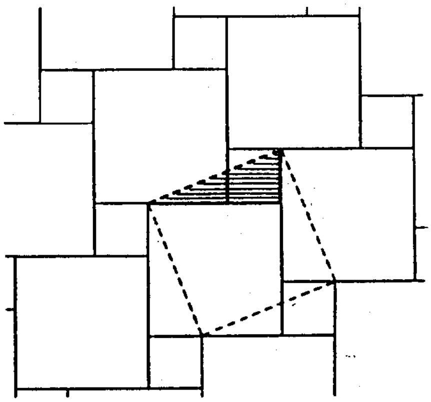


Figure 14

application of the theorem shows that V and W are equidistant from O. But if we apply the theorem to B, C, D, A on $\alpha$ instead of to P, Q, R, S on $\beta$, we see that U and V are equidistant from O. Hence U, V, W, X lie on a circle with centre O. The complete figure of twelve points, six lines and three circles now has a combinatorial symmetry group of order 48, and the twelve points all play identical rôles in the figure, as do the six lines and the three circles: the symmetry group is transitive on points, lines and circles. This configuration can be extended from concentric circles to coaxial circles, and there is a simple proof of its existence using coordinate geometry.

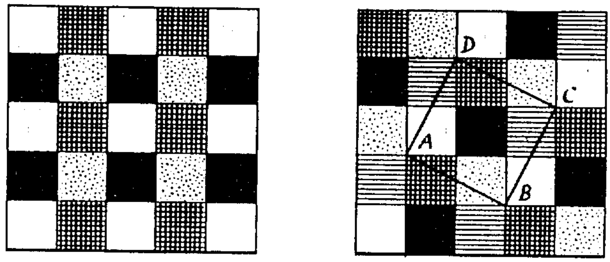
Figure 10 shows a tessellation composed of three sizes of equilateral triangles and congruent scalene triangles; the tessellation must be regarded as extending to infinity in the plane. It should be clear that the centre of each small equilateral triangle is equidistant from the centres of six other small equilateral triangles. Thus the centres of all the small equilateral triangles form an equilateral triangular lattice. In the centres of the triangles of this lattice are the centres of the remaining equilateral triangles; hence the centres of all the equilateral triangles form another equilateral triangular lattice. Thus we see that the centres of the equilateral triangles in Figure 11 form an equilateral triangle. This result is known as Napoleon's Theorem. The original tessellation can be built up using any shape of scalene triangle; hence Napoleon's Theorem is true for any triangle with equilateral triangles erected on its sides. This is not the only way of proving the theorem, but the tessellation, with its centres of threefold rotational symmetry and its translational symmetry, provides a simple visual way of perceiving the truth of the theorem. Unfortunately, tessellations cannot be used to prove various generalizations of the theorem (Rigby 1988).

The tessellation of Figure 12 can similarly be used to show that the centres of squares erected on the sides of a parallelogram form a square (Figure 13). The tessellation of Figure 14 illustrates clearly one of the proofs of Pythagoras' theorem (Friedrichs 1965); it shows how and why the square on the hypotenuse of a right-angled triangle can be cut into five pieces that can be fitted together again to form the squares on the other two sides.

## 2.

The subject of tessellations, patterns and designs in a plane, their different types of symmetry, and ways of colouring them, is a vast one; for extensive coverage of these topics in the Euclidean plane, and copious references, the reader is referred to Grünbaum and Shephard (1987). We shall consider three aspects of the subject here; the first is mathematical but the results can also be used as ornamental art, the second shows a mathematical idea being realized in an artistic manner, and the third is purely ornamental.

