Symmetry of STRUCTURE

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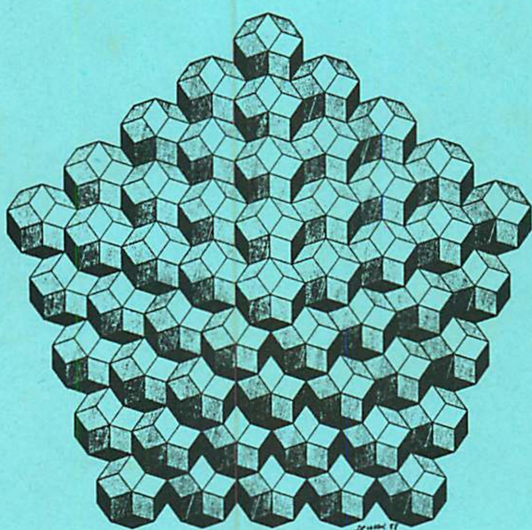
Abstracts

I.



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SIMPLE AND MULTIPLE ANTISYMMETRY

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As an intuitive concept, antisymmetry is present from the very beginnings of ornamental art, appearing with Neolithic "black-white" ceramics (Figure 1). As a scientific concept, it is an acquirement of the 20th century mathematics (Heesch, 1929, 1930).

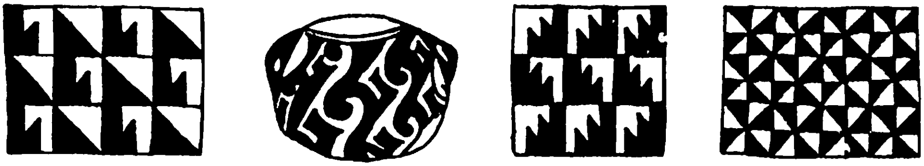

Figure 1. Antisymmetry groups of ornaments in Neolithic art: (a) p2/p1, Hacilar, ≈5200 B.C.; (b) p2/p1, Rahmani, ≈4000 B.C.; (c) pmg/pg, Hacilar, (d) p4m/p4g, Hajji Mohamad, ≈5000 B.C.

Let a symmetry group G and the permutation group $P=C_2$ generated by the antiidentity transformation $e_1=(0\ 1)$ satisfying the relation $e_1{}^2=E$ and commuting with all the elements of the group G, be given. If $S\in G$, $S^1=e_1S=Se_1$ is the antisymmetry transformation derived from S. Every group G^1 derived from G, which contains at least one antisymmetry transformation is called the antisymmetry group, and the group G is called its generating group. All the antisymmetry groups derived from G, consisting of a family, can be divided into the two types: senior groups of the form $G\times C_2$ and junior groups $G^1\cong G$. Every junior antisymmetry group G^1 is uniquely defined by its group/subgroup symbol G/H, where H is the symmetry subgroup of G^1, $G/H\cong C_2$ and $[G:H]=2$ (Shubnikov, Koptsik, 1972).

For denoting symmetry group categories, the Bohm symbols G_r... are used (Bohm, Dornberger-Schiff, 1966). Every category of symmetry groups of the space E^r is defined by the sequence r... of maximal invariant (sub)spaces inserted into one another in succession.

The antiidentity transformation introduced gives different possibilities for its interpretation. The first and most natural was a color-change "black"-"white", introducing in ornamental art a space component: visual representation of 3-D symmetry groups (e.g. symmetry groups of bands G_{321} or layers G_{32}) in a 2-D plane using black-white diagrams (Weber diagrams) of antisymmetry groups of friezes G_{21}^{1} and ornaments G_{2}^{1}. Namely such a interpretation was the origin of the theory of antisymmetry. Its mathematical generalization, the established relation between the antisymmetry groups $G_{r...}^{1}$ and symmetry groups $G_{(r+1)r...}$ of the (r+1)-dimensional space was introduced by H.Heesch for the derivation of four-dimensional symmetry groups G_{430} and for an approximate valuation of the number of symmetry groups G_{43}.

In a general sense, the antiidentity transformation e_1 can be interpreted as a change of any bivalent geometric or non-geometric property commuting with symmetries of the generating symmetry group G (e.g. (+ −), (S N), (yes no), (convex concave)...).

