Symmetry and the Crystallography of Logic

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The history of logic has been divided into three periods. The "traditional" period starts with Aristotle. The "algebraic" period starts with Boole. The "logistic" period starts when Russell successfully rescues the ideas of Frege. In what follows I will be going back to the middle period, back to the time of Boole, Schroeder, and Peirce.

More specifically, what follows will focus on a direct continuation of two of the main ideas that were emphasized by Peirce. In reference to logic, especially to what is called the propositional calculus. he understood the importance of symmetry. In reference to notation, he understood the importance of iconicity. But he did not go far enough. What follows will show that, when we start with a custom-designed notation, it is an easy step to go from symmetry-iconicity to the crystallography of logic.

In 1902 Peirce devised three iconic notations for the 16 binary connectives (and, or, if, etc.). Before we push his work a small notch ahead, let us look at a simple example in binary logic. It expresses the duality of "and" and "or," as in (1) and (2). A mental grasp of (1) and (2) can be maintained in three ways. Mem-

(1)	(A and B)	≡	Not(Not-A or Not-B)
(2)	(A . B)	≡	N(NA v NB)
(3)	N(A . B)	≡	(NA v NB)
(4)	N(A v B)	≡	(NA . NB)

orize it, derive it from something else that has been memorized, such as we see in De Morgan's laws [(3) and (4)], or work it out again and again, each time coming back to the truth table method.

See Table I, where (2) the duality of "and-dot" and "or-vee" has been worked out three times. Standard form has been used in (a), where the four 2-place entries (TT,TF,FT,FF) line up in a vertical column. Negation acts on A, on "or", and on B, so that it changes the truth table for "or" (A TTTF B), in such a way that it matches the truth table for "and" (A TFFF B), thereby justifying the presence and the validity of the equivalence sign.

Why is it a poor practice to place (TT,TF,FT,FF) in a vertical column? Because this arrangement is not sensitive to the equivalence relations between the connective relations between (A,B). In other words, it does not give us direct notational access to second-order relations. Instead, it forces us to memorize these relations, or work them out again and again, as in (a).

Table I

	1.	2.	3.	4.	5.	6.
	(A,B)	(A B)	(A V B)	(NA,NB)	(NA v NB)	N(NA v NB)
(a)						
	T T	T	T	F F	F	T
	T F	F	T	F T	T	F
	F T	F	T	T F	T	F
	F F	F	F	T T	T	F
(b)						
	FT TT	F T	T T	TF FF	T F	F T
	FF TF	F F	F T	TT FT	T T	F F
(c)						

__Symmetry form__ has been used in (b), where (TT,TF,FT,FF) now stand in the quadrants of Cartesian (A,B) coordinates. Rule 1 (R1) says that when N acts on A, as in NA, the 4-fold truth table for "or" (TTTF) is __flipped__ from left to right. R2 says that when N acts on "or" itself (Nv), all positions in the 4-fold are __mated__, reversed, or counterchanged (FFFT). R3 says the when N acts on B, as in NB, the 4-fold (TTTF) is __flipped__ from top to bottom.

Any order of (R1)(R2)(R3), that is, all six permutations of these rules, now called __flip-mate-flip__, can be applied to the "or" side of (b). When R1 and R3 act at the same time, the 4-fold for "or" (TTTF) is doubly flipped (rotated 180 degrees) (FTTT). This activates the pattern of symmetry transformations found in a Klein 4-group (K4). In (b), the mate (6.) of the rotate (5.) of the 4-fold for "or" (3.) repeats the 4-fold for "and" (2.); again, the equivalence sign has been justified. This activates the pattern in the 8-group known as (C2 x C2 x C2), also called (C2)3.

What makes (b) so much better than (a)? It is the ability of (b) to remind us that we have an acute world-wide need to come up with a much better notation. This notation should embody, abbreviate, and participate in the same symmetry transformations that are displayed in the truth table changes in (b). In other words, we need a notation that can do a dance called __flip-mate-flip__.

Now we are ready for the __logic alphabet form__ used in (c). Start with a 1-stemmed d-letter, when this shape is placed in four positions (p,b,q,d), and likewise for the same four positions of a 3-stemmed h-letter (h, ᑭ, ᑯ, ч). A s(T)em stands for (T)rue. The d-letter is an abbreviation for (A and B) and (A TFFF B). It has a stem in the upper-right quadrant of the 4-fold (T---) and