First we shall consider perfect and semi-perfect colourings of regular tessellations, and their relation to regular maps, a subject that I am still investigating. The colourings obtained are artistically pleasing. Figure 15 shows two colourings of the regular tessellation {4,4}. In the first colouring, using four colours, any symmetry of the underlying uncoloured tessellation (i.e. any isometry that maps the tessellation to itself) permutes the four colours; hence the colouring is called a *fully perfect* colouring. In the second colouring, using five colours, any direct symmetry of the underlying tessellation (i.e. any translation or rotation mapping the tessellation to itself) permutes the colours, but any opposite symmetry (for example a reflection) jumbles up the colours, mapping some black squares to black, some to white, some to shaded squares; this colouring is called a *chirally perfect* colouring: like a pair




(a) (b)

Figure 15

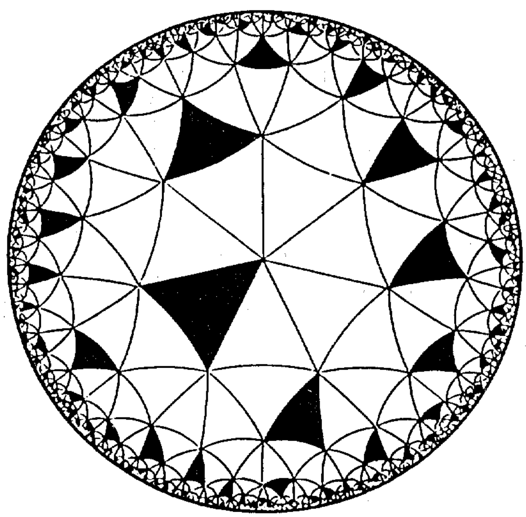


Figure 16

of hands, the colouring and its mirror image are different. Notice that to proceed from any black square (for instance) to the neighbouring black squares we make a "right-handed" knight's move as in chess. Suppose we cut out the square ABCD from Figure 15b, then join the edge AB to the edge DC to form a cylinder, and finally (assuming that we are dealing with an elastic material) join one circular edge of the cylinder, originally the edge BC of the square, to the other edge AD. We have now created a torus, and on this torus is a tessellation or regular map of five (distorted) squares meeting by fours at five points. Another way of visualizing the same situation is to start with Figure 15b and *identify squares of the same colour;* this means in effect that we roll up Figure 15b into a cylinder in such a way that AB coincides with DC, then push the cylinder into itself in such a way that BC coincides with AD.

The same procedure can be carried out in a hyperbolic place, but the situation is more complicated and more difficult to visualize. The regular maps to be described in the next few paragraphs, the notation used for them, and their symmetry groups, are discussed in Coxeter and Moser (1972), Chapter 8. Figure 16 shows a $\{3,7\}$ tessellation of a hyperbolic plane, illustrated by using a Poincaré model of the plane; the tessellation is composed of triangles meeting by sevens at each vertex. In "hyperbolic reality" all the triangles are equilateral and congruent to each other, but in a Euclidean plane we can only draw a distorted picture of this situation. The figure also illustrates the beginnings of a chirally perfect colouring of the tessellation in seven colours. All the black triangles have been coloured; the reader is encouraged to work out the two simple rules for proceeding from any black triangle to the neighbouring black triangles, and then to complete the colouring using a total of seven colours. If this is done, it will be found that the black triangles (for instance) are not all surrounded in the same way by triangles of other colours: there are in fact eight possible ways in which a black triangle can be surrounded. If two triangles are surrounded in the same way, they are said to be *equivalent;* there are therefore 56 inequivalent triangles in the tessellation. If we now "identify equivalent triangles" in the coloured tessellation, we obtain a surface with a regular map of 56 triangles meeting by sevens at 24 points. This surface is a "sphere with three holes" (in the same way that a torus is a sphere with one hole) and the regular map is denoted by $\{3,7\}_8$. It is possible to illustrate the surface in three-dimensional Euclidean space as a polyhedron with 56 plane triangular faces corresponding to the faces of the map, though the triangles are not of course regular. The polyhedron is shown in colour on the cover of an issue of the *Mathematical Intelligencer*, accompanying an article by Bokowski and Wills (1988), and its construction is described by Schulte and Wills (1984). The colouring of the tessellation also produces a chirally perfect colouring of the map. This colouring can be used to show that the symmetry group of the regular map $\{3,7\}_8$ is isomorphic to the group of collineations and correlations of the projective plane of seven points (Rigby 1989).