The natural extension of the (simple) antisymmetry is the multiple antisymmetry, introduced by A.M.Zamorzaev in 1957, where besides a generating symmetry group G we have the permutation group $P=C_2^{l}$ generated by l antiidentity transformations e_i (i=1,2,...,l) satisfying the relations $e_i^2=E$, commuting between themselves and with all elements of G. In a similar way, we have the senior (S^k-), middle (S^kM^m-) and junior (M^m-type) multiple antisymmetry groups, where only the last ones, isomorphic to G, are non-trivial in the sense of derivation.

During the 30 years, mostly by contribution of the Kishinev school, the theory of multiple antisymmetry has become an integral part of mathematical crystallography and acquired the status of a complete theory extended to all categories of isometric symmetry groups of the space E^r ($r \leq 3$), different kinds of non-isometric symmetry groups (of similarity symmetry, conformal symmetry...) and P-symmetry groups ((p)-, (p')-, ($p2$)-symmetry groups) (Zamorzaev, 1976; Zamorzaev, Galyarskij, Palistrant, 1978; Zamorzaev, Palistrant, 1980; Zamorzaev, Karpova, Lungu, Palistrant, 1986). The most important results from that period are: the derivation of the 1191 junior G_3^1, 9511 G_3^2, G_2^l and G_{32}^l (Zamorzaev, 1976). However, some problems (e.g. the derivation of G_3^l at $l \geq 3$), because of a large number of the multiple antisymmetry groups exceeding even possibilities of computers, remain unsolved.

This and many other problems are solved by the use of antisymmetric characteristic (AC) of a discrete symmetry group G (Jablan, 1986). Let a discrete symmetry group G be given by its presentation (Coxeter, Moser, 1980). The groups of simple and multiple antisymmetry can be derived by applying the general method of Shubnikov-Zamorzaev, i.e. by replacing the generators of the group G with antigenerators of one or several independent kinds of antisymmetry.

<u>Definition</u> 1: Let all products of generators of a group G, within which every generator participates once at the most, be formed and then subsets of transformations equivalent with regard to symmetry, be separated. The resulting system is called the antisymmetric characteristic of the group G ($AC(G)$).

A majority of AC permit the reduction, i.e. a transformation into the simplest form. The method for obtaining *AC* and reduced *AC* can be illustrated by example of symmetry group of ornaments pm, given by the set of generators $\{a,b\}(m)$, with the $AC(\mathrm{pm})=\{m,ma\}\{b\}\{mb,mab\}\{a\}\{ab\}$ and reduced $AC(\mathrm{pm})=\{m,ma\}\{b\}$.

Theorem 1: Two groups of simple or multiple antisymmetry G' and G'' of the M^m-type for fixed m, with common generating group G, are equal iff they possess equal *AC*.

Every $AC(G)$ completely defines the series $N_m(G)$, where by $N_m(G)$ is denoted the number of groups of the M^m-type derived from G, at fixed m $(1\leq m\leq l)$. For example, $N_1(\mathrm{pm})=5$, $N_2(\mathrm{pm})=24$, $N_3(\mathrm{pm})=84$.

Theorem 2: Symmetry groups possessing isomorphic *AC* generate the same number of simple and multiple antisymmetry groups of the M^m-type for every fixed m $(1\leq m\leq l)$, which correspond to each other with regard to structure.

Corollary: The derivation of all simple and multiple antisymmetry groups can be completely reduced to the construction of all non-isomorphic *AC* and derivation of simple and multiple antisymmetry groups of the M^m-type from these *AC*.

By the use of the *AC*-method and notion of the *AC*-type, the 109139 G_3^3, 1640955 G_3^4, 28331520 G_3^5 and 419973120 G_3^6 multiple antisymmetry groups of the M^m-type, are derived (Jablan, 1987).

The *AC*-method can be also used for a derivation of (P,l)-symmetry groups from *P*-symmetry groups. Let G^P be a junior group of *P*-symmetry derived from G (Zamorzaev, Galyarskij, Palistrant, 1978; Zamorzaev, Karpova, Lungu, Palistrant, 1986). By replacing in Definition 1 the term "transformations equivalent with regard to symmetry" with a more general notion "transformations equivalent with regard to *P*-symmetry", the transition from G to G^P induces the transition from $AC(G)$ to $AC(G^P)$, making possible the derivation of groups of (P,l)-symmetry of the M^m-type by the use of the *AC*-method.

