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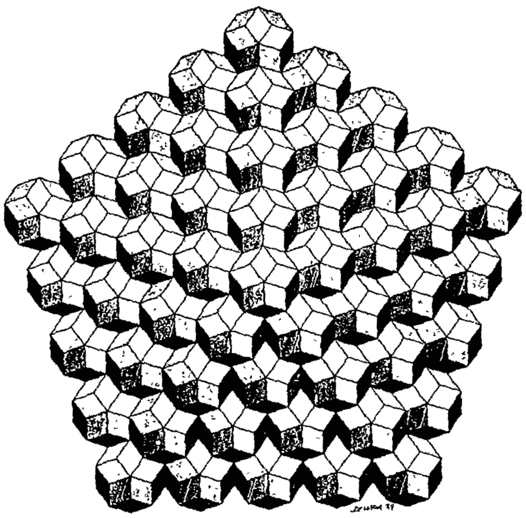
Symmetry of STRUCTURE

an interdisciplinary Symposium

Abstracts

II.



Edited by Gy. Darvas and D. Nagy

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hungary

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COLOURING REGULAR MAPS J. F. Rigby

This talk will be illustrated in colour with slides and OHP transparencies.

The five regular polyhedra can be blown up like balloons to form five regular maps on a sphere. For instance, the icosahedron is shown in Figure 1, and the icosahedral map (Figure 2) is made up of 20 spherical triangles meeting at 12 vertices; the map has 30 edges. The edges of a map do not need to be straight, and the faces do not need to be the same size; the regularity stems from the fact that each face has the same number of edges (three in the case of the icosahedral map) and each vertex is the meet of the same number of faces (five in the case of the icosahedral map), but we usually try to illustrate a regular map with as much visual regularity as possible. The icoshedron itself, with its 20 plane faces, illustrates the map just as well as the blown-up version on the sphere.

Regular maps on more complicated surfaces are more interesting; they can no longer be illustrated with their faces all of the same size. A famous simple example is the map on a torus with seven hexagonal faces, each face adjacent to all the others: take the rectangle in Figure 3 and glue the edge AE to DH thus forming a cylinder; then glue the two ends of the cylinder together, imagining it to be made of elastic material, so that A, B, C are glued to E, F, G. The resulting torus has a map of seven hexagonal faces meeting by threes at 14 vertices, and the map has 21 edges.

Another way of illustrating this map is by means of Figure 4: instead of cutting out and gluing the basic parallelogram (no longer a rectangle, but the material is elastic) we simply *identify hexagons labelled with the same number*. Think of the numbers as representing seven different colours; we then have, in Figure 4, a regular or "perfect" colouring (Grünbaum and Shephard 1987) of the hexagonal tessellation in a plane, which leads to the map on the torus when we identify hexagons of the same colour.

A word about regularity: if we apply the gluing process to the rectangle in Figure 5, we obtain a regular map of nine quadrangles (or squares) on a torus, but the map obtained from Figure 6 is not regular even though four squares meet at each vertex. The reason for this is that if we stand on square A and take a walk southwards we pass over two other squares (E and J) before arriving back at A; but if we walk eastwards we pass over three other squares (B, C and D) before arriving back at A. We shall not give a rigorous definition of regularity here (see Coxeter and Moser 1980), but this example shows that there is more to the idea than we mentioned earlier.

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Figure 7 shows a more complicated regular map: Coxeter's map {4,6|3}. Think of the figure as a solid object; the ten bars have triangular cross-section, and the bars meet by fours in the manner shown in Figure 8. The surface of the solid consists of 30 quadrangles meeting by sixes at 20 vertices, and there are 60 edges (Bokowski and Wills 1988). Can we illustrate this, as we did the hexagonal map on the torus, using a tessellation?

Figure 9 shows a regular tessellation in the hyperbolic plane, composed of regular quadrangles meeting six at each vertex. The hyperbolic plane may seem strange and alarming to those unfamiliar with it; suffice it to say that we can only illustrate it in a distorted manner, using the inside of a circle to depict the entire infinite hyperbolic plane. All the quadrangles in Figure 9 are regular and all have the same size in "hyperbolic reality", but in our picture they appear to get smaller as we approach the boundary. Now, imagine the 30 faces of Figure 7 to be labelled with the numbers 1 - 30; label the faces of Figure 9 so that the numbered faces fit together in the same way as in Figure 7 (compare Figure 8 with Figure 9 where only a few numbers have been inserted). Then the quadrangles of Figure 9 will be labelled with thirty numbers, or coloured with thirty colours, and when we identify quadrangles with the same colour we are led back to the regular map of Figure 7.

Thirty colours is a large number to use. Figure 10 shows a perfect colouring of the same tessellation in only five colours. If we examine it closely we find that the black tiles (for instance) are surrounded in six different ways by tiles of other colours. We say that two tiles are *equivalent* if they are surrounded in the same way; the colouring then contains 30 different types of inequivalent tile corresponding to the 30 numbers in Figure 9, and if we identify equivalent tiles we are led back once again to the regular map of Figure 7.