Figure 17 shows the black and the stippled regular quadrangles (or squares) in a fully perfect colouring of the hyperbolic tessellation $\{4,6\}$ in five colours; two of the colours are shown in this figure because the rules for colouring are not easy to see from one colour only. We can complete the colouring as before, but this time we shall extend the meaning of "equivalent": if the way in which one black square is surrounded is the exact mirror image of the way in which another black square is surrounded (and that situation does happen in this colouring) we also say that the squares are equivalent. If we now identify equivalent squares in the coloured tessellation, we obtain a surface with a regular map of 15 squares meeting by sixes at 10 points. Because of the "mirror image" situation just described, this surface is a one-sided (or non-orientable) surface; the Petrie polygons of the map have length 5, but the map must not be confused with another map denoted by $\{4,6\}_5$ which has 30

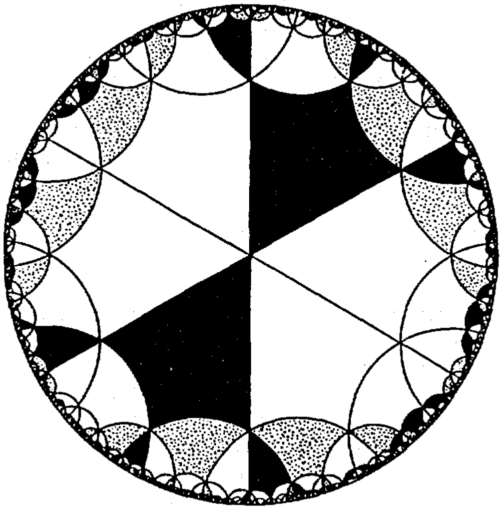


Figure 17

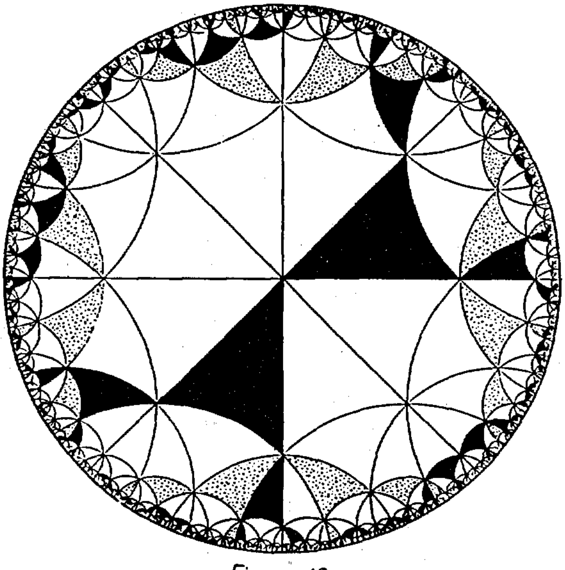


Figure 18

faces. We now also have a chirally perfect colouring of the map. It is worth noting that non-orientable surfaces sometimes arise in this way from fully perfect colourings, but they never arise from chirally perfect colourings. (The colouring in Figure 17 also provides a perfect colouring of $\{4,6\}_5$, which can be used to prove that the symmetry group of $\{4,6\}_5$ is $C_2 \times S_5$).

