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Duality, Nonlinearity, Asymmetry and Symmetry in Lattice Dynamics

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In my study of lattice dynamics, I have been helped by the ideas such as duality, nonlinearity, asymmetry and symmetry. I would like to present how I met these ideas, how they led me to some discoveries, and to see that many theories can be related by these ideas. I was first led to (1) the concept of duality which relates different one-dimensional lattices with the same frequencies of normal modes. In turn, it led me to (2) the discovery of integrable nonlinear lattice with asymmetric interaction potential. Recently I am interested in (3) a lattice with asymmetric potential and asymmetric kinetic energy (symmetric with respect to kinetic and potential energy terms), which describes asymmetric time sequence, such as stochastic process, chemical reaction, and some environment problem.

1) One-dimensional dual lattices

One of the problems I was interested in lattice dynamics was the effect of impurities on the normal mode frequencies (spectrum) of a linear lattice. We sometimes met different lattices with the same spectra. In this connection, I found the following duality theorem [Toda 1965]:

If we replace masses by springs, and springs by masses, we can obtain another lattice with the same spectrum as that of the original lattice, if certain relations are satisfied between the masses and the force constants of the springs.

These lattices can be symbolically as

A) .....K U K U .....  
B) .....U*K*U*K*.....
with kinetic energy terms $K, K^*$, and potential terms $U, U^*$:

$$K_j = \frac{P^2_{j}}{2m_j}, \quad U_j = \frac{k_j}{2}(x_j - x_{j-1})^2,$$
$$K_j^* = \frac{M^2_{j}}{2\mu_j}, \quad U_j^* = \frac{\mu_j}{2}(r_j - r_{j+1})^2.$$

We note that the kinetic energy is symmetric with respect to the momentum $p_j$ or $r_j$, and the potential energy is symmetric with respect to the relative displacement $x_j - x_{j-1}$ or $s_j - s_{j+1}$. The condition of duality (equivalence) of two lattices $A$ and $B$ is

$$\cdots m_1 k_1^* = m_1 k_1, m_1 k_2^* = m_2 k_2, \cdots.$$

The simplest equivalence of dual lattices can be achieved by the replacement

$$\frac{1}{m_j} = k_j^*, \quad \frac{1}{k_j} = m_j^*,$$

$$p_j = s_j - s_{j+1}, \quad x_j - x_{j-1} = r_j.$$

2) Nonlinear lattice

The idea of the dual lattice was extended to nonlinear lattices [Toda 1966]. For simplicity we consider a uniform lattice with the kinetic energy term $K_j = P^2_j/2m$, and the potential energy term

$$U_j = \phi(x_j - x_{j-1}).$$

The corresponding dual lattice has the potential energy term $U_j^* = (s_j - s_{j+1})^2/2m$, and the kinetic energy term

$$K_j^* = \phi(r_j).$$

$\phi$ means nonlinear. $\hat{K}$ is a kind of kinetic energy with a mass which depends on the momentum $r_j$ of the lattice $B$. We see that the lattices $A$ and $B$ symbolically expressed as

$$\begin{align*}
\text{A):} & \quad \cdots K \hat{K} \hat{K} \cdots \\
\text{B):} & \quad \cdots U^* \hat{K} U^* \hat{K} U^* \cdots
\end{align*}$$

are equivalent (behave the same).

Because the lattice $B$ seemed simpler than the lattice $A$, I worked on $B$, and was able to find an integrable lattice [Toda 1967].
This lattice has the asymmetric potential

\[ \phi(x) = e^{-p - 1 + p} \]

It was found that this nonlinear lattice is perfectly integrable [Lunsford and Ford 1972, Hénon 1974, Flaschka 1974].

3) Lattice asymmetric in time and space

Recently I am interested in a lattice with asymmetric kinetic energy as well as asymmetric potential energy. This lattice can be expressed symbolically as

\[ \ldots \hat{R} \hat{U} \hat{R} \hat{U} \hat{R} \ldots \]

The dual lattice is the same to the original lattice (self-dual). Motion in this lattice is asymmetric in time and space. That is, the motion to the right is different from the motion to the left. This strange behavior comes from the asymmetricity of the kinetic energy term.

Further, it turned out that this lattice is equivalent to a system considered by Kac and Moerbeke [Kac and Moerbeke 1975] as a model for a certain stochastic process, and to a special case of the Lotka-Volterra model [Lotka 1956, Volterra 1931] for conflicting species, or environment problems.

References
8) M. Toda 1988: RIMS Kyoto Univ. in press.