Symmetry of STRUCTURE

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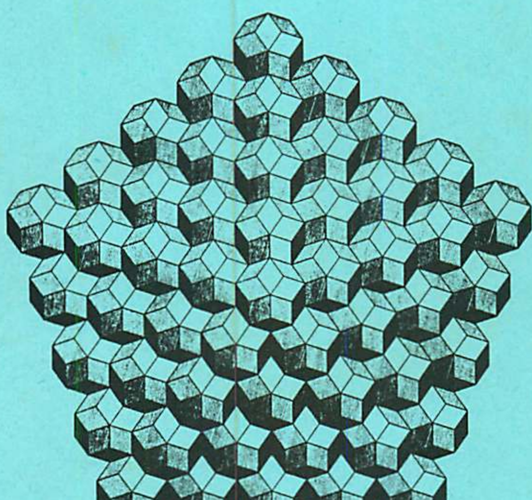
Abstracts

I.



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ON 3-PERIODIC MINIMAL SURFACES. II. TOPOLOGICAL PROPERTIES.

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Genus and Euler characteristic

A non-periodic surface in R^3 is said to be of **genus g** if it may topologically be deformed to a sphere with g handles. For 3-periodic minimal surfaces a modified definition must be used (cf. Schoen, 1970) counting only the number of handles per primitive unit cell. In other terms, the 3-periodic surface is embedded in a (flat) 3-torus T^3 to get rid of all translations, and then the conventional definition of the genus may be applied. This procedure corresponds to identifying the opposite faces of a primitive unit cell.

The genus of a 3-periodic minimal surface may be calculated in different ways, two of which will be discussed in the following:

(1) Labyrinth graphs: Each 3-periodic minimal surface without self-intersection subdivides R^3 into two infinite regions, called labyrinths, which are connected but not simply connected. Schoen (1970) proposed to represent the labyrinths by graphs in the following way: Each labyrinth graph is entirely located within its labyrinth; each branch of a labyrinth contains an edge of its graph; each circuit of one labyrinth graph encircles at least one edge of the other graph.

Any of the two labyrinth graphs may be used to represent topological properties of the surface. As each circuit of the graph corresponds to a handle of the surface the number of circuits per primitive unit cell may be counted to get the genus of the surface. In case of a **minimal balance surface** (intersection-free 3-periodic minimal surface that subdivides R^3 in two **congruent** labyrinths; cf. Fischer & Koch, 1987) the symmetry is best described by a group-subgroup pair G-H of space groups with index 2, and the genus has to refer to a primitive unit cell of the subgroup H. There exist two different possibilities to derive the genus with the aid of labyrinth graphs:

(a) In modification of a procedure proposed by Hyde (1989), a connected subgraph containing no translationally equivalent vertices may be separated from a labyrinth graph. Then the genus of the surface may be calculated as

$$g=\frac{p}{2}+q,$$

where p is the number of edges connecting the finite subgraph to the rest of the infinite labyrinth graph, and q is the number of edges that has to be omitted to make the subgraph simply connected. As p equals at least 6 the genus of a 3-periodic minimal surface without self-intersection is at least 3.

(b) Keeping in mind the embedding of the minimal surface in the torus T^3 a more crystallographic formula for the genus may be derived. g equals the difference between the number e of edges in the embedded labyrinth graph and the number e_s of edges in any simply connected subgraph with the same number v of vertices. With

$$e=\sum_i m_i e_i/2 \qquad \text{and} \qquad e_s=v-1=\sum_i m_i-1$$

it follows:

$$g=e-e_s=1+\sum_i m_i(e_i/2-1).$$

Here m_i means the multiplicity of the i-th kind of vertices referred to a primitive unit cell of H, and e_i is the number of edges meeting in this vertex. The summation runs over all kinds of symmetrically equivalent vertices of the labyrinth graph.

Details on the labyrinth graphs and the genera of the known minimal balance surfaces are tabulated by Fischer & Koch (1989c).

(2) Euler characteristics: An intersection-free surface in R^3 may also be characterized by a number χ, its Euler characteristic. χ is related to g by

$$g=1-\chi/2.$$

The Euler characteristic of an intersection-free surface may be derived in a simple way by defining a tiling on the surface, i.e. by subdividing the surface into tiles (disk-like surface patches). For such an arbitrary tiling the equation

$$\chi=f-e+v$$

holds, where f, e and v are the numbers of tiles (faces), edges and vertices, respectively, in the tiling. For a 3-periodic surface the tiling must be compatible with the translations of the surface, and the tiles, edges and vertices have to be counted per primitive unit cell of H (cf. Fischer & Koch, 1989c).

For a minimal balance surface generated from **disk-like-surface patches** spanned by skew polygons of 2-fold axes (cf. Fischer & Koch, 1987; Koch & Fischer, 1988) these surface patches may be used as tiles. Then χ may be calculated as

$$\chi=f(1-e_p/2)+\sum_i v_i,$$

where e_p is the number of edges of such a skew polygon, f is the number of skew polygons and v_i the multiplicity for the i-th kind of symmetrically equivalent vertices. f and v_i are both referred to a primitive unit cell of H.