The derivation of (P,l)-symmetry groups of the M^m-type from *P*-symmetry groups by use of the *AC*-method can be reduced to a series of successive transitions

$$G\longrightarrow G^P\longrightarrow G^{P,1}\longrightarrow\ldots\longrightarrow G^{P,l}$$

and induced transitions

$$AC(G)\longrightarrow AC(G^P)\longrightarrow AC(G^{P,1})\longrightarrow\ldots\longrightarrow AC(G^{P,l}).$$

Every induced *AC* consists of the same number of generators. Since every transition $G^{P,k-1}\longrightarrow G^{P,k}$, $(1\leq k\leq l)$, is a derivation of simple antisymmetry groups using $AC(G^{P,k-1})$, for the derivation of all multiple antisymmetry groups the catalogue of all non-isomorphic *AC* formed by l generators and simple antisymmetry groups derived from these *AC* is completely sufficient.

One of the most important results, obtained jointly with A.F.Palistrant, is the derivation of the junior M^m-type groups $G_3^{l,p}$ from the groups G_3^p $(p=3,4,6)$: 4840(4134) $G_3^{1,p}$, 40996(29731) $G_3^{2,p}$, 453881(260114) $G_3^{3,p}$, 5706960(2048760) $G_3^{4,p}$ and 59996160(1249920) $G_3^{5,p}$, where the numbers of complete (p,l)-symmetry are given in parentheses.

Finaly, the use of such a generalized AC makes possible the reduction of the theory of multiple antisymmetry to the theory of simple antisymmetry. The basis of this reduction is the transition $G \longrightarrow G^p$ and induced transition $AC(G) \longrightarrow AC(G^p)$, where $AC(G)$ and $AC(G^p)$ consist of the same number of generators. This means that every step in the derivation of multiple antisymmetry groups: $G \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \ldots \longrightarrow G^{k-1} \longrightarrow G^k \longrightarrow \ldots \longrightarrow G^l$, i.e. the transition $G^{k-1} \longrightarrow G^k$, $(1 \leq k \leq l)$, is a derivation of simple antisymmetry groups by the use of $AC(G^{k-1})$, followed by the induced transition $AC(G^{k-1}) \longrightarrow AC(G^k)$, $(1 \leq k \leq l-1)$. All the AC of the induced series consist of the same number of generators.

The said can be illustrated by example of the derivation of multiple antisymmetry groups from the symmetry group of ornaments pm: $\{a,b\}(m)$ with the AC: $\{m,ma\}\{b\} \cong \{A,B\}\{C\}$. At $m=1$, the five junior simple antisymmetry groups, are obtained:

$\{A,B\}\{C\}$ $\{E,E\}\{e_1\} \longrightarrow \{A,B\}\{C\}$
$\{e_1,e_1\}\{E\} \longrightarrow \{A,B\}\{C\}$
$\{e_1,e_1\}\{e_1\} \longrightarrow \{A,B\}\{C\}$
$\{E,e_1\}\{E\} \longrightarrow \{A\}\{B\}\{C\}$
$\{E,e_1\}\{e_1\} \longrightarrow \{A\}\{B\}\{C\}$.

In the first three cases AC remains unchanged, but in two other cases AC is transformed into the new AC: $\{A\}\{B\}\{C\}$. To continue the derivation of multiple antisymmetry groups of the M^m-type from the symmetry group pm, only the derivation of simple antisymmetry groups from the AC: $\{A\}\{B\}\{C\}$ is indispensable. This AC is trivial and gives the seven groups of simple antisymmetry. If the AC: $\{A,B\}\{C\}$ is denoted by 3.2 and AC: $\{A\}\{B\}\{C\}$ by 3.1, then the result obtained can be denoted in a symbolic form by $3.2 \longrightarrow 2(3.1)+3(3.2)$. Knowing that $N_1(\text{pm})=N_1(3.2)=5$, $N_1(3.1)=7$, we can simply conclude that $N_2(\text{pm})=24$ and $N_3(\text{pm})=84$ (Yablan, 1988).

So, after the 30 years we are coming back to the roots of the theory of multiple antisymmetry – to the simple antisymmetry, but knowing today some more about the first.

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