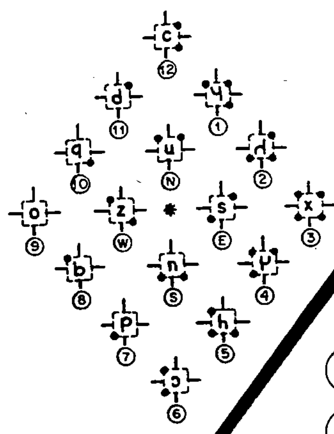

Fig. 1

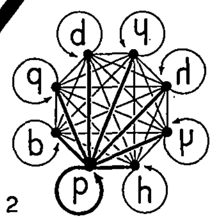

Fig. 2

Table II

(???)	1 o	2 p	3 b	4 q	5 d	6 c	7 u	8 s	9 z	10 n	11 ɔ	12 h	13 µ	14 ɹ	15 ɥ	16 x
OOO	o	p	b	q	d	c	u	s	z	n	ɔ	h	µ	ɹ	ɥ	x
OON	o	b	p	d	q	c	n	z	s	u	ɔ	µ	h	ɥ	ɹ	x
NOO	o	q	d	p	b	ɔ	u	z	s	n	c	ɹ	ɥ	h	µ	x
NON	o	d	q	b	p	ɔ	n	s	z	u	c	ɥ	ɹ	µ	h	x
NNN	x	h	µ	ɹ	ɥ	c	u	z	s	n	ɔ	p	b	q	d	o
NNO	x	µ	h	ɥ	ɹ	c	n	s	z	u	ɔ	b	p	d	q	o
ONN	x	ɹ	ɥ	h	µ	ɔ	u	s	z	n	c	q	d	p	b	o
ONO	x	ɥ	ɹ	µ	h	ɔ	n	z	s	u	c	d	q	b	p	o

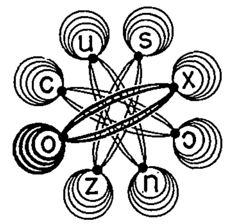

Fig. 3

no where else (-FFF). Likewise. the h-letter stands for (Not-A or Not-B) and (A FTTT B). This 4-fold has three stems (-TTT). And so forth, for a full set of 16 letter-shapes, when each of them is placed inside of an all-common standard square and when all of them are placed in the 4-by-4 clock-compass in Fig. 1.

Now for what happens in (c), in isomorphism with (b). The symbol for the mate (d) of the rotate (h) of "or" (ɥ) is the same as the symbol for "and" (d). Notice that this is like what happens for De Morgan's laws, which are both cases of "split duality." In (3), the mate of "and" and the rotate of "or" both become an h-letter. In (4), the mate of "or" and the rotate of "and" both become a p-letter. Also compare and contrast the up and down columns of Table I, with respect to (a), (b), and (c).

We have come to the moment when all of the letter-shapes, built to exist at several levels of symmetry-asymmetry, will find themselves trapped in the same 8-group frame of symmetry transformations (C2)3. Let the asterisk in (A * B) stand for <u>any</u> of the 16 letter-shapes, and let O stand for the <u>absence</u> of negation. See that N can act on only three places in (A * B): at NA, at N*, and at NB. This brings us back to flip-mate-flip and all combinations of (R1)(R2)(R3): OOO, OON, NOO, NON, NNN, NNO, ONN, and ONO. When all of these negation triplets act on all of the letter-shapes, we arrive at the (8 x 16) table of transformational negation in Table II. Please realize that, when the flip-mate-flip rules are applied in Table II, they are rich enough to absorb (2), (3), and (4), along with the 125 cells not mentioned.

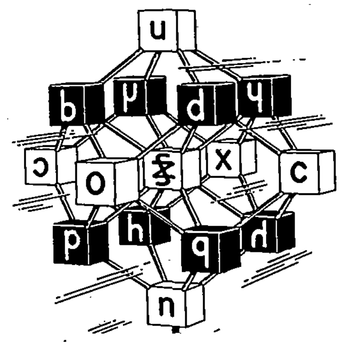

Fig. 4

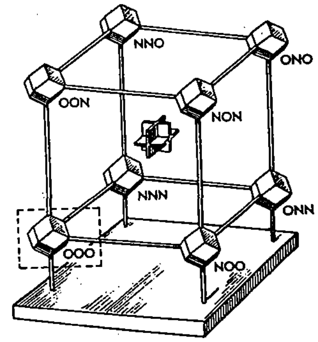

Fig. 5

More about relations between relations. When flip-mate-flip acts on the eight, tall, odd-stemmed letter-shapes along the top of Table II, it generates 64 cells (a half-table) that obey the network of cubic symmetry relations in Fig. 2. When flip-mate-flip acts on the eight, squat, even-stemmed letter-shapes, it generates 64 cells that obey the octahedral symmetry relations in Fig. 3. Remember that an ordinary rhombic dodecahedron is the interpenetration of a cube and an octahedron, such that the black vertices in Fig. 4 absorb the symmetry relations in Fig. 2, and likewise for the white ones and Fig. 3. Also realize that, more accurately, Fig. 4 is a shadow rhombic dodecahedron. A Boolean 4-cube of letter-shapes has been shadowed into 3-space. After that, when a Boolean 3-cube of negations triplets, in the form of three mutually perpendicular mirrors, acts on Fig. 4, it generates the 8-cell of logical garnets in Fig. 5. This symmetry model absorbs all and exactly all of the 128 cells in Table II.

More, there is much more! We have only touched on the logic of two atoms (A,B). No mention has been made of a whole family of hand-held models that can be constructed for (A,B). This approach can also be used for three atoms (A,B,C), for n atoms (A,B,C . . n), for an extension into 3-valued logic, and especially for a consideration of the symmetry structures that are activated by compound atomic forms. In general, the crystallography of logic becomes especially interesting when all of this is extended into n-dimensional geometry, at least enough so that we will be able to activate complex structures, including hyperstructures, when they are themselves located on hyperstructures.

My first example (2) is very simple. Hyperstructures on hyperstructures can become very complex. In brief, the logic alphabet is not only robust enough to stay alive at both extemes. There is also much more in between that has not been mentioned.