If a minimal balance surface consists of **catenoid-like surface patches** spanned between parallel plane nets of 2-fold axes (Koch & Fischer, 1988) its Euler characteristic is given by

$$\chi=v_N-e_N,$$

where v_N and e_N refer to the plane nets of 2-fold axes. v_N means the number of vertices, e_N the number of edges counted for all nets of 2-fold axes and per primitive unit cell of H. Making use of the relation $f_N-e_N+v_N=0$ for nets, the genus may be calculated from χ as

$$g=k+1$$

where $k=f_N/2$ gives the number of catenoids per primitive unit cell of H. The same formulae hold for minimal balance surfaces generated from **infinite strips** spanned between plane nets of 2-fold axes (Fischer & Koch, 1989b). Then k must be understood as the number of original catenoids (per primitive unit cell of H) that have been united to infinite rows.

Similar formulae have been derived (Fischer & Koch, 1989c) for three other kinds of minimal balance surfaces spanned also between parallel plane nets of 2-fold axes. Here k means the number of surface patches per primitive unit cell of H.

If a minimal balance surface is made up from **multiple catenoids** (Koch & Fischer, 1989a; Karcher, 1988)

$$g=km+1$$

holds. m gives the number of catenoids that must be united to form one multiple catenoid.

The genus of a minimal balance surface built up from **branched catenoids** (Fischer & Koch, 1989a) is given by

$$g=\frac{k(1+b)}{2}+1.$$

b is the number of branches at one of the ends of a branched catenoid.

If a minimal balance surface may be generated from **catenoids with s spouts attached** (Koch & Fischer, 1989b), its genus may be calculated as

$$g=ks+1.$$

Flat points

For each point of a minimal surface the defining condition

$$K_1+K_2=0$$

must be fulfilled, where K_1 and K_2 are the main curvatures in that point. Normally $K_1=-K_2\neq0$ holds, i.e. the point is a saddle point. For exceptional points, however,

$$K_1=K_2=0$$

may be fulfilled. Such points are called **flat points** of the surface. In contrast to normal saddle points, the surrounding of a flat point shows n valleys separated by n ridges ($n\geq3$). The simplest example with n=3 is the "monkey saddle".

For any point on an intersection-free minimal surface its degree of flatness may be characterized by an integer number ß, called its **order**. The order of a point P_o with normal vector $\vec{n}_o$ can be derived as follows: A second point P with normal vector $\vec{n}$ is moved on the surface around P_o. If P_o is a normal point, $\vec{n}$ rotates once around $\vec{n}_o$ during one revolution of P. If, however, P_o is a flat point, $\vec{n}$ rotates more than once (e.g. p times) around $\vec{n}_o$ per revolution of P. Then the **order** ß of P_o is defined as

$$\beta=p-1.$$

Accordingly, a normal point has order $\beta=0$, and the order of a flat point may equal any positive integer. For 3-periodic minimal surfaces flat-point orders up to $\beta=4$ have been observed so far. The number n of valleys (or ridges) in the surrounding of a flat point is

$$n=\beta+2.$$

Each order of a (flat) point corresponds to a maximal site symmetry of such a point. This symmetry is $\bar{4}m2$ for $\beta=0$, $\bar{3}m$ for $\beta=1$, $\bar{8}m2$ for $\beta=2$, $\bar{5}m$ for $\beta=3$ and $\overline{12}m2$ for $\beta=4$. Therefore, most site symmetries of points on a minimal surface enforce the existence of a flat point. Conversely, only points with site symmetry $\bar{4}$, 222, 2mm, 2, m, or 1 can be non-flat points of a minimal surface.

There exists a relation between the genus of an intersection-free minimal surface and the order of its flat points (cf. Hyde, 1989; Hopf, 1983).

$$g=1+\frac{1}{4}\sum_i \beta_i .$$

The sum runs over all flat points within a primitive unit cell of H. This formula can be used in different ways:

(1) If all flat points with their orders are known the genus of a minimal surface may be calculated.

(2) If the symmetry of a minimal surface and its genus are known the relation between flat-point symmetry and flat-point order in combination with the above formula may be used to derive a complete list of flat points (cf. Koch & Fischer, 1989c).

References

Fischer, W. & Koch, E.: Z. Kristallogr. **179**, 31-52 (1987).
Fischer, W. & Koch, E.: Acta Crystallogr. **A45**, 166-169 (1989a).
Fischer, W. & Koch, E.: Acta Crystallogr. **A45** (in the press) (1989b).
Fischer, W. & Koch, E.: Acta Crystallogr. **A45** (submitted) (1989c).
Hopf, H.: Springer Lecture Notes in Mathematics No. 1000 (1983).
Hyde, S.T.: Z. Kristallogr. (in the press) (1989).
Koch, E. & Fischer, W.: Z. Kristallogr. **183**, 129-152 (1988).
Koch, E. & Fischer, W.: Acta Crystallogr. **A45**, 169-174 (1989a).
Koch, E. & Fischer, W.: Acta Crystallogr. **A45** (in the press) (1989b).
Koch, E. & Fischer, W.: Acta Crystallogr. **A45** (submitted) (1989c).
Schoen, A.H.: NASA Technical Note D-5541 (1970).