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Abstracts

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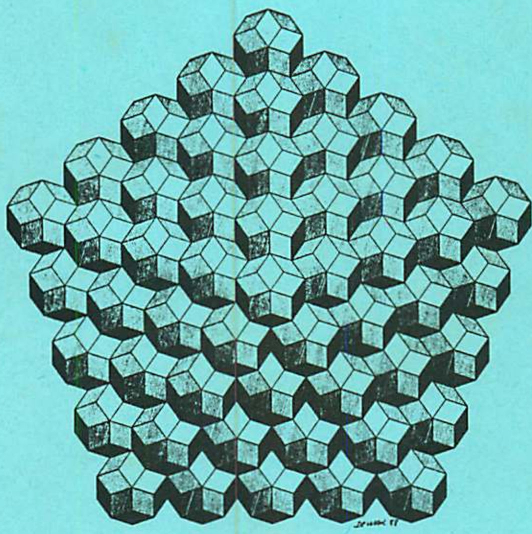

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CRYSTALLOGRAPHIC STRUCTURES AND SPACE PARTITIONS: NEW TRENDS OF THE THEORY

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1. LOCAL THEORY OF REGULAR POINT SYSTEMS AND SPACE PARTITIONS.

What convex polytopes can tile Euclidean space in a regular way? What way can we enumerate all their combinatorial types ?

Answers to these principal questions are related very closely with important results obtained by collaboraters of Delone's geometrical school in the middle of 1970-s.

First of all let's remember the definition of the *regular system* (crystallographic monostructure) as a point set S in the d-dimensional Euclidean space $\mathbf{R}^d$ which fulfils next conditions:

(i) The point set S is Delone (r,R)-system. That's here r is a lower limit of distances between any two points of S and R is such a value that any ball of radius R contains at least one point of S.

(ii) The point set S "looks" the same if seen from every point of S.

The first condition is of a general kind and is fulfilled in a rather wide class of point sets. It's important for instance in the theory of amorphous maters. The second one characterises the class of regular point systems only. Indeed that condition predetermines the set S to be a point orbit with respect to some Fedorov space group.

The condition (ii) of a regularity has a "global" sense: *infinite* sets of *all* straight line segments, drawn from any point of system S to its remaining points, must be congruent. Therefore it was naturally trying to find some "local" substitution of that condition. In 1974 a problem of such a kind was supposed by Delone and Galiulin. Soon the grounding answer was found [1]:

Theorem 1. For a given dimension d there exists number c_d (it can be calculated effectively) such that if for every point of the (r,R)-system S its neighbourhood of radius c_dR is the same, then the system S is regular one.

The main idea of a proof of the theorem (Dolbilin & Stogrin) consists in using of the fact that a sequence of finite groups of neighbourhoods of fixed point, when they are being extended, is stabilizing. As further investigations have shown the notion of the stable group had been introduced in [1] is very essential for that problem. Here we are thinking about the problem of a determination of regular systems by means of a comparison of point neighbourhoods of the discrete system with themselves. An underestimation of this fact reduces to an appearance of probable confirmations but for the time been unfounded ones, for instance that $C_3 \leqslant 6$.

It's clear that the value of c_d can't be taken too small. The main idea of a proof of the Theorem 1 allows to express it in the next more concrete form:

Theorem 1a. Let S be a (r,R)-system. Assume that for any points A, $A' \in S$ the sets of straight segments, drawn from points A and A' respectively, are congruent within a finite sphere of radius $(\nu + 2)2R$. Here ν is the number of prime factors in order of group H_0, where $.H_0$ is the complete group of rotations around the point A (or point A' no importance) moving the point subset of the set S laying inside a sphere of radius $2R$ and with its center at the A (or A' respectively).