Another well known regular map on a non-orientable surface is $\{3,8\}_7$, whose combinatorial symmetry group is isomorphic to that of $\{3,7\}_8$ as described above, so it is natural to ask whether there is a perfect colouring of $\{3,8\}_7$ in seven colours. An equivalent question is whether there is a fully perfect colouring of {3,8} in seven colours from which we can construct $\{3,8\}_7$ by identifying equivalent triangles. In fact there is no perfect colouring of {3,8} in seven colours, either fully perfect or chiral. But there is a *semi-perfect* colouring of {3,8} in seven colours; "semi-perfect" means that half the direct symmetries and half the opposite symmetries of {3,8} permute the colours, whilst the other symmetries jumble them up. This semi-perfect colouring is indicated in Figure 18, where the black and the stippled triangles are shown. We must be more careful now about the rules for colouring. Imagine each triangle to be designated as positive or negative, adjacent triangles having opposite designations; then there is one set of rules for proceeding from a positive triangle to the adjacent triangles of the same colour, and the mirror image of those rules must be used to proceed from a negative triangle to the adjacent triangles of the same colour. If we identify equivalent triangles in this colouring we obtain the regular map $\{3,8\}_7$. The reader may like to find a fully perfect colouring of {3,8} in 28 colours that gives rise to $\{3,8\}_7$ when we identify equivalent triangles.

The Dutch artist M.C. Escher used hyperbolic tessellations as the basis for his "Circle Limit" designs, which are illustrated in the various books about his work; see also the article "Creating Hyperbolic Escher Patterns" by D.J. Dunham in Coxeter et al. (1986), where many other articles relevant to tessellations and their colouring can also be found. An excellent illustration of a perfect colouring of {7,3} in eight colours, by C. Leger, formed part of a travelling exhibition "Horizons Mathématiques", and is illustrated in colour in the accompanying booklet "Mosaïque Mathematique" which was originally obtainable from M. Darche (1981).

Now let us consider wallpaper designs and their symmetry types and pattern types. For our present discussion, without being too technical, we can define a *wallpaper design* to be a design in a Euclidean plane that repeats regularly in more than one direction. A detailed discussion of the definitions and ideas used in this section can be found in Grünbaum and Shephard (1987), Chapters 1 and 5. There are seventeen different symmetry types of wallpaper design. One of these types, denoted by *cmm*, is shown in Figure 19a; this design has horizontal and vertical mirror lines forming a rectangular grid, with centres of two-fold rotational symmetry at the centres of the rectangles of the grid (Figure 19b). Figures 19c, d and e show other designs with the same symmetry type. The basic motif in Figure 19a has no non-trivial symmetries: it is called a *primitive* motif for the symmetry type. The motifs in c, d and e show the different symmetry transformations in this symmetry type: mirror symmetry, twofold rotational symmetry, and d2 symmetry respectively ("d2 symmetry" is the symmetry possessed by a motif that has two perpendicular mirror lines; the symmetry group of such a motif is denoted by d2 or $D_2$). Figure 19a, c, d and e therefore show the four different *pattern types* associated with symmetry type *cmm*, denoted in Grünbaum and Shephard (1987) by PP17, PP19, PP18 and PP20; the first of these is a *primitive* pattern type. The problem that occurred to me on reading about the concept of pattern types was: "can we find, adapt or create artistic designs that incorporate all the different imprimitive pattern types associated with a particular symmetry type?"