There are many interesting regular maps; most of them cannot be represented by plane faces in 3-space as in Figure 7. If a regular map has N faces, we can obtain it from a perfect colouring of a regular tessellation with N colours, but we can often reduce the number of colours to a divisor of N in the way just described. For instance, there is a colouring of the tessellation {3,8} in only ten colours that leads to a regular map with 604800 faces. The number of colours can sometimes be reduced even further if we use instead a semi-regular map.

Of more mathematical interest is the fact that these colourings can sometimes be used to investigate the symmetry group of a map. For instance, the colouring in five colours leading to the map {4,6|3} can be used to show that its symmetry group is $C_2 \times S_5$, and from the colouring in Figure 1 we can show that the symmetry group of the icosahedral map is $C_2 \times A_5$. There is a perfect colouring of {3,7} in seven colours, which

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leads to Klein's map $\{3,7\}_8$ and can be used to show that the symmetry group of this map is PGL(2,7), the group of collineations and correllations of the projective plane of seven points and lines.

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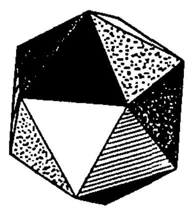

Figure 1

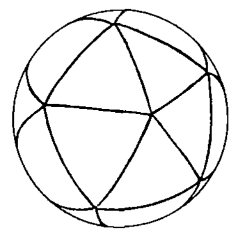

Figure 2

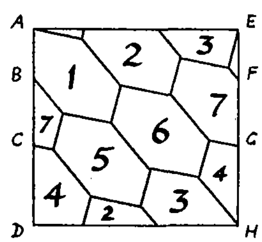

Figure 3

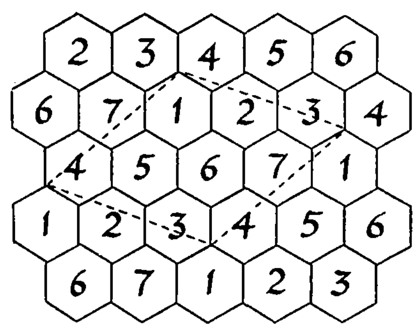

Figure 4

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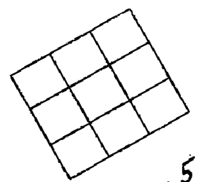

Figure 5

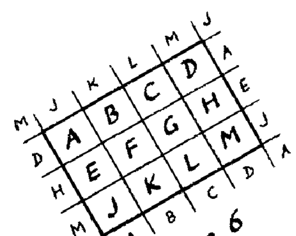

Figure 6

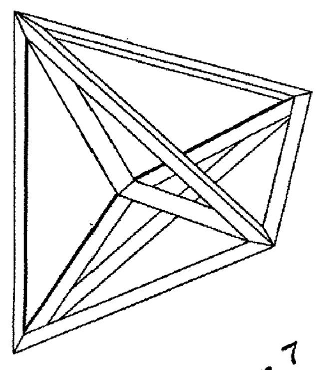

Figure 7

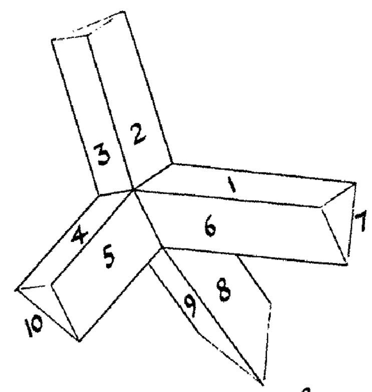

Figure 8

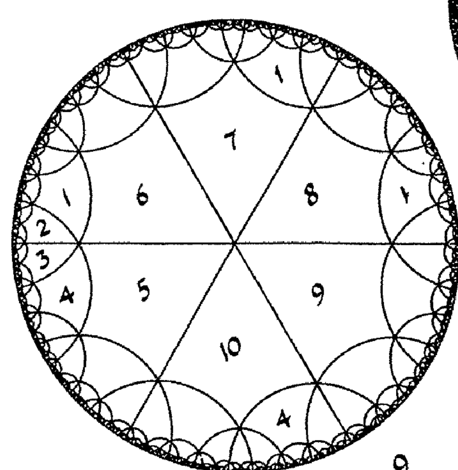

Figure 9

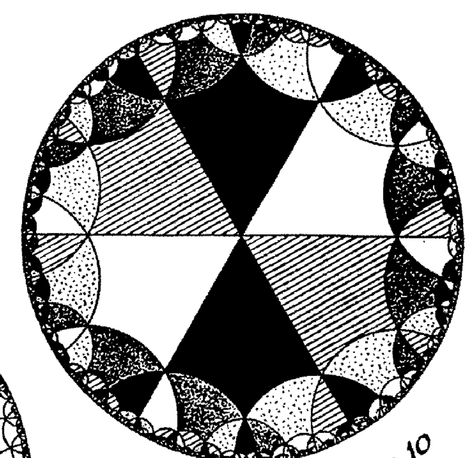

Figure 10