The final result in the case $d = 2$ belongs to Stogrin. He has proved that $c_2 = 4$ and the value can't be diminished. The analogous result but under more weak assumption as compared to the condition of congruency has been obtained by Dolbilin:

Let S be a (r,R)-system in the plane, and for $A \in S$ let $S_A(\rho) \subset S$ be a set of all points $A' \in S$ with $dist(A,A') \leqslant \rho$. Let also $N_A(\rho) = \{ dist(A',A''),\ A'A'' \in S_A(\rho) \}$, that is, the number set $N_A(\rho)$ is the set of all distances between each two points of $S_A(\rho)$.

Let's now assume that the (r,R)-system S is such that for each their two points A and $B \in S$ the sets of distances $N_A(\rho)$, $N_B(\rho)$ coincide:

$$N_A(\rho) \equiv N_B(\rho)$$

for any ρ with $0 \leqslant \rho \leqslant 2R$. Then the system S is regular.

This at the first vue very dificulte condition indeed is much weeker than the condition of congruency of neighbourhoods $S_A(4R)$ and $S_B(4R)$, which as it's enough easily to see implies the condition $N_A(\rho) \equiv N_B(\rho)$ for each $\rho \in [0, 4R]$. The controverse affirmation isn't trivial and its proof is based on a revise of monstrous number of combinations and we haven't full assurance in their completness.

Let $d = 3$. With the help of one Stogrin's lemma it's not dificulte to prove that we can take as a value of c_3 a number 14. It was more dificulte to prove that it has been proved that $c_3 \leqslant 10$. P.Engel [6] gave an example of the partition disproving the probable conjecture ($c_3 = 4$) and demonstrating $c_3 > 4$.

Recently **there was** announced the generalization of the local theorem for any crystallographic structure, that's, for multiregular systems but not only simple regular ones (Dolbilin & Stogrin [5]). The results have been received by authors it has to say more ten years ago.

On a base of the idea of a stabilization of finite symmetry groups and with the help of topological reasonnings similar to Poincare's ones used by him in his works on the theory of automorphic functions, there has been obtained a geometrical criterion of the convex polytope being a stereotope (Dolbilin [3]). Let's remind that a *stereotope* is called a convex polytope being a tile of some regular partition with transitively acting on it Fedorov group.

Let P be a convex polytope and $C_1, C_2, \dots, C_n$ be its coronas (if there exists they) consisted of congruent to P polytopes (for the definition of corona see, for instance [6]).

Let also $H_0 \supseteq H_1 \supseteq H_2 \supseteq \dots \supseteq H_n$ be a sequence of symmetry groups of the polytope P and of its following envelopes

$$E_1 = P \cup C_1,\ E_2 = P \cup C_1 \cup C_2, \dots,\ E_n = P \cup C_1 \cup C_2 \cup \dots \cup C_n$$

respectively.

Theorem 2 (the criterion of a stereotope). Assume that for a given convex polytope P there exists such a sequence of its coronas $C_1, C_2, \dots, C_n$, which fulfils next condition:

in the corresponding sequence of symmetry groups the first equality occurs at the n-th step of the chain

$$H_0 \supset H_1 \supset \dots \supset H_{n-1} = H_n \quad n=1,2,\dots$$

and for each polytope $P_i \in C_1$ there exists such an isometry g_i that $P \xrightarrow{g_i} P_i$ and the intersection $E_n \cap g_i E_n$ of the complex E_n and its image $g_i E_n$ with respect to g_i represents a subcomplex of the complex E_n. Then the polytope P is a stereotope.

It's evident that the conditions are necessary. The proof of their sufficiency isn't an easy problem. It's important to point out that an upper bound for the number n of coronas depends of the estimate for the coefficient c_d in the Theorems 1 and 1a. In particular for $d = 3$ we may set the next limit: $n < 5$. It's too probab that this limit can be diminished to 2, but at present this conjecture hasn't been settled yet. Also we note that the sence of the theorem is similar to Venkov's criterion of polytope, rediscovered recently by McMullen.