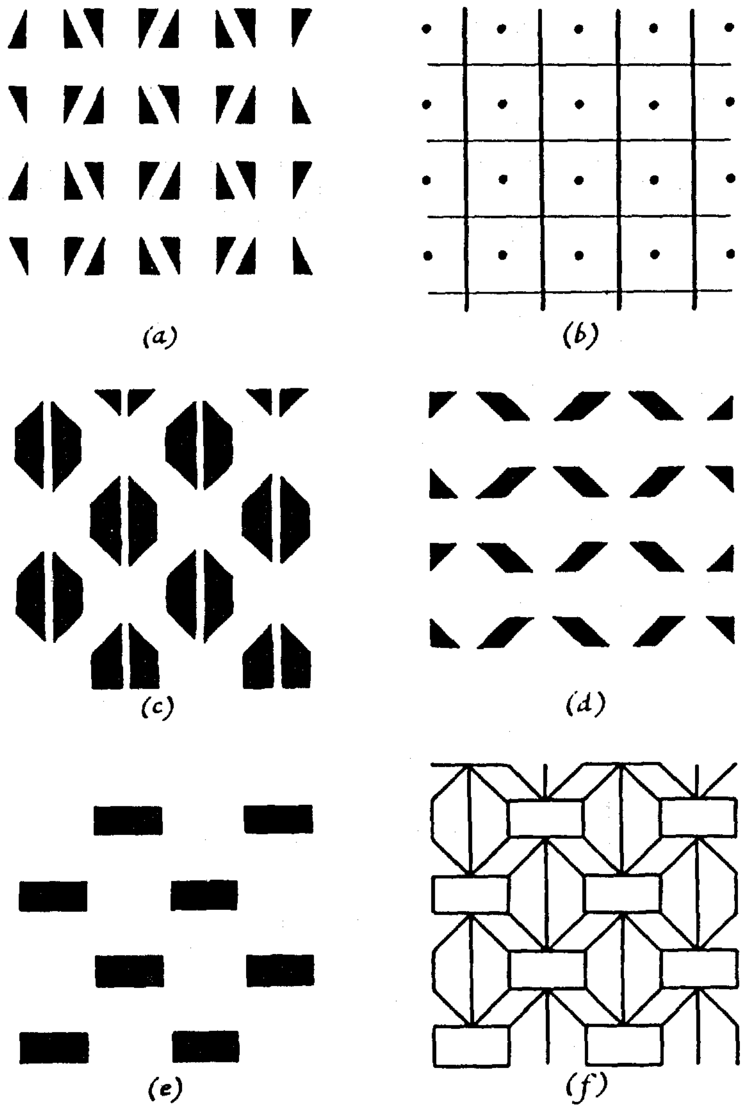



Figure 19

Figure 19f shows the patterns of Figure 19c, d and e merged together, the motifs now being white instead of black. I found this design in leaded window-panes at Haddon Hall, Derbyshire, England, just a day or two after formulating the problem, and this is partially what prompted me to leave out the primitive pattern type from the composite design.

Figure 20 shows a design of symmetry type *p4m*, combining four patterns of types PP38, PP39, PP40 and PP41; the basic motifs are now either black or white, and we must imagine them as tiles separated by cement because it is important that tiles should not coalesce to form larger tiles with a different type of symmetry. This design was adapted from Figure 21, a tile design from Fountains Abbey, Yorkshire, England; see also Grünbaum and Shephard (1987), p.7.

Figure 22 shows an Islamic-style design by the author, of symmetry type *p6m*, combining six patterns of types PP47, PP48A, PP48B, PP49, PP50 and PP51. Figure 23 shows a variation of this design, of symmetry type *p6*.

Figure 10 has symmetry type *p3*; the black tiles are primitive motifs forming a pattern of type PP21, and the equilateral triangles of any one size form an imprimitive pattern of type PP22. Figure 12 has symmetry type *p4*; the black tiles form an imprimitive pattern of type PP31, and the squares of either size form an imprimitive pattern of type PP32.

Finally, Figure 24 and 25 show two practical contributions by the author to tapestry work or cross-stitch embroidery. The fish design is based on a design by Escher (Coxeter et al., 1986, pp. 387, 395), but I created the carnation design some years ago before becoming aware of Escher's work. The fishes can be perfectly coloured with five colours, creating a more interesting effect than Escher's four colours; see G.C. Shephard's article in Coxeter et al. (1986). Both these designs have actually been embroidered, on a church kneeler and a footstool.