From this fundamental theorem with the help of the well-known Tarskii theorem it follows an existence (together with its description) of an algorithm listing for each given dimension d all combinatorial types of d-dimensional stereotopes (Dolbilin [4]). So far there was a similar algorithm for a listing only of types of Diriclet-Voronoi stereotopes (Delone & Sandakova [2]).

2. MONOTYPIC SPACE PARTITIONS.

Let's go over to a consideration of more general space partitions. We shall consider a face-to-face and normal partition with convex tiles on condition that all tiles of a given partition are isomorphic polytopes. We call such partitions *monotypic*.

Let's remind that a partition $\mathcal{T}$ is called *normal* if there exist two positive numbers ρ_1 and ρ_2 such that for each tile $T \in \mathcal{T}$ it holds that $B_1 \subset T \subset B_2$, where B_1 and B_2 are suitable balls of radius ρ_1 and ρ_2 respectively. For instance, tiles of normal partitions of the Euclidean plane can be only 3-, 4-, 5- and 6-gons.

An investigation of monotypic partitions is connected with the interesting conjecture of McMullen which says that a tile (a facet) of interesting conjecture of McMullen which says that a tile (a facet) of any monotypic d-dimensional sphere partitions has the same type as a tile of some monotypic d-dimensional space (euclidean) partition.

In the case $d = 2$ the McMullen conjecture is true evidently, for $d \geqslant 3$ it hasn't been settled yet. We point out that an extremely interesting cycle of works in this field belongs to E. Schulte.

In particular due to one simple idea which was expressed and used as far às we know for the first tyme by Danzer and Schulte, one can construct a wide class of monotypic partitions in the d-dimensional Euclidean space with the help of monotypic partitions of the d-dimensional sphere S^d of a special kind.

The matter is that if a monotypic partition of S^d contents at least one primitive vertex, that's $(d+1)$-valent one, then there exists a monotypic partition of R^d with tiles of the same combinatorial type. The construction of the mentioned partition is realized for two steps. At first with the help of a special projection almost all tiles of the sphere partition are projectered onto a d-dimensional simplex T^d yielding a face-to-face tiling of T^d. At the second step the simplex T^d together with its tiling is mapped affinely onto a simplex T' which is a fundamental simplex of a Fedorov group generated with reflections, that's some Coxeter space group. The Coxeter group extends the local tiling of T' through the space. Due to the reflections in faces of fundamental simplex T' the local tilings in adjoining images of T' are connected one with another one by a face-to-face fashion.

In due time there was obtained a rather rich class of regular 3-dimensional sphere partitions (particularly these results were exposed in Dolbilin's dissertation). Not having here a possibility to expose them we formulate just two thesis bearing relation to monotypic partitions:

1. A combinatorial type of 3-dimensional space monotypic tile can have very complicated structure, for instance, with any large number of faces. In particular n-gon prisme is a 3-dimensional tile for any n.

2. An infinite series of 3-dimensional tiles with infinitely increasing quantity of faces has been found. An interest in the series is determined by unnexpected its tie with such geometry number objects as Klein's and Newton's polygons in the 2-dimensional lattice. In particular, structures of these polygons determine the structures of the tiles of the mentioned series by a constructive and one-to-one way.

[1] Delone B.N., Dolbilin N.P., Stogrin M.I., Galiulin R.V.. Doklady Akad. Nauk SSSR = Soviet Math. Dokl. 17 (1976) 319-322.
[2] Delone B.N., Sandakova N.N.. Trudy Math. Inst. Akad. Nauk SSSR V.A. Steklova, 64 (1961).
[3] Dolbilin N.P.. Doklady Akad. Nauk SSSR = Soviet Math. Dokl. 17 (1976) 1333-1337.
[4] Dolbilin N.P.. Short Communications of the Intern. Math. Congress, p.III. Warsaw, (1982) (in Russian).
[5] Dolbilin N.P., Stogrin M.I.. Abstracts of IX All Soviet Union Geometrical Conference. Kishinev, (1988) (in Russian).
[6] Engel P.. Geometric Crystallography. D. Reidel P. Co., Dordecht(1986)