## REFERENCES


Bokowski, J. and Wills, J.M. (1988), Regular polyhedra with hidden symmetries. *Mathematical Intelligencer* 10, 27-32

Coxeter, H.S.M. and Moser, W.O.J. (1972) *Generators and Relations for Discrete Groups*. 3rd ed., Berlin, Springer

Coxeter, H.S.M. et al. (ed.) (1986), *M.C. Escher: Art and Science*, Amsterdam. North Holland

Darche, M. et al. (1981), *Mosaïque Mathematique*. Musée de la Villette, 211 Avenue Jean-Jaurès, 75019 Paris

Friedrichs, K.O. (1965), *From Pythagoras to Einstein*. New York, Random House, New Mathematical Library 16

Grünbaum, B. and Shephard, G.C. (1987), *Tilings and Patterns*. New York, W.H. Freeman and Company

Kavanau, J.L. (1982), *Curves and Symmetry*. Vol.1, Los Angeles, Science Software Systems Inc.

Rigby, J.F. (1977), Half turns and Clifford configurations in the inversive plane. *Journal of the London Mathematical Society* (2) 15, 521-533

Rigby, J.F. (1983), The inversive symmetry of curves (unpublished competition entry)

Rigby, J.F. (1987), Solution to Problem 71.C, *Mathematical Gazette* 71, 309-310

Rigby, J.F. (1988), Napoleon revisited. *Journal of Geometry* 33, 129-146

Rigby, J.F. (1989), Compound tessellations and perfect colourings. *Leonardo* (to appear)

Schulte, E. and Wills, J.M. (1984), *A polyhedral realization of Felix Klein's map {3,7}$_8$ on a Riemannian manifold of genus 3*. Mathematische Forschungsberichte, Universität-Gesamthochschule-Siegen

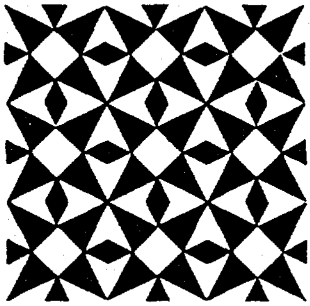


Figure 20

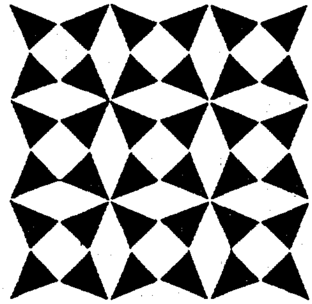


Figure 21

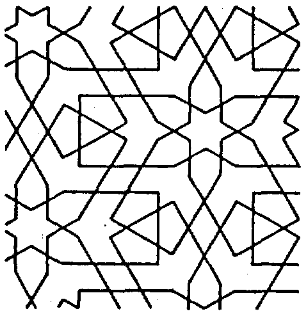


Figure 22

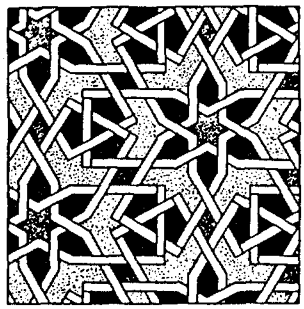


Figure 23

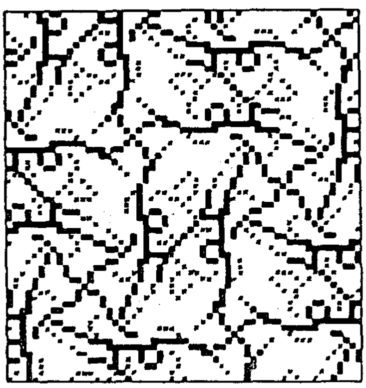


Figure 24

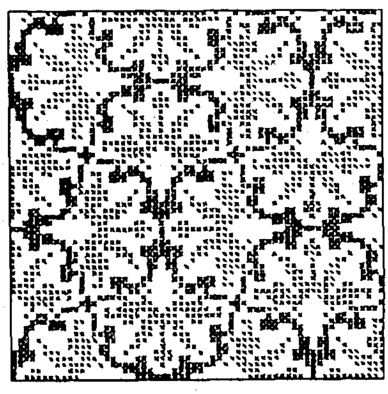


Figure 25